Singly-even self-dual codes with minimal shadow

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Abstract
In this note we investigate extremal singly-even self-dual codes with minimal shadow. For particular parameters we prove non-existence of such codes. By a result of Rains [8], the length of extremal singly-even self-dual codes is bounded. We give explicit bounds in case the shadow is minimal.

Index Terms: self-dual codes, singly-even codes, minimal shadow, bounds

1 Introduction
Let \( C \) be a singly-even self-dual \([n, \frac{n}{2}, d]\) code and let \( C_0 \) be its doubly-even subcode. There are three cosets \( C_1, C_2, C_3 \) of \( C_0 \) such that \( C_0^\perp = C_0 \cup C_1 \cup C_2 \cup C_3 \), where \( C = C_0 \cup C_2 \). The set \( S = C_1 \cup C_3 = C_0^\perp \setminus C \) is called the shadow of \( C \). Shadows for self-dual codes were introduced by Conway and Sloane [3] in order to derive new upper bounds for the minimum weight of singly-even self-dual codes and to provide restrictions on their weight enumerators.

According to [7] the minimum weight \( d \) of a self-dual code of length \( n \) is bounded by \( 4[n/24] + 4 \) for \( n \not\equiv 22 \pmod{24} \) and by \( 4[n/24] + 6 \) if \( n \equiv 22 \pmod{24} \). We call a self-dual code meeting this bound extremal. Note that for some lengths, for instance length 34, no extremal self-dual codes exist.

Some properties of the weight enumerator of \( S \) are given in the following theorem.

Theorem 1 [3] Let \( S(y) = \sum_{r=0}^{n} B_r y^r \) be the weight enumerator of \( S \). Then

\begin{itemize}
  \item \( B_r = B_{n-r} \) for all \( r \),
  \item \( B_r = 0 \) unless \( r \equiv n/2 \pmod{4} \),
  \item \( B_0 = 0 \),
\end{itemize}

*Supported by the Humboldt Foundation. On leave from Faculty of Mathematics and Informatics, Veliko Tarnovo University, 5000 Veliko Tarnovo, Bulgaria
• $B_r \leq 1$ for $r < d/2$,
• $B_{d/2} \leq 2n/d$,
• at most one $B_r$ is nonzero for $r < (d + 4)/2$.

In this note we study singly-even self-dual codes for which the minimum weight of the shadow is the smallest possible.

**Definition 1** We say that a self-dual code $C$ of length $24m + 8l + 2r$ with $r = 1, 2, 3$ and $l = 0, 1, 2$ is a code with minimal shadow if $\text{wt}(S) = r$. For $r = 0$, $C$ is called of minimal shadow if $\text{wt}(S) = 4$.

Self-dual codes of length $24m + 8l + 2$ with $\text{wt}(S) = 1$ were considered in [2] and their structure has been used to characterize access groups in a secret sharing scheme based on codes. The performance of the extremal self-dual codes of length $24m + 8l$ where $l = 1, 2$ has been studied in [1]. In particular different types of codes with the same parameters are compared with regard to the decoding error probability.

This article is organized as follows. In Section 2 we prove that extremal self-dual codes with minimal shadow of length $24m + 2t$ for $t = 1, 2, 3, 5, 11$ do not exist. Moreover, for $t = 4, 6, 7$ and 9, we obtain upper bounds for the length. We also prove that if extremal doubly-even self-dual codes of length $n = 24m + 8$ or $24m + 16$ do not exist then extremal singly-even self-dual codes with minimal shadow do not exist for the same length. The only case for which we do not have a bound for the length is $n = 24m + 20$.

All computations are carried out with Maple.

## 2 Extremal self-dual codes with minimal shadow

Let $C$ be a singly-even self-dual code of length $n = 24m + 8l + 2r$ where $l = 0, 1, 2$ and $r = 0, 1, 2, 3$. The weight enumerator of $C$ and its shadow are given by [3]:

$$W(y) = \sum_{j=0}^{12m+4l+r} a_j y^{2j} = \sum_{i=0}^{3m+l} c_i (1 + y^2)^{12m+4l+r-4i} (y^2(1-y^2)^2)^i$$

$$S(y) = \sum_{j=0}^{6m+2l} b_j y^{4j+r} = \sum_{i=0}^{3m+l} (-1)^i c_i 2^{12m+4l+r-6i} y^{12m+4l+r-4i} (1 - y^4)^{2i}$$

