On Type IV self-dual codes over $\mathbb{Z}_4$

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Abstract

Recently, Type IV self-dual codes over rings of order 4 have been introduced as self-dual codes over the rings with the property that all Hamming weights are even. All Type IV self-dual codes over $\mathbb{Z}_4$ of lengths up to 16 are known. In this paper, the classification of such codes of length 20 is given. The highest minimum Hamming, Lee and Euclidean weights of Type IV $\mathbb{Z}_4$-codes of lengths up to 40 and length 56 are also determined. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Type IV codes; Codes over $\mathbb{Z}_4$; Self-dual codes and binary doubly even self-complementary codes

1. Introduction

Recently, Type IV self-dual codes over rings of order 4 have been introduced as self-dual codes over the rings with the condition that all Hamming weights are even [7]. A number of properties of Type IV $\mathbb{Z}_4$-codes are studied in [7]. For example, it is shown that a Type IV self-dual $\mathbb{Z}_4$-code is closely related to a class of binary doubly even self-complementary codes. An upper bound on the minimum Lee weight of a Type IV self-dual code over $\mathbb{Z}_4$ is also established. All Type IV self-dual codes of lengths up to 12 and Type IV–II codes of lengths up to 16 are also known. All Type IV–I $\mathbb{Z}_4$-codes of length 16 are classified in [9], establishing a classification method based on the classification binary doubly even self-dual codes (see Table 1 for the known classification). More recently the first author [2,3] has introduced an improved upper bound for the minimum Lee weight and upper bounds for the minimum Hamming and Euclidean weights (see Tables 8 and 9 for the known highest minimum weights). For a

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Table 1
Classification of Type IV $\mathbb{Z}_4$-codes of lengths up to 16

<table>
<thead>
<tr>
<th>Lengths</th>
<th>Codes</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$K_4$</td>
<td>[7]</td>
</tr>
<tr>
<td>8 (Type I)</td>
<td>$K_2^2$</td>
<td>[7]</td>
</tr>
<tr>
<td>8 (Type II)</td>
<td>$K_8$</td>
<td>[7]</td>
</tr>
<tr>
<td>12</td>
<td>$K_2^2, K_4 + K_8, K_{12}, [12,3]-3d4b$</td>
<td>[7]</td>
</tr>
<tr>
<td>16 (Type I)</td>
<td>$K_4, K_4^2 + K_8, K_4 + K_{12}, K_4 + [12,3]-3d4b, C_{16,5}, C_{16,9}$</td>
<td>[9] (see also [2])</td>
</tr>
<tr>
<td>16 (Type II)</td>
<td>$K_4, K_{16}, 3, f_3, 4, f_4, 5, f_5$</td>
<td>[7]</td>
</tr>
</tbody>
</table>

fixed class of codes, it is a fundamental problem to classify these codes and determine the highest minimum weights.

In this paper, we deal with this problem for Type IV self-dual $\mathbb{Z}_4$-codes. In Section 2, definitions used in this paper are given. Basic properties and known characterizations of Type IV self-dual $\mathbb{Z}_4$-codes are also described. In particular, Theorem 2.1 is a powerful tool for our study of Type IV self-dual $\mathbb{Z}_4$-codes. In Section 3, we give the classification of Type IV codes of length 20. In order to classify such codes, binary doubly even self-complementary codes $C$ of length 20 satisfying the condition $w_H(x+y) \equiv 0 \pmod{4}$ for any $x$ and $y \in C$ are classified, where $x \ast y$ denotes the Hadamard product of $x$ and $y$, and $w_H(x)$ denotes the Hamming weight of $x$. The highest minimum Hamming, Lee and Euclidean weights of Type IV self-dual $\mathbb{Z}_4$-codes of lengths up to 24 are determined in [2,7,9]. In Sections 4–8, we determine the highest minimum weights of Type IV–I codes of lengths up to 40 and length 56, and of Type IV–II codes of lengths up to 64. It is worthwhile to note that there is a Type IV–I code of length 40 such that the minimum Hamming, Lee and Euclidean weights are higher than any Type IV–II code of that length. For binary self-dual codes, it is not still known if there is a Type I code with higher minimum weight than any Type II code of that length (cf. [5]). Section 7 also gives a construction method of Type IV self-dual codes. In Section 8, we investigate the highest minimum weights for larger lengths. Using the above method, it is also shown that there are Type IV self-dual codes with minimum Euclidean weight 16 for lengths $n \geq 64$ and $n \equiv 0 \pmod{8}$.

2. Definitions and known results

2.1. Self-dual codes

A code $C$ of length $n$ over $\mathbb{Z}_4$ (or a $\mathbb{Z}_4$-code of length $n$) is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. Let $x$ be a codeword of $C$ and let $n_0(x)$, $n_1(x)$, $n_2(x)$ and $n_3(x)$ be the numbers of 0’s, 1’s, 2’s and 3’s in $x$, respectively. The Hamming weight $w_H(x)$, the Lee weight $w_L(x)$ and the Euclidean weight $w_E(x)$ of $x$ are $n_1(x) + n_2(x) + n_3(x)$, $n_1(x) + 2n_2(x) + n_3(x)$ and $n_1(x) + 4n_2(x) + n_3(x)$, respectively. The minimum Hamming, Lee and Euclidean
weights \( d_H \), \( d_L \) and \( d_E \) of \( C \) are the smallest Hamming, Lee and Euclidean weights among all non-zero codewords of \( C \), respectively.

Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be two elements of \( \mathbb{Z}_4^n \). The dual code \( \overline{C} \) of \( C \) is defined as \( \overline{C} = \{ x \in \mathbb{Z}_4^n | x \cdot y = 0 \text{ for all } y \in C \} \) where \( x \cdot y = x_1 y_1 + \cdots + x_n y_n \). \( C \) is self-dual if \( C = \overline{C} \). We say that two codes are equivalent if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates. Codes differing by only a permutation of coordinates are called permutation-equivalent. The automorphism group \( \text{Aut}(C) \) of \( C \) consists of all permutations and sign changes of the coordinates which preserve \( C \). The symmetrized weight enumerator of a code \( C \) over \( \mathbb{Z}_4 \) is

\[
\text{sw}e_C(a, b, c) = \sum_{x \in C} a^{n_0(x)} b^{n_1(x)} c^{n_2(x)}.
\]

The Lee weight enumerator of \( C \) is defined as \( \text{sw}e_C(1, y, y^2) \).

Any code \( C \) is permutation-equivalent to a code with generator matrix of the form

\[
\begin{pmatrix}
I_{k_1} & A & B_1 + 2B_2 \\
0 & 2I_{k_2} & 2D
\end{pmatrix},
\]

where \( A, B_1, B_2 \) and \( D \) are (1, 0)-matrices and \( I_n \) is the identity matrix of order \( n \). We say that a code with generator matrix in this form (1) is of type \( 4^{k_1}2^{k_2} \) (cf. [6]). The binary \( \mathbb{Z}_4 \) code \( C_1 \) with generator matrix

\[
\begin{pmatrix}
I_{k_1} & A \\
0 & B_1
\end{pmatrix}
\]

(2)

is called the residue code of \( C \). The binary \( \mathbb{Z}_4 \) code \( C_2 \) with generator matrix

\[
\begin{pmatrix}
I_{k_1} & A & B_1 \\
0 & 2I_{k_2} & D
\end{pmatrix}
\]

(3)

is called the torsion code of \( C \). Throughout this paper, \( d_1 \) and \( d_2 \) denote the minimum weights of \( C_1 \) and \( C_2 \), respectively.

2.2. Type IV codes and Type II codes

A self-dual code is called Type IV if all Hamming weights are even (see [7] for their properties). A Type IV self-dual code of length \( n \) exists if and only if \( n \equiv 0 \pmod{4} \) [7]. Type II codes over \( \mathbb{Z}_4 \) are self-dual codes with the property that all Euclidean weights are divisible by eight. A self-dual code which is not Type II is called Type I.

If a Type IV code is Type II (resp. Type I) then it is called Type IV–II (resp. Type IV–I). It is known that a Type II code of length \( n \) exists if and only if \( n \equiv 0 \pmod{8} \). An upper bound on \( d_E \) for a Type II code of length \( n \) was given in [1] as \( d_E \leq 8(\lfloor n/24 \rfloor + 1) \).
Now, we present infinite families of Type IV self-dual codes over \( \mathbb{Z}_4 \).

