The Shell Element Method for Laplace's Equation

In this document we consider the solution method of the Laplace problems exterior to *thin* discontinuities or 'shells' by the *shell element method* which is viewed as an extension to the traditional boundary element method[1]. For example this technique has been for example to model capacitors[2] (Laplace) acoustic shields[3] (Helmholtz) and has been used to extend the boundary element method for Laplace[4] and Helmholtz[5] problems.

The purpose is to solve the boundary-value problem[6] consisting of the two-dimensional Laplace equation[7]

$$\nabla^2 \varphi(\boldsymbol{p}) = 0 \qquad (\boldsymbol{p} \in E) \tag{1a}$$

in the domain E exterior to an open boundary H with a Robin boundary condition of the form

$$\mathrm{a}(\mathbf{p})\,\delta(\mathbf{p}) + \mathrm{b}(\mathbf{p})\,\nu(\mathbf{p}) = \mathrm{f}(\mathbf{p}) \tag{1b}$$

and

$$\mathrm{A}(\mathbf{p})\,\Phi(\mathbf{p}) + \mathrm{b}(\mathbf{p})\,\mathrm{V}(\mathbf{p}) = \mathrm{F}(\mathbf{p}) \tag{1c}$$

for $\mathbf{p} \in H$. Where, in equations (1b) and (1c), we are using the notation in Integral Equation Formulation for Laplace's Equation surrounding thin shells[8], an 'upper' and 'lower' surface is defined and in which $\delta(\mathbf{p})$ and $\Phi(\mathbf{p})$ are the difference in and average potential for corresponding points on the upper and lower surfaces. Similarly, $\nu(\mathbf{p})$ and $\mathrm{V}(\mathbf{p})$ are the difference and average for the normal derivative of the potential.

In order to apply the boundary element method the boundary is approximated by a set of n_H panels[9]

$$H \approx \tilde{H} = \sum_{j=1}^{n_H} \Delta \tilde{H}_j$$

and the boundary functions are approximated or represented by a constant value on each panel[10]. The integral equations within the boundary element method are solved by collocation[11]. By approximating the operators[12] the boundary integral equations are reduced to a linear system of equations[13]. The resulting linear system of equations is solved in order to find the solution on the boundary and this is used in turn in order to find the solution in the exterior domain.

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The relevant boundary integral equations are[7]

$$\Phi(\mathbf{p}) = \varphi^{i}(\mathbf{p}) + \{M\,\delta\}_H(\mathbf{p}) - \{L\,\nu\}_H(\mathbf{p}) \quad (\mathbf{p}\in H),$$

$$V(\mathbf{p}) = v^{i}(\mathbf{p}) + \{N\,\delta\}_H(\mathbf{p}) - \{M^t\,\nu\}_H(\mathbf{p}) \quad (\mathbf{p}\in H),$$

where L, M, M^t and N are the Laplace integral operators[4] $\varphi^{i}(\mathbf{p})$ is the possible incident potential and $v^{i}(\mathbf{p})$ is its normal derivative.

The application of the collocation method to these equations, that is applying the equation to every collocation point **p** on H gives the following linear systems of equations:

$$\underline{\hat{\Phi}}_H = \underline{\varphi}_H{}^{i} + \mathrm{M}_{HH}\underline{\hat{\delta}}_H - \mathrm{L}_{HH}\underline{\hat{\nu}}_H,$$

$$\underline{\hat{V}}_H = \underline{v}_H{}^{i} + \mathrm{N}_{HH}\underline{\hat{\delta}}_H - \mathrm{M}^t_{HH}\underline{\hat{\nu}}_H,$$

where $\underline{\varphi}_H{}^{i}$ and $\underline{v}_H{}^{i}$ list the incident potential and its normal derivative at the collocation points and $\underline{\hat{\Phi}}_H, \underline{\hat{V}}_H, \underline{\hat{\delta}}_H$ and $\underline{\hat{\nu}}_H$ list the (approximate) values of $\Phi(\mathbf{p})$, $V(\mathbf{p})$, $\delta(\mathbf{p})$ and $\nu(\mathbf{p})$ at the collocation points. The $n_H \times n_H$ matrices L_{HH}, M_{HH}, M^t_{HH} and N_{HH} are the discrete equivalent of the releant Laplace integral operator on H.

This is $2n_H$ equations in $4n_H$ unknowns. The remaining $2n_H$ equations are obtained by applying the boundary conditions at the n_H collocation points:

$$a_{H_i}\hat{\delta}_{H_i} + b_{H_i}\hat{\nu}_{H_i} = f_{H_i},$$

$$A_{H_i}\hat{\Phi}_{H_i} + B_{H_i}\hat{V}_{H_i} = F_{H_i},$$

for $i = 1,..,n_H$. On solution, approximations to the boundary functions are obtained.

The solution in the domain can then be found by using the discrete equivalent of the following equation[7] for the n_E exterior points:

$$\varphi(\mathbf{p}) = \varphi^{i}(\mathbf{p}) + \{M\,\delta\}_H(\mathbf{p}) - \{L\,\nu\}_H(\mathbf{p}) \quad (\mathbf{p}\in E).$$

The discrete equivalent of this equation is as follows:

$$\underline{\hat{\varphi}}_E = \underline{\varphi}_E{}^{i} + \mathrm{M}_{EH}\underline{\hat{\delta}}_H - \mathrm{L}_{EH}\underline{\hat{\nu}}_H,$$

where the terms $\underline{\hat{\varphi}}_E$ lists the approximations to the solution at the domain points and $\underline{\varphi}_E{}^{i}$ similarly lists the incident potential at the domain points. The $n_E \times n_H$ matrices L_{EH} and M_{EH} are the discrete equivalent of the releant Laplace integral operator for the exterior points.