Using these expressions we can write $c_i$ as a linear combination of the $a_j$ and as a linear combination of the $b_j$ in the following way [7]:

$$c_i = \sum_{j=0}^{i} \alpha_{ij} a_j = \sum_{j=0}^{3m+l-i} \beta_{ij} b_j. \quad (1)$$
Suppose $C$ is an extremal singly-even self-dual code with minimal shadow, hence $d = 4m + 4$ and $\text{wt}(S) = r$ if $r = 1, 2, 3$ and $\text{wt}(S) = 4$ if $r = 0$. Obviously in this case $a_0 = 1, a_1 = a_2 = \cdots = a_{2m+1} = 0$. According to [7] we have $\alpha$ coefficients this equation, which turns out to be crucial in the following, we need to consider the $t$ and if $u \in S$ then $u + v \in C$ with $\text{wt}(u + v) \leq 4m + 2r - 4 \leq 4m + 2$, a contradiction to the minimum distance of $C$. Similarly, if $r = 0$ and $m \geq 2$ then $b_2 = \cdots = b_{m-1} = 0$.

**Remark 1** For extremal self-dual codes of length $24m + 8l + 2$ we furthermore have $b_m = 0$. Otherwise $S$ would contain a vector $v$ of weight $4m + 1$, and if $u \in S$ is the vector of weight 1 which exists since $\text{wt}(S) = 1$, then $u + v \in C$ with $\text{wt}(u + v) \leq 4m + 2$ contradicting the minimum distance of $C$.

If $m \geq 2$ we have by (1)

$$c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,\epsilon} + \sum_{j=m}^{m+l-1} \beta_{2m+1,j} b_j,$$

where $\epsilon = 1$ for $r = 0$ and $\epsilon = 0$ otherwise, since $3m + l - 2m - 1 = m + l$. To evaluate this equation, which turns out to be crucial in the following, we need to consider the coefficients $\alpha_{i0}$ in details. In order to do this we denote by $\alpha_i(n)$ the coefficient $\alpha_{i0}$ if $n$ is the length of the code. According to [7] we have

$$\alpha_i(n) = \alpha_{i0} = -\frac{n}{2i^2} [\text{coeff. of } y^{i-1} \text{ in } (1 + y)^{-n/2-1+4i}(1 - y)^{-2i}].$$

Let $t = 4l + r$ and $n = 24m + 8l + 2r = 24m + 2t$. Then

$$\alpha_{2m+1}(n) = -\frac{12m + t}{2m + 1} [\text{coeff. of } y^{2m} \text{ in } (1 + y)^{-12m-t-1+8m+4}(1 - y)^{-4m-2}]$$

$$= -\frac{12m + t}{2m + 1} [\text{coeff. of } y^{2m} \text{ in } (1 + y)^{-4m-t+3}(1 - y)^{-4m-2}]$$

For $t > 5$ we obtain

$$\alpha_{2m+1}(n) = -\frac{12m + t}{2m + 1} [\text{coeff. of } y^{2m} \text{ in } (1 - y^2)^{-4m-t+3}(1 - y)^{t-5}],$$

and if $t \leq 5$ then

$$\alpha_{2m+1}(n) = -\frac{12m + t}{2m + 1} [\text{coeff. of } y^{2m} \text{ in } (1 - y^2)^{-4m-2}(1 + y)^{5-t}].$$
Since
\[(1 - y^2)^{-a} = \sum_{0 \leq j} \binom{-a}{j} (-1)^j y^{2j} = \sum_{0 \leq j} \binom{a + j - 1}{j} y^{2j}\]
for \(a > 0\),
it follows in case \(t \leq 5\) that
\[\alpha_{2m+1}(n) = -\frac{12m + t}{2m + 1} \text{[coeff. of } y^{2m} \text{ in } (1 + y)^{5-t} \sum_{j=0}^{m} \binom{4m + j + 1}{j} y^{2j}]\]
\[= -\frac{12m + t}{2m + 1} \sum_{s=0}^{\left\lfloor \frac{t-5}{2} \right\rfloor} \binom{5 - t}{2s} \binom{5m + 1 - s}{m - s},\]
and in case \(t > 5\) that
\[\alpha_{2m+1}(n) = -\frac{12m + t}{2m + 1} \text{[coeff. of } y^{2m} \text{ in } (1 - y)^{t-5} \sum_{j=0}^{m} \binom{4m + t + j - 4}{j} y^{2j}]\]
\[= -\frac{12m + t}{2m + 1} \sum_{s=0}^{\left\lfloor \frac{t-5}{2} \right\rfloor} \binom{t - 5}{2s} \binom{5m + t - 4 - s}{m - s}.\]