- **Klemm codes**: The Klemm codes \( K_n \) of length \( n = 4m \) are constructed from the repetition code \( R_n \) and its dual code \( P_n \) as follows:
  \[
  K_n = R_n + 2P_n.
  \]
  The code \( K_n \) is a Type IV self-dual code [7].

- **\( \mathcal{C}_{m,r} \)**: These codes \( \mathcal{C}_{m,r} \) were introduced in [1] as constructions from binary Reed–Muller codes. For \( 3r \leq m - 1 \), the code
  \[
  \mathcal{C}_{m,r} = RM(r, m) + 2RM(m - r - 1, m),
  \]
  is a Type IV self-dual code [7].

We present some characterization of Type IV self-dual codes.

**Theorem 2.1** (Dougherty et al. [7]). Let \( C \) be a code over \( \mathbb{Z}_4 \). Suppose that \( C_1 \) and \( C_2 \) have generator matrices given by (2) and (3), respectively. If \( C \) is Type IV, then there exists a unique \((1,0)\)-matrix \( B \) such that
\[
\begin{pmatrix}
I_{k_1} + 2B & A & B_1 \\
0 & 2I_{k_2} & 2D
\end{pmatrix}
\]
is a generator matrix of \( C \). Moreover, we have

1. \( C_2 = C_1^\perp \),
2. \( C_1 \) contains the all-ones vector, and \( w_H(x + y) \equiv 0 (\mod 4) \) for any \( x \) and \( y \in C_1 \),
3. the number of 2’s in each row of \( I_{k_1} + 2B \) is even, and the matrix \( B \) is symmetric.

Conversely, if \( C_1 \) and \( C_2 \) are binary codes with generator matrices given by (2) and (3), respectively, and if conditions (1)–(3) are satisfied, then the \( \mathbb{Z}_4 \)-code \( C \) with generator matrix (4) is a Type IV self-dual code.

By the above theorem, Type IV self-dual codes are closely related to binary doubly even self-complementary codes satisfying condition (2) and binary symmetric matrices \( B \). Here we say that such a binary code is **Type IV-residue**. In this paper, the above theorem is a power tool for our study of Type IV self-dual \( \mathbb{Z}_4 \)-codes.

In [7], an upper bound on the minimum Lee weights of Type IV self-dual codes was given. Recently, an improved upper bound on the minimum Lee weights and upper bounds on other minimum weights have been given in [2,3].

**Theorem 2.2** (cf. Bouyuklieva [3]). For a Type IV self-dual code of length \( n \), we have that
\[
d_L \leq 4 \left\lfloor \frac{n}{12} \right\rfloor, \quad 2d_H = d_L \quad \text{and} \quad d_E \leq 4d_H.
\]

**Remark.** It is shown in [10] that \( d_2 = d_H \).
In this paper, we show that the above improved bounds are not still tight in general (compare the bounds with the highest minimum weights in Tables 8 and 9).

2.3. Known classifications and highest minimum weights

All Type IV self-dual $\mathbb{Z}_4$-codes are known for lengths up to 16 [2,7,9]. We describe in Table 1 what is known for the classification of Type IV self-dual codes of these lengths. In this paper, Type IV self-dual codes of length 20 are classified.

Throughout this paper, let $d_{H}(n)$, $d_{L}(n)$ and $d_{E}(n)$ denote the highest minimum Hamming, Lee and Euclidean weights, respectively, of Type IV–I codes of length $n$, and let $d_{H}'(n)$, $d_{L}'(n)$ and $d_{E}'(n)$ denote the highest minimum Hamming, Lee and Euclidean weights, respectively, of Type IV–II codes of length $n$. For lengths up to 24, the highest minimum weights are determined. For the highest minimum weights of Type IV–I (resp. Type IV–II) codes, the known results are listed in Table 8 (resp. Table 9). In this paper, we determine the highest minimum weights of Type IV–I codes of lengths up to 40 and length 56, and of Type IV–II codes of lengths up 64.

3. Classification of length 20

In this section, we give the classification of Type IV self-dual codes of length 20.

3.1. Mass formula and binary residue codes

We first give the mass formula to check that our classification completes.

**Theorem 3.1** (Dougherty et al. [7]). Let $N(n)$ be the number of distinct Type IV self-dual codes of length $n$ and let $\tau(n,k)$ be the number of distinct binary Type IV-residue codes $C'$ of length $n$ and dimension $k$, then

$$N(n) = \sum_{k \leq n/2} \tau(n,k)2^{1+k(k-1)/2}.$$ 

**Remark.** This mass formula can be used for each $k$. We denote $\tau(n,k)2^{1+k(k-1)/2}$ by $N_k(n)$.

By Theorem 2.1, in order to classify Type IV self-dual codes of length 20, we first need the classification of binary Type IV-residue codes of length 20.

By a direct argument, our computer search shows the following classification.

**Proposition 3.2.** There are exactly 13 inequivalent Type IV-residue codes of length 20.
Table 2
Type IV-residue codes of length 20

| $k_1$ | Codes   | $d_1$ | $|\text{Aut}(D_{k_1,1})|$ |
|-------|---------|-------|--------------------------|
| 1     | $D_1$   | 20    | 20!                      |
| 2     | $D_{2,1}$ | 4     | 502146957312000          |
|       | $D_{2,2}$ | 4     | 1931344512000           |
| 3     | $D_{3,1}$ | 8     | 3344302080                |
|       | $D_{3,2}$ | 4     | 78033715200             |
|       | $D_{3,3}$ | 4     | 551809843200          |
| 4     | $D_{4,1}$ | 8     | 15482880                 |
|       | $D_{4,2}$ | 4     | 191102976               |
|       | $D_{4,3}$ | 4     | 3344302080            |
| 5     | $D_{5,1}$ | 8     | 2419200                  |
|       | $D_{5,2}$ | 4     | 8257536                  |
|       | $D_{5,3}$ | 4     | 955514880               |
| 6     | $D_6$   | 4     | 7741440                 |

For each $k_1$, the minimum weight $d_1$ and the orders $|\text{Aut}(D_{k_1,1})|$ of the automorphism groups of the 13 codes $D_{k_1,1}$ are listed in Table 2. Note that generator matrices of the codes can be obtained from generator matrices of corresponding Type IV $\mathbb{Z}_4$-codes given below.

From the classification of Type IV-residue codes of lengths up to 16, it follows that there is no Type IV-residue $[20, k \geq 7, 4]$ code since such a code is decomposable. Moreover, there is no Type IV-residue $[20, k \geq 7, 8]$ code.

From the classification, we have the following values $\tau(20, k)$ where $1 \leq k \leq 6$:

\[
\begin{align*}
\tau(20, 1) &= 1, & \tau(20, 2) &= 130815, & \tau(20, 3) &= 763063275, \\
\tau(20, 4) &= 170593297875, & \tau(20, 5) &= 1302838111575 \\
\text{and} & & \tau(20, 6) &= 314269956000.
\end{align*}
\]

3.2. Type IV self-dual codes of length 20

To check that our classification is complete, we use the above mass formula. Thus, we need to compute the order of the automorphism group of a given Type IV $\mathbb{Z}_4$-code. Instead of computing directly the automorphism group of a $\mathbb{Z}_4$-code, we use Proposition 3 in [9] since the automorphism group of a binary code is easily computed by MAGMA or GAP.

- Decomposable codes: We give all decomposable codes of length 20 in Table 3 for each $k_1$. There are 13 decomposable codes of length 20. We shall show that the 13 codes are inequivalent.
Table 3
Decomposable Type IV $Z_4$-codes of length 20

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>Codes</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$K_4 + K_{16}, K_8 + K_{12}$</td>
</tr>
<tr>
<td>3</td>
<td>$K_2^4 + K_{12}, K_4 + K_8^2$</td>
</tr>
<tr>
<td>4</td>
<td>$K_4^1 + K_8, K_4 + 3, f_3, K_4 + C_{16.5}, K_8 + [12, 3] - 3d4b$</td>
</tr>
<tr>
<td>5</td>
<td>$K_2^2, K_3^2 + [12, 3] - 3d4b, K_4 + 4, f_4, K_4 + C_{16.9}$</td>
</tr>
<tr>
<td>6</td>
<td>$K_4 + 5, f_5$</td>
</tr>
</tbody>
</table>

- $k_1 = 1$: There is a unique Type IV self-dual code, namely $K_{20}$ [7].
- $k_1 = 2$: We first give a characterization of Type IV self-dual codes with $k_1 = 2$.