For the different lengths \(n\) the values of \(\alpha_{2m+1}(n)\) are listed in Table 1.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(24m + 2)</th>
<th>(24m + 10)</th>
<th>(24m + 18)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_{2m+1})</td>
<td>(\frac{(12m + 1)(56m + 4)}{2m + 1}(5m - 1))</td>
<td>(-\frac{12m + 5}{m}(5m + 1))</td>
<td>(-\frac{12(7m + 5)(4m + 3)}{m(m - 1)}(5m + 3))</td>
</tr>
<tr>
<td>(n)</td>
<td>(24m + 4)</td>
<td>(24m + 12)</td>
<td>(24m + 20)</td>
</tr>
<tr>
<td>(\alpha_{2m+1})</td>
<td>(-\frac{2(6m + 1)(8m + 1)}{m(2m + 1)})</td>
<td>(-\frac{6m + 2}{m})</td>
<td>(-\frac{20(6m + 5)(4m + 3)}{m(m - 1)}(5m + 4))</td>
</tr>
<tr>
<td>(n)</td>
<td>(24m + 6)</td>
<td>(24m + 14)</td>
<td>(24m + 22)</td>
</tr>
<tr>
<td>(\alpha_{2m+1})</td>
<td>(-\frac{3(4m + 1)(6m + 1)}{m(2m + 1)}(5m - 1))</td>
<td>(-\frac{3(12m + 7)}{m}(5m + 2))</td>
<td>(-\frac{6(12m + 11)(6m + 5)(8m + 7)}{m(m - 1)(m - 2)}(5m + 4))</td>
</tr>
<tr>
<td>(n)</td>
<td>(24m + 8)</td>
<td>(24m + 16)</td>
<td></td>
</tr>
<tr>
<td>(\alpha_{2m+1})</td>
<td>(-\frac{4(3m + 1)}{2m + 1}(5m + 1))</td>
<td>(-\frac{16(3m + 2)}{m}(5m + 3))</td>
<td></td>
</tr>
</tbody>
</table>

To evaluate equation (2) we also need \(\beta_{ij}\) which are known due to [7]. Here we have
\[\beta_{ij} = (-1)^i 2^{-n/2+6i} \frac{k - j}{i} \binom{k + i - j - 1}{k - i - j},\]
where $k = \lfloor n/8 \rfloor = 3m + l$. In particular,
\[ \beta_{2m+1,j} = -2^{6-t} \frac{3m + l - j}{2m + 1} \left( \frac{5m + l - j}{m + l - 1 - j} \right) \quad \text{and} \quad \beta_{2m+1,m+l-1} = -2^{6-t}. \]

Now we are prepared to prove:

**Theorem 2** Extremal self-dual codes of lengths $n = 24m + 2$, $24m + 4$, $24m + 6$, $24m + 10$ and $24m + 22$ with minimal shadow do not exist.

**Proof.** According to [7] any extremal self-dual code of length $24m + 22$ has minimum distance $4m + 6$ and the minimum weight of its shadow is $4m + 7$. Thus the shadow is not minimal since a minimal shadow must have minimum weight $3$. (There is a misprint in [7] where it is stated that the minimum weight of the shadow is $4m + 6$. But actually the weights in this shadow are of type $4j + 3$).

In the other four cases we have
\[ c_{2m+1} = \alpha_{2m+1,0} = \beta_{2m+1,0} \]  \hspace{1cm} (5)
by (2). In case $n = 24m + 10$ we use the fact that $b_m = 0$, according to Remark 1.

Simplifying equation (5) according to Table 1 we obtain
\[ 48m^2 + 26m + 1 = 0, \quad \text{if} \quad n = 24m + 2 \]
\[ 24m^2 + 14m + 1 = 0, \quad \text{if} \quad n = 24m + 4 \]
\[ 48m^2 + 30m + 3 = 0, \quad \text{if} \quad n = 24m + 6 \]
\[ 6m + 3 = 0, \quad \text{if} \quad n = 24m + 10. \]

Since all these equations have no solutions $m \geq 0$ extremal self-dual codes with minimal shadow do not exist for $n \equiv 2, 4, 6, 10 \pmod{24}$. \hfill \Box

**Remark 2** So far no extremal self-dual codes of length $24m + 2t$ are known for $t = 1, 2, 3, 5$. According to [5] extremal self-dual codes of length $24m + 2r$ do not exist for $r = 1, 2, 3$ and $m = 1, 2, \ldots, 6, 8, \ldots, 12, 16, \ldots, 22$. Thus if there is (for instance) a self-dual $[170, 85, 32]$ code it will not have minimal shadow, by Theorem 2.