**Proposition 3.3.** Any Type IV self-dual code of length $n$ and $k_1 = 2$ is decomposable.

**Proof.** There is only one possibility for the matrix $B$, namely the zero matrix. Moreover, any binary doubly even self-complementary code of dimension 2 is decomposable. Therefore a Type IV self-dual code is decomposable. $\square$

Thus $K_4 + K_{16}$ and $K_8 + K_{12}$ are the Type IV self-dual codes with $k_1 = 2$. Since

$$|\text{Aut}(K_4 + K_{16})| = 2^{36} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$$

and

$$|\text{Aut}(K_8 + K_{12})| = 2^{35} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 11,$$

we have that

$$2^{20} \cdot 20! / |\text{Aut}(K_4 + K_{16})| + 2^{20} \cdot 20! / |\text{Aut}(K_8 + K_{12})| )$$

$$= 523260 = 4 \cdot \tau(20, 2) = N_2(20).$$

This shows that our classification for $k_1 = 2$ completes.

- $k_1 = 3$: There are two possible matrices for $B$, namely

$$B_{3.1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$B_{3.2} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

For $k_1 \geq 3$, we only give the inequivalent Type IV self-dual codes. We define the codes $C_{3,i}$ in Table 4. The orders $|\text{Aut}(C_{3,i})|$ of the automorphism groups are also listed.

It is easy to see that $C_{3,2}$ is equivalent to $K_4 + K_8^2$ and $C_{3,3}$ is equivalent to $K_4^2 + K_{12}$. Only the two codes are decomposable codes with $k_1 = 3$. 
Table 4
Type IV–I $\mathbb{Z}_4$-codes with $k_1 = 3$

| Codes   | Residue codes | Matrices $B$ | $|\text{Aut}(C_{3,i})|$ |
|---------|---------------|-------------|--------------------------|
| $C_{3,1}$ | $D_{3,1}$     | $B_{3,1}$   | $2^{33} \cdot 3^6 \cdot 5 \cdot 7$ |
| $C_{3,2}$ | $D_{3,2}$     | $B_{3,1}$   | $2^{35} \cdot 5^5 \cdot 7^2$ |
| $C_{3,3}$ | $D_{3,3}$     | $B_{3,1}$   | $2^{34} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11$ |
| $C_{3,4}$ | $D_{3,2}$     | $B_{3,2}$   | $2^{35} \cdot 5^5 \cdot 7^2$ |
| $C_{3,5}$ | $D_{3,3}$     | $B_{3,2}$   | $2^{34} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11$ |

From Table 4, we have that
$$
\sum_{i=1,2,3,4,5} 2^{20} \cdot 20! / |\text{Aut}(C_{3,i})| = 12209012400 = 2^4 \cdot \tau(20,3) = N_3(20).
$$

Hence $C_{3,1}$, $C_{3,2}$, $C_{3,3}$, $C_{3,4}$ and $C_{3,5}$ complete the classification for $k_1 = 3$.

Generator matrices of $C_{3,1}$, $C_{3,2}$ and $C_{3,3}$ are as follows:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
Table 5
Type IV–I Z₄-codes with k₁ = 4

| Codes   | Residue codes | Matrices B | |Aut(C₄,i)| |
|---------|---------------|------------|---|-----------|
| C₄,1    | D₄,1          | B₄,1       | 2²⁷·3³·5·7 |
| C₄,2    | D₄,2          | B₄,1       | 2³³·3⁶    |
| C₄,3    | D₄,2          | B₄,3       | 2³³·3⁵    |
| C₄,4    | D₄,3          | B₄,1       | 2³³·3⁶·5·7 |
| C₄,5    | D₄,3          | B₄,2       | 2³³·3⁶·5·7 |
| C₄,6    | D₄,3          | B₄,3       | 2³³·3⁵·5·7 |
| C₄,7    | D₄,3          | B₄,4       | 2³³·3⁵·5·7 |

respectively. Generator matrices of C₃,4 and C₃,5 can be obtained from these matrices using B₃,1 and B₃,2.

• k₁ = 4: We first define the following matrices for B:

\[
B_{4,1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad B_{4,2} = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
B_{4,3} = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]

and

\[
B_{4,4} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]

We give the inequivalent codes C₄,i in Table 5 with the orders |Aut(C₄,i)| of the automorphism groups.

From Table 5, we have that

\[
\sum_{i=1}^{7} 2^{20} \cdot 20!/|\text{Aut}(C₄,i)| = 21835942128000 = 2⁷ \cdot \tau(20, 4) = N₄(20).
\]

Hence there are exactly seven inequivalent Type IV codes with k₁ = 4. This completes the classification for k₁ = 4.

It is not hard to see that C₄,2 is equivalent to K₄ + 3 · f 3, C₄,4 is equivalent to K₄³ + K₈, C₄,5 is equivalent to K₈ + [12, 3]-3d4b and C₄,7 is equivalent to K₄ + C₁₆,₅.
Generator matrices of $C_{4,1}$, $C_{4,2}$ and $C_{4,4}$ are as follows:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

and

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

respectively. Generator matrices for the other codes can be obtained from these matrices using $B_{4,i}^2$ or $B_{4,i}^3$.

- $k_1 = 5$: We give the inequivalent codes $C_{5,i}$ in Table 6 with the orders $|\text{Aut}(C_{5,i})|$ of the automorphism groups where the matrices $B$ are as follows:

$$
B_{5,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{5,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{5,3} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{5,4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad B_{5,5} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.
$$

-
Table 6
Type IV–I \(\mathbb{Z}_4\)-codes with \(k_1 = 5\)

<table>
<thead>
<tr>
<th>Codes</th>
<th>Residue codes</th>
<th>Matrices (B)</th>
<th>([\text{Aut}(C_{5,i})])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_{5,1})</td>
<td>(D_{5,1})</td>
<td>(B_{5,1})</td>
<td>(2^{18} \cdot 3^4 \cdot 5^2 \cdot 7)</td>
</tr>
<tr>
<td>(C_{5,2})</td>
<td>(D_{5,2})</td>
<td>(B_{5,1})</td>
<td>(2^{20} \cdot 3^2 \cdot 7)</td>
</tr>
<tr>
<td>(C_{5,3})</td>
<td>(D_{5,2})</td>
<td>(B_{5,4})</td>
<td>(2^{20} \cdot 3^2)</td>
</tr>
<tr>
<td>(C_{5,4})</td>
<td>(D_{5,3})</td>
<td>(B_{5,1})</td>
<td>(2^{33} \cdot 3^6 \cdot 5)</td>
</tr>
<tr>
<td>(C_{5,5})</td>
<td>(D_{5,3})</td>
<td>(B_{5,2})</td>
<td>(2^{32} \cdot 3^6)</td>
</tr>
<tr>
<td>(C_{5,6})</td>
<td>(D_{5,3})</td>
<td>(B_{5,3})</td>
<td>(2^{33} \cdot 3^5)</td>
</tr>
<tr>
<td>(C_{5,7})</td>
<td>(D_{5,3})</td>
<td>(B_{5,4})</td>
<td>(2^{33} \cdot 3^5)</td>
</tr>
<tr>
<td>(C_{5,8})</td>
<td>(D_{5,3})</td>
<td>(B_{5,5})</td>
<td>(2^{33} \cdot 3^5)</td>
</tr>
<tr>
<td>(C_{5,9})</td>
<td>(D_{5,3})</td>
<td>(B_{5,6})</td>
<td>(2^{31} \cdot 3^5 \cdot 5)</td>
</tr>
<tr>
<td>(C_{5,10})</td>
<td>(D_{5,3})</td>
<td>(B_{5,7})</td>
<td>(2^{33} \cdot 3^6 \cdot 5)</td>
</tr>
</tbody>
</table>

\[
B_{5,6} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B_{5,7} = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

From Table 6, we have that

\[
\sum_{i=1}^{10} 2^{20} \cdot 20!/|\text{Aut}(C_{5,i})| = 2668212452505600 = 2^{11} \cdot \tau(20, 5) = N_5(20).
\]

Hence there are exactly 10 inequivalent Type IV codes with \(k_1 = 5\). This completes the classification for \(k_1 = 5\).

It is not hard to check that \(K_2\) is equivalent to \(C_{5,4}\), \(K_2 + [12, 3]\)-3d4b is equivalent to \(C_{5,5}\), \(K_4 + 4_f4\) is equivalent to \(C_{5,2}\) and \(K_4 + C_{16,9}\) is equivalent to \(C_{5,6}\).