The next result is a crucial observation in order to prove explicit bounds for the existence of extremal singly-even self-dual codes.

**Theorem 3** Extremal singly-even self-dual codes with minimal shadow of lengths $n = 24m+8$, $24m+12$, $24m+14$ and $24m+18$ have uniquely determined weight enumerators.
Proof. For \( m = 0 \) and \( m = 1 \) see Remark 3 and the examples at the end of the paper. Now let \( m \geq 2 \).

In case \( n = 24m + 12 \) or \( n = 24m + 14 \) we have

\[
c_i = \alpha_{i0} = \beta_{i0} + \sum_{j=m}^{3m+1-i} \beta_{ij}b_j \quad \text{for } i \leq 2m + 1 \quad \text{and}
\]

\[
c_i = \alpha_{i0} + \sum_{j=2m+2}^{i} \alpha_{ij}a_j = \beta_{i0} \quad \text{for } i > 2m + 1.
\]

Therefore \( c_i = \alpha_{i0} \) for \( i = 0, 1, \ldots, 2m + 1 \) and \( c_i = \beta_{i0} \) for \( i = 2m + 2, \ldots, 3m + 1 \).

In the case \( n = 24m + 8 \) we have \( b_0 = 0, b_1 = 1 \) and \( b_2 = \cdots = b_{m-1} = 0 \). Hence \( c_i = \alpha_{i0} \) for \( i = 0, 1, \ldots, 2m + 1 \) and \( c_i = \beta_{i1} \) for \( i = 2m + 2, \ldots, 3m + 1 \).

Similarly, if \( n = 24m + 18 \) we obtain \( c_i = \alpha_{i0} \) for \( i = 0, 1, \ldots, 2m + 1 \) and \( c_i = \beta_{i0} \) for \( i = 2m + 2, \ldots, 3m \). In both cases the weight enumerator can be computed as above.

By \( 3 \) and \( 4 \), the values of \( c_i \) can be calculated and they depend only on the length \( n \). Thus the weight enumerators are unique in all cases.

\[
\square
\]

In \cite{12}, Zhang obtained upper bounds for the lengths of the extremal binary doubly-even codes. He proved that extremal doubly-even codes of length \( n = 24m + 8l \) do not exist if \( m \geq 154 \) (for \( l = 0 \)), \( m \geq 159 \) (for \( l = 1 \)) and \( m \geq 164 \) (for \( l = 2 \)). For extremal singly-even codes there is also a bound due to Rains \cite{8}. Unfortunately, he only states the existence of a bound. In the next corollary we give explicit bounds for extremal singly-even self-dual codes with minimal shadow for lengths congruent 8, 12, 14 and 18 mod 24.

In the proof we need the value of \( c_{2m} = \alpha_{2m,0} \). According to \cite{7} we have

\[
\alpha_{2m}(n) = -\frac{24m + 2t}{4m} \left[ \text{coeff. of } y^{2m-1} \text{ in } (1 + y)^{-4m-t-1}(1 - y)^{-4m} \right]
\]

\[
= -\frac{12m + t}{2m} \left[ \text{coeff. of } y^{2m-1} \text{ in } (1 - y)^{t+1}(1 - y^2)^{-4m-t-1} \right]
\]

\[
= -\frac{12m + t}{2m} \left[ \text{coeff. of } y^{2m-1} \text{ in } (1 - y)^{t+1} \sum_{j=0}^{m} \binom{4m + t + j}{j} y^{2j} \right]
\]

\[
= -\frac{12m + t}{2m} \sum_{s=1}^{\left\lfloor \frac{t+1}{2} \right\rfloor} \binom{t + 1}{2s - 1} \binom{5m + t - s}{m - s}
\]

where \( t = 4l + r \) and \( n = 24m + 8l + 2r = 24m + 2t \). The values for \( \alpha_{2m}(n) \) are listed in Table \ref{table2}.