Generator matrices of \(C_{5,1}\), \(C_{5,2}\) and \(C_{5,4}\) are as follows:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
and

\[
\begin{pmatrix}
100011110000000000000000 \\
100000001110000000000000 \\
001000000011100000000000 \\
000100000000001110000000 \\
00001000000000000011100000 \\
000002020000000000000000 \\
000000022000000000000000 \\
000000002020000000000000 \\
000000022200000000000000 \\
000000002020000000000000 \\
000000000220000000000000 \\
000000000002020000000000 \\
000000000000220000000000 \\
000000000000002020000000 \\
000000000000000220000000 \\
000000000000000002020000 \\
000000000000000000220000 \\
000000000000000000002020 \\
000000000000000000002200 \\
000000000000000000000202 \\
000000000000000000000022
\end{pmatrix},
\]

respectively. Generator matrices for the other codes can be obtained from these matrices using \(B_{5,i}, i = 2, \ldots, 7\).

- \(k_1 = 6\): We have verified that \(K_4 + 5_5f5\) has the automorphism group of order \(2^{21} \cdot 3^3 \cdot 5\cdot 7\). Let \(C_6\) be the Type IV self-dual code with the following generator matrix:

\[
\begin{pmatrix}
100022111000000000000000 \\
1000000001110111010101 \\
0012020001101011010101 \\
0021200001011011001111 \\
2002100001110000111111 \\
2020010000000011111111 \\
0000002020000000000000 \\
0000000220000000000000 \\
0000000020200000000000 \\
0000000002200000000000 \\
0000000000202000000000 \\
0000000000022000000000 \\
0000000000002020000000 \\
0000000000000220000000 \\
0000000000000020200000 \\
0000000000000002200000 \\
0000000000000000222000 \\
00000000000000000022222
\end{pmatrix},
\]

We have obtained that the order of the automorphism group of \(C_6\) is \(2^{21} \cdot 3^2 \cdot 7\). Thus,

\[
2^{20} \cdot 20! / |\text{Aut}(K_4 + 5_5f5)| + 2^{20} \cdot 20! / |\text{Aut}(C_6)| = 20595995836416000
\]

\[
= 2^{16} \cdot \tau(20, 6) = N_6(20).
\]

Hence there are exactly two inequivalent Type IV self-dual codes with \(k_1 = 6\).

Therefore we have the following classification:

**Theorem 3.4.** There are exactly 27 inequivalent Type IV \(\mathbb{Z}_4\)-codes of length 20.

In Table 7, we summarize the numbers of the codes for each \(k_1\).
The number of Type IV $\mathbb{Z}_4$-codes of length 20

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>2</td>
</tr>
</tbody>
</table>

We end this section by listing few terms of the symmetrized weight enumerators $\text{swe}_C$ of the indecomposable codes $C$:

\[
\text{swe}_{C_{3,1}} = a^{20} + 46a^{18}c^2 + 1197a^{16}c^4 + 9768a^{14}c^6 + 192a^{12}b^8 + 31314a^{12}c^8 + 12672a^{10}b^8c^2 + 46420a^{10}c^{10} + 3072a^8b^{12} + 95040a^8b^8c^4 + \cdots,
\]

\[
\text{swe}_{C_{3,4}} = a^{20} + 62a^{18}c^2 + 1261a^{16}c^4 + 512a^{14}b^4c^2 + 9576a^{14}c^6 + 7168a^{12}b^8c^4 + 31250a^{12}c^8 + 8192a^{10}b^8c^2 + 32256a^{10}b^6c^6 + 46772a^{10}c^{10} + \cdots,
\]

\[
\text{swe}_{C_{3,5}} = a^{20} + 78a^{18}c^2 + 1325a^{16}c^4 + 768a^{14}b^4c^2 + 9384a^{14}c^6 + 64a^{12}b^8 + 14848a^{12}b^4c^4 + 31186a^{12}c^8 + 4224a^{10}b^8c^2 + 64768a^{10}b^4c^6 + \cdots,
\]

\[
\text{swe}_{C_{4,1}} = a^{20} + 22a^{18}c^2 + 589a^{16}c^4 + 4936a^{14}c^6 + 224a^{12}b^8 + 15538a^{12}c^8 + 14784a^{10}b^8c^2 + 23364a^{10}c^{10} + 3584a^{8}b^8c^4 + 110880a^{8}b^8c^4 + \cdots,
\]

\[
\text{swe}_{C_{4,3}} = a^{20} + 30a^{18}c^2 + 621a^{16}c^4 + 256a^{14}b^4c^2 + 4840a^{14}c^6 + 128a^{12}b^8 + 3584a^{12}b^4c^4 + 15506a^{12}c^8 + 12544a^{10}b^8c^2 + 16128a^{10}b^4c^6 + \cdots,
\]

\[
\text{swe}_{C_{4,5}} = a^{20} + 46a^{18}c^2 + 685a^{16}c^4 + 640a^{14}b^4c^2 + 4648a^{14}c^6 + 64a^{12}b^8 + 11008a^{12}b^4c^4 + 15442a^{12}c^8 + 8320a^{10}b^8c^2 + 48512a^{10}b^4c^6 + \cdots,
\]

\[
\text{swe}_{C_{5,1}} = a^{20} + 10a^{18}c^2 + 285a^{16}c^4 + 2520a^{14}c^6 + 240a^{12}b^8 + 7650a^{12}c^8 + 15840a^{10}b^8c^2 + 11836a^{10}c^{10} + 3840a^{8}b^8c^4 + 118800a^{8}b^8c^4 + \cdots,
\]

\[
\text{swe}_{C_{5,3}} = a^{20} + 14a^{18}c^2 + 301a^{16}c^4 + 128a^{14}b^4c^2 + 2472a^{14}c^6 + 192a^{12}b^8 + 1792a^{12}b^4c^4 + 7634a^{12}c^8 + 14720a^{10}b^8c^2 + 8064a^{10}b^4c^6 + \cdots,
\]

\[
\text{swe}_{C_{5,5}} = a^{20} + 30a^{18}c^2 + 365a^{16}c^4 + 384a^{14}b^4c^2 + 2280a^{14}c^6 + 256a^{12}b^8 + 9472a^{12}b^4c^4 + 7570a^{12}c^8 + 10752a^{10}b^8c^2 + 40576a^{10}b^4c^6 + \cdots,
\]

\[
\text{swe}_{C_{5,7}} = a^{20} + 30a^{18}c^2 + 365a^{16}c^4 + 512a^{14}b^4c^2 + 2280a^{14}c^6 + 128a^{12}b^8 + 9216a^{12}b^4c^4 + 7570a^{12}c^8 + 10496a^{10}b^8c^2 + 40448a^{10}b^4c^6 + \cdots,
\]
Lemma 4.3. Let $\text{wec}_{C,9} = a^{20} + 30a^{18}c^2 + 365a^{16}c^4 + 640a^{14}b^4c^2 + 2280a^{14}c^6 + 8960a^{12}b^4c^4 + 7570a^{12}c^8 + 10240a^{10}b^8c^2 + 40320a^{10}b^4c^6 + 12276a^{10}c^{10} + \cdots$

Lemma 4.2. Let $\text{wec}_{C,10} = a^{20} + 30a^{18}c^2 + 365a^{16}c^4 + 2280a^{14}c^6 + 640a^{12}b^4c^4 + 10240a^{10}b^8c^2 + 7570a^{12}c^8 + 11520a^{10}b^8c^2 + 40960a^{10}b^4c^6 + 12276a^{10}c^{10} + \cdots$

Lemma 4.1. Let $\text{wec}_C = a^{20} + 6a^{18}c^2 + 141a^{16}c^4 + 64a^{14}b^4c^2 + 1288a^{14}c^6 + 224a^{12}b^8 + 896a^{12}b^4c^4 + 3696a^{12}c^8 + 15808a^{10}b^8c^2 + 4032a^{10}b^4c^6 + \cdots$

4. Length 28

In Sections 4, 5, 6 and 7, we determine the highest minimum weights of lengths 28, 32, 36 and 40, respectively. We begin with elementary lemmas.

Lemma 4.1. Let $C$ be a binary $[n,k,d]$ code with dual distance $d^\perp$, and $c \in C$. If the residual code $\text{Res}(C,c)$ with respect to $c$ has dual distance $d_{\text{res}}^\perp$, then $d_{\text{res}}^\perp \geq d^\perp$ or $\text{Res}(C,c)$ is trivial.

Proof. Without loss of generality, we can assume that $c = (0,\ldots,0,1,\ldots,1)$. Let $v$ be a vector such that $v \cdot x = 0$ for all $x \in \text{Res}(C,c)$. If $u \in C$ then $u = (x,y)$ for some $x \in \text{Res}(C,c)$, and $(v,0) \cdot u = (v,0) \cdot (x,y) = v \cdot x + 0 \cdot y = 0$. Hence $(v,0) \in C^\perp$. It follows that $w_H(v) \geq d^\perp$ or $w_H(v) = 0$. \(\square\)

Lemma 4.2. Let $C_1$ be the residue code of a Type IV self-dual $\mathbb{Z}_4$-code $C$ and let $c$ be a codeword of weight $w$ in $C_1$. Then $\text{Res}(C_1,c+1)$ is a doubly even self-complementary code of length $w$, where $1$ denotes the all-ones vector.

Proof. Since $C_1$ contains the all-ones vector, we can consider the residual code $\text{Res}(C_1,c+1)$. By Theorem 2.1, $w_H(x \ast y) \equiv 0 \pmod{4}$ for all codewords $x$ and $y$ in $C_1$. Thus $\text{Res}(C_1,c+1)$ is doubly even. Moreover $C_1$ contains the vector $c$. Hence $\text{Res}(C_1,c+1)$ contains the all-ones vector. \(\square\)

Lemma 4.3. If $C_1$ contains a codeword of weight 12 then $d_2 = 2$.

Proof. Let $w \in C_1$ is a codeword of weight 12. Without loss of generality, we can assume that $w = (1,1,1,1,1,1,1,1,1,0,0,\ldots,0,0)$. By Lemma 4.2, the residual code $\text{Res}(C_1,w+1)$ is a doubly even self-complementary $[12,s]$ code with weight enumerator

$$W(z) = 1 + (2^{s-1} - 1)z^4 + (2^{s-1} - 1)z^8 + z^{12}.$$
Using the MacWilliams identities, we obtain that its dual code contains $2^{7-s} + 2$ codewords of weight 2. By Lemma 4.1, the dual code of $C_1$ contains a codeword of weight 2. □

By Theorem 2.2, $d_H(28) \leq 4$.

**Proposition 4.4.** If $C$ is a Type IV code of length 28 then the minimum distance $d_2$ of its torsion code is 2.

**Proof.** Suppose that there is a Type IV self-dual code $C$ with $d_2 = 4$. According to Proposition 4.1 in [2], the residue code $C_1$ must be a doubly even self-orthogonal [28, $k_1, d_1 \geq 8$] code whose dual code $C_2$ has the parameters [28, 28 - $k_1, 4]$. Using the Brouwer’s table [4], we have $6 \leq k_1 \leq 14$ and $d_1 = 8$ or 12. By Lemma 4.3, if $C_1$ contains a codeword of weight 12 then $d_2 = 2$. Hence $d_1 = 8$ and the weight enumerator of this code is $W(z) = 1 + (2^{k_1-1} - 1)z^8 + (2^{k_1-1} - 1)z^{20} + z^{28}$. It follows that its dual distance is 2 which contradicts $d_2 = 4$. □

$K_{28}$ has the following symmetrized weight enumerator

\[
a^{28} + 13421728a^{26}c^2 + 3784572c^4 + 376740a^{22}c^6 + 3108105a^{20}c^8 + 13123110a^{18}c^{10} + 30421755a^{16}c^{12} + 40116600a^{14}c^{14} + 30421755a^{12}c^{16} \\
+ 13123110a^{10}c^{18} + 3108105a^{8}c^{20} + 376740a^{6}c^{22} + 20475a^{4}c^{24} + 378a^{2}c^{26} + c^{28}.
\]

Moreover, since there are Type IV–I codes with $d_E = 8$ for lengths 12 and 16, the direct sum of the codes is a Type IV self-dual code of length 28 and $d_E = 8$. Thus we determine the highest minimum weights for length 28.

**Corollary 4.5.** $d_L(28) = 4$, $d_H(28) = 2$ and $d_E(28) = 8$.

5. Length 32

By Theorem 2.2, $d_H(32) \leq 4$ and $d'_H(32) \leq 4$.

**Proposition 5.1.** If $C$ is a Type IV self-dual code of length 32 with $d_2 = 4$, then $C$ is a Type IV–II code.

**Proof.** Suppose that there is a Type IV self-dual code $C$ with $d_2 = 4$. Then the residue code $C_1$ must be a doubly even self-orthogonal [32, $k_1, d_1 \geq 8$] code with dual
distance 4. By the tables in [4], \(6 \leq k_1 \leq 16\) and \(d_1 = 8, 12\) or 16.

(1) Suppose that there is a Type IV self-dual code \(C\) with \(d_1 = 16\). Then \(k_1\) must be 6 by the tables in [4]. Since \(C_1\) contains the all-ones vector, its weight enumerator is \(W(z) = 1 + (2^6 - 2)z^{16} + z^{32}\). All weights of codewords of \(C_1\) are divisible by 8. By Proposition 3.2 in [7], the corresponding \(\mathbb{Z}_4\)-code is Type IV–II.

(2) Suppose that there is a Type IV self-dual code \(C\) with \(d_1 = 12\). By Lemma 4.3, if \(C_1\) contains a codeword of weight 12 then \(d_2 = 2\).

(3) Suppose that there is a Type IV self-dual code \(C\) with \(d_1 = 8\). Since \(C_1\) contains the all-ones vector and contains no codeword of weight 12, \(C_1\) has the weight enumerator \(W(z) = 1 + A_8z^8 + A_{16}z^{16} + A_8z^{24} + z^{32}\). Since all weights of codewords of \(C_1\) are divisible by 8, the corresponding \(\mathbb{Z}_4\)-code is Type IV–II.

Therefore the result follows. \(\square\)

\(C_{5,1}\) is a Type IV–II code. The symmetrized weight enumerator is found in [1]. From the weight enumerator it follows that \(d_L = 8, d_H = 4\) and \(d_E = 16\). Therefore we have the following corollaries.

**Corollary 5.2.** \(d'_L(32) = 8, d'_H(32) = 4\) and \(d'_E(32) = 16\).

**Corollary 5.3.** The smallest length for which there is a Type IV–II code with \(d_E = 16\) is 32.

By Proposition 5.1, \(d_H(32) \leq 2, d_L(32) \leq 4\) and \(d_E(32) \leq 8\). Thus every Type IV–I code has \(d_H = 2\) and \(d_L = 4\). There are Type IV–I codes of length 16 and \(d_E = 8\), namely \(C_{16,5}\) and \(C_{16,9}\) in [9]. Thus, the direct sum of the codes is a Type IV–I code of length 32 and \(d_E = 8\). This determines the highest minimum weights of length 32.

**Corollary 5.4.** \(d_L(32) = 4, d_H(32) = 2\) and \(d_E(32) = 8\).

### 6. Length 36

**Proposition 6.1.** \(d_H(36) = 2\).

**Proof.** By Theorem 2.2, \(d_H(36) \leq 6\). First, suppose that there is a Type IV self-dual code \(C\) with \(d_2 = 6\). In this case \(C_2\) is a \([36, 36 - k_1, 6]\) code. According to tables in [4], \(k_1 \geq 11\). Then \(C_1\) is a doubly even \([36, k_1, d_1 \geq 12]\) code. By Lemma 4.3, if \(C_1\) contains a codeword of weight 12 then \(d_2 = 2\). Thus \(C_1\) is a doubly even self-complementary \([36, k_1, d_1 = 16]\) code. From the tables in [4], \(k_1 \leq 8\) which contradicts \(k_1 \geq 11\). Therefore \(d_2 \neq 6\).

Now suppose that there is a Type IV self-dual code \(C\) with \(d_2 = 4\). In this case \(d_1 \geq 8\). Since the code \(C_1\) does not contain codewords of weight 12, its weight
enumerator is

\[ W(z) = 1 + \alpha z^8 + \beta z^{16} + \gamma z^{20} + \alpha z^{28} + z^{36}. \]

Then the number of the vectors of weight 2 in the dual code \( C_2 \) is \((1280 + 384/\gamma) = 2^{k_1 - 10} \). It follows that \( \gamma = (5 \cdot 2^{k_1 - 10} - 10)/3 \) and \( \beta = (91 \cdot 2^{k_1 - 6} + 7)/3 \).