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Table 2: The values $\alpha_{2m}(n)$ for an extremal self-dual $[n = 24m + 2t, \frac{n}{2}, 4m + 4]$ code

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha_{2m}(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$24m + 8$</td>
<td>$\frac{8(4m + 1)(11m + 3)(3m + 1)}{m(m - 1)(m - 2)} \begin{pmatrix} 5m + 1 \ m - 3 \end{pmatrix}$</td>
</tr>
<tr>
<td>$24m + 12$</td>
<td>$\frac{24(116m^2 + 79m + 15)(1 + 2m)^2}{m(m - 1)(m - 2)(m - 3)} \begin{pmatrix} 5m + 2 \ m - 4 \end{pmatrix}$</td>
</tr>
<tr>
<td>$24m + 14$</td>
<td>$\frac{24(1 + 2m)(12m + 7)(28m^2 + 22m + 5)}{m(m - 1)(m - 2)(m - 3)} \begin{pmatrix} 5m + 3 \ m - 4 \end{pmatrix}$</td>
</tr>
<tr>
<td>$24m + 16$</td>
<td>$\frac{16(3m + 2)(2m + 1)(1216m^3 + 1956m^2 + 1073m + 210)}{m(m - 1)(m - 2)(m - 3)(m - 4)} \begin{pmatrix} 5m + 3 \ m - 5 \end{pmatrix}$</td>
</tr>
<tr>
<td>$24m + 18$</td>
<td>$\frac{120(2m + 1)(4m + 3)(176m^3 + 308m^2 + 189m + 42)}{m(m - 1)(m - 2)(m - 3)(m - 4)} \begin{pmatrix} 5m + 4 \ m - 5 \end{pmatrix}$</td>
</tr>
<tr>
<td>$24m + 20$</td>
<td>$\frac{16(6m + 5)(2m + 1)(4m + 3)(1592m^3 + 3280m^2 + 2363m + 630)}{m(m - 1)(m - 2)(m - 3)(m - 4)(m - 5)} \begin{pmatrix} 5m + 4 \ m - 6 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

Furthermore, $\beta_{2m,j} = 2^{-t} \frac{3m + l - j}{2m} \begin{pmatrix} 5m + l - 1 - j \\ m + l - j \end{pmatrix}$. Hence $\beta_{2m,m+l} = 2^{-t}$ and $\beta_{2m,m+l-1} = 2^{1-t}(2m + 1)$.

**Corollary 4** There are no extremal singly-even self-dual codes of length $n$ with minimal shadow if

(i) $n = 24m + 8$ and $m \geq 53$,

(ii) $n = 24m + 12$ and $m \geq 142$,

(iii) $n = 24m + 14$ and $m \geq 146$,

(iv) $n = 24m + 18$ and $m \geq 157$.

**Proof.** Using the equation

$$c_i = \alpha_{i0} = \beta_{i\epsilon} + \sum_{j=m}^{3m+l-i} \beta_{ij} b_j \quad \text{for } i \leq 2m + 1,$$

where $\epsilon = 1$ if $n = 24m + 8$ and $\epsilon = 0$ in the other cases, we see that

$$b_{m+l-1} = -2^{t-6}(\alpha_{2m+1,0} - \beta_{2m+1,\epsilon}).$$

The values of $b_{m}$ for $n = 24m + 8$, $24m + 12$ and $24m + 14$ are given in Table 3.
If $n = 24m + 18$ we have

$$b_m = 0 \quad \text{and} \quad b_{m+1} = \frac{(24m + 17)(17m + 10)}{(2m + 1)(4m + 5)} \left( \frac{5m + 2}{m + 1} \right).$$

In the first three cases we compute

$$b_{m+1} = \frac{\alpha_{2m,0} - \beta_{2m,0} - \beta_{2m,m} b_m}{\beta_{2m,m+1}}.$$

If $n = 24m + 8$ we obtain

$$b_{m+1} = \frac{16(6m + 1)(-4m^3 + 209m^2 + 141m + 24)}{5m(m + 1)(4m + 3)} \left( \frac{5m + 1}{m - 1} \right).$$

In case $m \geq 53$ the polynomial $-4m^3 + 209m^2 + 141m + 24$ takes negative values, hence $b_{m+1} < 0$, a contradiction. For $24m + 12$ we have

$$b_{m+1} = \frac{2(12m + 5)(-32m^4 + 4496m^3 + 4242m^2 + 1257m + 117)}{(5m + 1)(4m + 3)(4m + 5)(2m + 3)} \left( \frac{5m + 2}{m + 1} \right)$$

If $m \geq 142$ the polynomial $-32m^4 + 4496m^3 + 4242m^2 + 1257m + 117$ takes negative values, hence $b_{m+1} < 0$, a contradiction. For $24m + 14$ the calculations lead to

$$b_{m+1} = \frac{2(-5376m^6 + 772352m^5 + 1663728m^4 + 1386448m^3 + 557970m^2 + 107643m + 7875)}{(4m + 3)(4m + 5)(2m + 3)(4m + 7)(5m + 1)} \left( \frac{5m + 2}{m + 1} \right)$$

which is negative if $m \geq 146$.