Hence \( k_1 \) is odd. If \( k_1 = 7 \), the code \( C_1 \) is a doubly even \([36, 7, 16]\) code with weight enumerator \( 1 + 63z^{16} + 63z^{20} + z^{36} \). There are exactly four such inequivalent \([36, 7, 16]\) codes [11]. Note that the codewords of weight 16 form a quasi-symmetric 2-(36, 16, 12) design with intersection numbers 4 and 6. By Theorem 2.1 these codes cannot be Type IV-residue codes. Therefore \( k_1 \geq 9 \). It follows that \( \alpha > 0 \) and \( d'_1 = 8 \). Without loss of generality, we can assume that \( \alpha = (1, 1, 1, 1, 1, 1, 1, 0, \ldots, 0) \in C_1 \). According to Lemmas 4.1 and 4.2, \( \text{Res}(C, x + 1) \) is a doubly even self-complementary code of length 8 with dual distance at least 4. There exists a unique code with these properties and it is the extended Hamming code \( e_8 \). Hence we can take a generator matrix of \( C_1 \) in the form

\[
G_1 = \begin{pmatrix}
11111111 & 0 & \cdots & 0 \\
11110000 & y_2 \\
11001100 & y_3 \\
10101010 & y_4 \\
O & G'_1
\end{pmatrix},
\]

where \( G'_1 \) generates a doubly even self-complementary \([28, k_1 - 4, d'_1] \) code with nonzero weights 8, 20 and 28. It follows that \( d'_1 = 8 \) and up to equivalence,

\[
G_1'' = \begin{pmatrix}
11111111 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
11110000 & v_2 & w_2 \\
11001100 & v_3 & w_3 \\
10101010 & v_4 & w_4 \\
00000000 & u_2 & z_2 \\
00000000 & u_3 & \vdots \\
00000000 & u_l & z_l \\
O & O & G''_1
\end{pmatrix},
\]

where \( G''_1 \) generates a doubly even self-complementary \([20, k'' \geq 1, d'' \geq 8] \) code. Hence \( k'' = 1 \) and \( d'' = 20 \). Since the vectors \( 1, u_2, \ldots, u_l, v_2, v_3, v_4 \) generate \( e_8 \), we have \( l \leq 4 \).
and \( k_1 - 4 = l + 1 \leq 5 \). It follows that \( k_1 \leq 9 \). Hence \( k_1 = 9 \) and

\[
G_1 = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & w_2 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & w_3 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & w_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & z_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & z_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & z_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1
\end{pmatrix}.
\]

Let \( w \) be the last row of this matrix. Then the code \( \text{Res}(C_1, w + 1) \) is a doubly even self-complementary code of length 20 with weight enumerator

\[
W(z) = 1 + A_4 z^4 + A_8 z^8 + A_{32} z^{12} + A_{44} z^{16} + z^{20}.
\]

If the dimension of this code is \( s \) then \( A_8 = 2^{s-1} - 1 - A_4 \). By the MacWilliams identities, the dual code of \( \text{Res}(C_1, w + 1) \) contains \( 2^{7-s} A_4 + 2^{7-s} \cdot 3 - 2 \) vectors of weight 2. It follows that the dual distance of this residual code is 4 only when \( A_4 = 2^{s-6} - 3 \). Since the vectors \( \mathbf{1}, w_2, w_3, w_4, z_2, z_3, z_4 \) generate \( \text{Res}(C_1, w + 1) \), its dimension is at most 7 and we have \( A_4 \leq -1 \) which is impossible. Therefore we have \( d_2 = 2 \).

By the above proposition, \( d_L(36) = 4 \) and \( d_E(36) \leq 8 \). \( K_{36} \) has the following symmetrized weight enumerator:

\[
a^{36} + 34359738368 b^{36} + 630 a^{34} c^2 + 58905 a^{32} c^4 + 1947792 a^{30} c^6 \\
+ 30260340 a^{28} c^8 + 254186856 a^{26} c^{10} + 1251677700 a^{24} c^{12} + 3796297200 a^{22} c^{14} \\
+ 7307872110 a^{20} c^{16} + 9075135300 a^{18} c^{18} + 7307872110 a^{16} c^{20} \\
+ 3796297200 a^{14} c^{22} + 1251677700 a^{12} c^{24} + 254186856 a^{10} c^{26} + 30260340 a^{8} c^{28} \\
+ 1947792 a^{6} c^{30} + 58905 a^{4} c^{32} + 630 a^{2} c^{34} + c^{36}.
\]

Hence \( K_{36} \) has minimum Euclidean weight 8. Moreover, since there are Type IV self-dual codes of lengths 12, 16, 20, 24 and \( d_E = 8 \), a Type IV self-dual code of length 36 and \( d_E = 8 \) is constructed by considering the direct sum. Therefore we have the following:

**Corollary 6.2.** \( d_L(36) = 4 \) and \( d_E(36) = 8 \).

7. **Length 40**

**Lemma 7.1.** \( d_H(40) \leq 4 \) and \( d'_H(40) = 2 \).
Proof. By Theorem 2.2, \( d_H(40) \leq 6 \) and \( d_H'(40) \leq 6 \). Suppose that there is a Type IV self-dual code \( C \) with \( d_2 = 6 \). By Lemma 4.3, \( d_1 \geq 16 \). By the tables in [4], there is no binary \([40,k,d_1 \geq 16]\) code with dual distance \( \geq 6 \).

Let \( C_1 \) be the residue code of a Type IV self-dual code \( C \) of length 40 with minimum Hamming weight 4. Then \( C_1 \) is a doubly even self-complementary \([40,k_1,d_1 \geq 8]\) code with weight enumerator

\[
W(z) = 1 + A_8 z^8 + A_{16} z^{16} + A_{20} z^{20} + A_{16} z^{24} + A_8 z^{32} + z^{40},
\]

where \( A_{20} = 2^{k_1} - 2 - 2A_8 - 2A_{16} \). Since the dual distance of \( C_1 \) is 4, then

\[
A_{16} = \frac{5}{16} 2^{k_1} - 25 - 9A_8, \quad A_{20} = 2^{k_1} - 2A_8 - 2A_{16} = \frac{3}{8} 2^{k_1} + 48 + 16A_8 > 0.
\]

It follows that \( C_1 \) contains a codeword of weight 20 and hence \( C \) is Type IV–I. \( \Box \)

We give a method to construct binary Type IV-residue codes.

**Theorem 7.2.** Suppose that \( n \equiv 0 \pmod{4} \). If \( B \) is a binary doubly even self-complementary \([n \geq 8,k>1,d]\) code with generator matrix \( G \). Then the following matrix

\[
\tilde{G} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ G & 0 & 0 & \cdots & 0 & G \end{pmatrix}
\]

generates a binary Type IV-residue \([2n,k+1,2d]\) code \( \tilde{B} \). Moreover, the weight enumerator is \( W_{\tilde{B}}(z) = W_B(z^2) + 2^k z^n \) and dual distance is \( \min\{d^\perp,4\} \), where \( d^\perp \) is the dual distance of \( B \).

**Proof.** It is easy to see that

\[
\tilde{B} = \{(x,x) | x \in B\} \cup \{(x+1,x) | x \in B\}.
\]

Since \( w_H(x,x) = 2w_H(x) \) and \( w_H(x+1,x) = n - w_H(x) + w_H(x) = n \), the weight enumerator of \( \tilde{B} \) is \( W_{\tilde{B}}(z) = W_B(z^2) + 2^k z^n \). Hence the minimum weight of \( \tilde{B} \) is 2d.

Let \( x \) be a codeword in \( B^\perp \) then \((x|0,\ldots,0)\) is a codeword of \( \tilde{B}^\perp \). Thus the dual distance of \( \tilde{B} \) is at most \( d^\perp \). Moreover \((1,1,0,\ldots,0|1,1,0,\ldots,0)\) is a codeword of \( \tilde{B}^\perp \).

Hence the dual distance is \( \min\{d^\perp,4\} \).

A codeword in \( \tilde{B} \) has forms \((x,x)\) or \((1+x,x)\), where \( x \in B \).

\[
w_H((x,x) \ast (y,y)) = 2w_H(x \ast y) \equiv 0 \pmod{4},
\]

\[
w_H((1+x,x) \ast (y,y)) = w_H((1+x) \ast y) + w_H(x \ast y)
\]

\[= w_H(y) - w_H(x \ast y) + w_H(x \ast y)
\]

\[= w_H(y) \equiv 0 \pmod{4},
\]

\[
w_H((1+x,x) \ast (1+y,y)) = w_H((1+x) \ast (1+y)) + w_H(x \ast y)
\]

\[= n - w_H(x) - w_H(y) + w_H(x \ast y) + w_H(x \ast y)
\]
\[ n - w_H(x) - w_H(y) + 2w_H(x * y) = n - w_H(x + y) \equiv 0 \pmod{4}. \]

Thus \( \tilde{B} \) satisfies the above condition. \( \square \)

**Remark.** If \( n \equiv 0 \pmod{8} \) then the weights of codewords in \( \tilde{B} \) are divisible by eight. By Proposition 3.2 in [7], the corresponding \( \mathbb{Z}_4 \)-code is Type II.

We investigate the minimum weights of \( \tilde{B} + 2\tilde{B}^\perp \) where \( \tilde{B} \) is the binary code constructed by Theorem 7.2.

**Proposition 7.3.** Let \( B \) and \( \tilde{B} \) be the codes given in Theorem 7.2. If the dual distance of \( B \) is greater than or equal to 4 and the minimum weight of \( B \) is greater than or equal to 8 then the Type IV self-dual code \( \tilde{B} + 2\tilde{B}^\perp \) has minimum Hamming weight 4, minimum Lee weight 8 and minimum Euclidean weight 16.

**Proof.** By Theorem 7.2, the dual distance of \( \tilde{B} \) is 4. Thus, the minimum Hamming weight and the minimum Lee weight of \( \tilde{B} + 2\tilde{B}^\perp \) are 4 and 8, respectively. Since the minimum weight of \( \tilde{B} \) is greater than or equal to 16, a codeword \( x \) of Hamming weight \( \leq 14 \) in \( \tilde{B} + 2\tilde{B}^\perp \) satisfies \( n_1(x) + n_3(x) = 0 \). Therefore \( \tilde{B} + 2\tilde{B}^\perp \) has minimum Euclidean weight 16. \( \square \)

Let \( B_{20} \) be the code with generator matrix

\[
G_{20} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}.
\]

By Theorem 7.2, \( \overline{B_{20}} \) is a binary Type IV-residue \([40,10,8]\) code. Moreover, it has the following weight enumerator:

\[ 1 + 5z^8 + 250z^{16} + 512z^{20} + 250z^{24} + 5z^{32} + z^{40}. \]

Hence \( \overline{B_{20}} + 2\overline{B_{20}}^\perp \) is a Type IV self-dual code. The Lee weight enumerator is

\[ 1 + 310y^8 + 9600y^{12} + 322605y^{16} + \cdots. \]

Hence \( d_H = 4 \) and \( d_L = 8 \). Since \( \overline{B_{20}}^\perp \) has the following weight enumerator:

\[ 1 + 230z^4 + 8320z^6 + 149165z^8 + 1645952z^{10} + \cdots, \]
there are codewords \( c \) with \( n_0(c) = 32, n_1(c) + n_3(c) = 8 \) and \( n_2(c) = 0 \). Thus \( B_{20} \perp 2B_{20} \perp \) has minimum Euclidean weight 8.

In order to construct a Type IV self-dual code with \( d_E = 16 \), we consider another matrix as \( B \). Let \( C_{40} \) be the code with the following generator matrix. The matrix is written using the form \( g_1, g_2, \ldots, g_{30} \) where \( g_i \) is the \( i \)th row:

\[
\begin{align*}
g_1 &= 10000000022101101111101111100110111000, \\
g_2 &= 01000002201010111111010000010100110110, \\
g_3 &= 0010002200100111101010110000001101010101, \\
g_4 &= 00010220000101111011000010010011011010110, \\
g_5 &= 0001200020110011110101000101011000110101, \\
g_6 &= 002210000011011010110000001101101010111, \\
g_7 &= 02200010001110000000000111100000000000, \\
g_8 &= 022000010000000111000000000011110000000, \\
g_9 &= 22000000100000000000111100000000001111, \\
g_{10} &= 20002000100000000000011111111111111111, \\
g_{11} &= 0000000000000200000000200000000000002, \\
g_{12} &= 0000000000020000000020000000000000002, \\
g_{13} &= 0000000000000200000000200000000000002, \\
g_{14} &= 0000000000000200000000200000000000002, \\
g_{15} &= 0000000000000200000000200000000000002, \\
g_{16} &= 0000000000000200000000200000000000002, \\
g_{17} &= 0000000000000200000000200000000000002, \\
g_{18} &= 0000000000000200000000200000000000002, \\
g_{19} &= 0000000000000200000000200000000000002, \\
g_{20} &= 0000000000000200000000200000000000002, \\
g_{21} &= 0000000000000200000000200000000000002, \\
g_{22} &= 0000000000000200000000200000000000002, \\
g_{23} &= 0000000000000200000000200000000000002, \\
g_{24} &= 0000000000000200000000200000000000002, \\
g_{25} &= 0000000000000200000000200000000000002, \\
g_{26} &= 0000000000000200000000200000000000002, \\
g_{27} &= 0000000000000200000000200000000000002, \\
g_{28} &= 0000000000000200000000200000000000002, \\
g_{29} &= 0000000000000200000000200000000000002, \\
g_{30} &= 0000000000000200000000200000000000002.
\]
The code $C_{40}$ has the following Lee weight enumerator:

$$1 + 230y^8 + 10112y^{12} + 270509y^{16} + \cdots.$$  

Comparing the weight enumerator of $B_{20}^\perp$ and the Lee weight enumerator of $C_{40}$, the codewords of Lee weight 8 have Euclidean weight 16. Moreover, we have verified that the codewords of Lee weight 12 have Euclidean weight $\geqslant 16$. Thus $C_{40}$ has minimum Euclidean weight 16. Therefore we have the following:

**Proposition 7.4.** $d_H(40) = 4$, $d_L(40) = 8$ and $d_E(40) = 16$.

**Remark.** By Construction $A_4$ (cf. [1]), the lattice obtained from $C_{40}$ is an optimal odd unimodular lattice with minimum norm 4.

**Corollary 7.5.** The smallest length for which there is a Type IV–I code with $d_H = 4$ is 40. The smallest length for which there is a Type IV–I code with $d_L = 8$ is 40. The smallest length for which there is a Type IV–I code with $d_E = 16$ is 40.

Combined with Lemma 7.1, we have the following:

**Corollary 7.6.** There is a Type IV–I code of length 40 such that the minimum Hamming, Lee and Euclidean weights are higher than any Type IV–II code of that length.

**Remark.** For binary self-dual codes, it is not still known if there is a Type I code with higher minimum weight than any Type II code of that length (cf. [5]).

8. Larger lengths

**Lemma 8.1.** If $C_1$ contains a codeword of weight 16 or 20 then $d_2 \leqslant 4$.

**Proof.** Suppose that $C$ is a Type IV self-dual code with $d_2 \geqslant 6$ and $w \in C_1$ is a codeword of weight 20. According to Lemma 4.2, the residual code $\text{Res}(C_1, w + 1)$ is a doubly even self-complementary code. Moreover, since the Gray map image of a Type IV $\mathbb{Z}_4$-code is linear, $x \ast y \in C_2$ for any $x$ and $y \in C_1$ [8]. Thus the residual code has minimum weight $\geqslant 6$. Hence it is a $[20, s, 8]$ code with weight enumerator

$$W(z) = z + (2^{s-1} - 1)z^8 + (2^{s-1} - 1)z^{12} + z^{20}.$$  

Using the MacWilliams identities, we obtain that its dual code contains $3 \cdot 2^{8-s} - 2 \geqslant 1$ codewords of weight 2. Therefore, the dual code of $C_1$ contains a codeword of weight 2, which contradicts the dual distance of $C_1$.

Suppose now that there is a codeword $u \in C_1$ of weight 16. Similarly, the residual code $\text{Res}(C_1, u + 1)$ is a doubly even self-complementary $[16, s, 8]$ code. But all codes with such parameters have dual distance 2 or 4. Hence the dual distance of $C_1$ must be 2 or 4. $\square$
Proposition 8.2. If $C$ is a Type IV code of length 44, 48, 52 or 56 then the minimum distance $d_2$ of its torsion code is $\leq 4$.