In the last case we have to compute

$$b_{m+2} = \frac{\alpha_{2m,0} - \beta_{2m,0} - \beta_{2m,m+1} b_{m+1}}{\beta_{2m,m+2}}.$$

The computations yield

$$b_{m+2} = \frac{2(24m + 17)(-544m^5 + 83696m^4 + 184210m^3 + 149089m^2 + 52809m + 6930)}{(4m + 5)(2m + 3)(4m + 7)(4m + 9)(5m + 2)} \left( \frac{5m + 3}{m + 2} \right)$$

which is negative for $m \geq 157$. \hfill $\square$
Proposition 5 If there are no extremal doubly-even self-dual codes of length \( n = 24m + 8 \) or \( 24m + 16 \) then there are no extremal singly-even self-dual codes of length \( n \) with minimal shadow.

Proof. We shall prove the contraposition. Let \( C \) be a singly-even self-dual \([n = 24m + 8l, 12m + 4l, 4m + 4]\) code and suppose that the coset \( C_1 \) contains the vector \( u \) of weight 4. If \( v \in C_3 \) then \( u + v \in C_2 \) and hence \( \text{wt}(u + v) \geq 4m + 6 \). It follows that

\[
\text{wt}(v) \geq 4m + 6 - 4 + 2\text{wt}(u \ast v) \geq 4m + 4.
\]

Thus \( \text{wt}(C_3) \geq 4m + 4 \). Therefore \( C_0 \cup C_3 \) is an extremal doubly-even code with parameters \([24m + 8l, 12m + 4l, 4m + 4]\).

\[\square\]

Corollary 6 There are no extremal singly-even self-dual codes with minimal shadow of length \( n = 24m + 16 \) for \( m \geq 164 \).

Proof. This follows immediately from the Zhang bound \([12]\) for doubly-even codes in connection with Proposition 5.

\[\square\]

Summarizing the results in Theorem 2, Corollary 4 and Corollary 6 we have proved either the non-existence or an explicit bound for the length \( n \) of an extremal singly-even self-dual code unless \( n \equiv 20 \pmod{24} \). To find an explicit bound for \( n = 24m + 20 \) seems to be difficult since the weight enumerator is not unique in this case.

Remark 3 Extremal singly-even self-dual codes of length \( 24m + 8 \) are constructed only for \( m = 1 \), i.e. \( n = 32 \). There are exactly three inequivalent singly-even self-dual \([32, 16, 8]\) codes. Yorgov proved that there are no extremal singly-even self-dual codes with minimal shadow of length \( 24m + 8 \) in the case \( m \) is even and \( \binom{5m}{m} \) is odd \([11]\).

Examples. Extremal singly-even self-dual codes of lengths \( 24m + 12, 24m + 14 \) and \( 24m + 18 \):

\( m = 0 \): There are unique extremal singly-even codes of lengths 12, 14 and 16, and they have minimal shadows. There are two inequivalent self-dual \([18, 9, 4]\) codes, but only one of them is a code with minimal shadow (see \([3]\)).

\( m = 1 \): Extremal self-dual codes of lengths 36, 38 and 42 with minimal shadow are constructed. Only for the length 36 there is a complete classification \([6]\). There are 16 inequivalent self-dual \([36, 18, 8]\) codes with minimal shadow and their weight enumerator is \( W = 1 + 225y^8 + 2016y^{10} + 9555y^{12} + \cdots \) (see \([4]\)).

\( m = 2 \): There exists a doubly circulant code with parameters \([60, 30, 12]\) and shadow of minimum weight 2, denoted by \( D_{13} \) in \([3]\). The first examples for extremal self-dual codes with minimal shadow of lengths 62 and 66 are constructed in \([9]\) and \([10]\), respectively.
Finally, we would like to mention that similar to the case of extremal doubly-even self-dual codes there is a large gap between the bounds for extremal singly-even self-dual codes and what we really can construct.

References