Proof. By Theorem 2.2, $d_2 \leq 8$. Suppose that there is a Type IV self-dual code $C$ of length $n = 44, 48, 52$, or 56, and $d_2 \geq 6$. By Lemma 8.1, the residue code $C_1$ must be a doubly even self-complementary $[n,k_1,d_1 \geq 24]$ code. We use the tables from [4]. Let $d(n,k)$ be the highest minimum distance for which a linear binary $[n;k;d(n,k)]$ code exists.

1. Let $n=44$. Since $d(44,4) = 23$ and $d(44,3) = 24$, we have $k_1 \leq 3$. But $d(44,41) = 2$ and hence the dual distance of any linear binary $[44,k_1 \geq 24]$ code is at most 2.

2. Let $n=48$. Since $d(48,7) = 22$ and $d(48,6) = 24$, we have $k_1 \leq 6$. But $d(48,42) = 3$ and hence the dual distance of any linear binary $[48,k_1 \geq 24]$ code is at most 3.

3. Let $n=52$.

4. Let $n=56$. Since $d(56,12) = 22$ and $d(56,8) = 24$, we have $k_1 \leq 8$. But $d(56,44) = 4$ and hence the dual distance of any linear binary $[52,k_1 \geq 24]$ code is at most 4.

Therefore the result follows.

By Proposition 7.3, we construct a Type IV self-dual $\mathbb{Z}_4$-code of length 48, $d_H = 4$, $d_L = 8$ and $d_E = 16$. Let $G_{24}$ be the binary extended Golay code of length 24. It is well known that $G_{24}$ is self-dual and its minimum weight is 8. Let $\overline{G_{24}}$ be the binary code of length 48 constructed by Theorem 7.2 from $G_{24}$. By Proposition 7.3, the Type IV self-dual $\mathbb{Z}_4$-code $\overline{G_{24}} + 2\overline{G_{24}} \perp$ has $d_H = 4$, $d_L = 8$ and $d_E = 16$. Since all the weights of $\overline{G_{24}}$ are divisible by eight, $\overline{G_{24}} + 2\overline{G_{24}} \perp$ is Type IV–II. The Lee weight enumerator is

$$1 + 276y^8 + 302082y^{16} + \cdots.$$ 

Therefore we have the following:

Corollary 8.3. $d'_H(48) = 4$, $d'_L(48) = 8$ and $d'_E(48) = 16$.

Similarly, for larger lengths, Proposition 7.3 is very useful to construct Type IV self-dual codes with $d_H = 4$, $d_L = 8$ and $d_E = 16$. It is known that there are binary self-dual codes with minimum weight 8 for lengths $\geq 32$ (cf. [5]). Therefore we have the following:

Proposition 8.4. For lengths $n \geq 64$ and $n \equiv 0 \pmod{16}$, there are Type IV–II $\mathbb{Z}_4$-codes with $d_H = 4$, $d_L = 8$ and $d_E = 16$. For lengths $n \geq 72$ and $n \equiv 8 \pmod{16}$, there are Type IV–I $\mathbb{Z}_4$-codes with $d_H = 4$, $d_L = 8$ and $d_E = 16$.

Corollary 8.5. $d'_H(56) = 2$, $d'_L(56) = 4$ and $d'_E(56) = 8$. 


Proof. Let $C_1$ be the residue code of a Type IV self-dual code $C$ of length 56 with minimum Hamming weight 4. Then $C_1$ is a doubly even self-complementary $[56, k_1, d_1 \geq 8]$ code with weight enumerator
\[ 1 + A_8 z^8 + A_{16} z^{16} + A_{20} z^{20} + A_{24} z^{24} + A_{28} z^{28} + A_{24} z^{32} + A_{20} z^{36} + A_{16} z^{40} + A_{8} z^{48} + z^{56}, \]
where $A_{28} = 2^{k_1 - 2} - 2A_8 - 2A_{16} - 2A_{20} - 2A_{24}$. Since the dual distance of $C_1$ is 4, then
\[ A_{24} = 7 \cdot 2^{k_1 - 4} - 4A_{20} - 9A_{16} - 25A_8 - 49 \]
and
\[ A_{28} = 2^{k_1 - 3} + 96 + 48A_8 + 16A_{16} + 6A_{20} \geq 0. \]
It follows that $C_1$ contains a codeword of weight 28 and hence $C$ is Type IV–I.

The highest minimum weight of binary self-dual codes of length 28 is 6 (cf. [5]). However, by considering the doubly even subcode of some binary self-dual code, a Type IV–I code with \(d_H = 4, d_L = 8\) and \(d_E = 16\) is constructed as follows. Let $B_0$ be the doubly even subcode of the self-dual $[28, 14, 6]$ code $B_{28}$ (see [5]). Its weight enumerator is $W_{28} = 1 + 42z^6 + 378z^8 + 1624z^{10} + 3717z^{12} + 4680z^{14} + \cdots$. The weight enumerators of $B_0$ and $B_0^\perp$ are
\[ 1 + 378z^8 + 3717z^{12} + 3717z^{16} + 378z^{20} + z^{28} \]
and
\[ 1 + 126z^6 + 378z^8 + 4872z^{10} + 3717z^{12} + 14580z^{14} + \cdots, \]
respectively. Let $\overline{B_0}$ be the binary code of length 56 constructed by Theorem 7.2 from $B_0$. By Proposition 7.3, the Type IV self-dual $\mathbb{Z}_4$-code $\overline{B_0} + 2\overline{B_0^\perp}$ has $d_H = 4$, $d_L = 8$ and $d_E = 16$. Its Lee weight enumerator is
\[ 1 + 378y^8 + 4032y^{12} + 254331y^{16} + \cdots. \]

Corollary 8.6. \(d_H(56) = 4, d_L(56) = 8\) and \(d_E(56) = 16\).

Remark. In order to construct a Type IV code with \(d_E = 16\), we need to compute only the minimum weight of $B_0^\perp$. The minimum weight is also obtained considering the minimum weight of the shadow of $B_{28}$.

Corollary 8.7. \(d_H'(64) = 4, d_L'(64) = 8\) and \(d_E'(64) = 16\).

Proof. Suppose that $C$ is a Type IV–II self-dual code with $d_2 \geq 6$. By Lemma 8.1, $C_1$ must be a doubly even self-complementary $[64, k_1, d_1 \geq 24]$ code with weight enumerator
\[ 1 + A_{24} z^{24} + A_{32} z^{32} + A_{24} z^{40} + z^{64}, \]
where \( A_{32} = 2^{k_i} - 2 - 2A_{24} \). Then the number of the codewords of weight 2 in \( C_2 \) is 
\[ 2^{12-k_i} \cdot A_{24} - 32 \] and since it must be 0, we have

\[ A_{24} = 2^{k_i-3} - 16 \quad \text{and} \quad A_{32} = 3 \cdot 2^{k_i-2} + 30. \]

Hence \( C_2 \) contains \( 5 \cdot 2^{18-k_i} + 176 > 0 \) codewords of weight 4 which contradicts \( d_2 \geq 6 \).

It follows that the highest value of \( d_2 \) is 4. By Proposition 8.4, a Type IV–II code with \( d_H = 4, \ d_L = 8 \) and \( d_E = 16 \) exists. \( \square \)

As a summary, the highest minimum weights of Type IV–I (resp. Type IV–II) codes are listed in Table 8 (resp. Table 9).

**Theorem 8.8.** The highest minimum Hamming, Lee and Euclidean weights of Type IV–I codes of lengths up to 40 and length 56 are determined. The highest minimum Hamming, Lee and Euclidean weights of Type IV–II codes of lengths up to 64 are determined.
We end this paper with the following problems:

1. For lengths 44, 48 and 52, determine the highest minimum weights of Type IV–I codes.

2. The known upper bounds on minimum weights of Type IV self-dual codes are not tight in general. Hence find improved upper bounds.

3. It is not known if there is a Type IV self-dual code with minimum Hamming weight $\geq 6$. In particular, determine the smallest lengths for which such a code exists. In this paper and [7], the smallest length for which there is a Type IV–I and Type IV–II code with minimum Hamming weight 4 was determined, respectively.

4. Find a construction of binary Type IV-residue codes and Type IV self-dual $\mathbb{Z}_4$-codes with higher minimum weights. In particular, find a construction of binary Type IV-residue codes for lengths $n \equiv 4 \pmod{8}$.

5. Our classification of Type IV self-dual codes of length 20 given in Section 3 has been verified by the mass formula. However, the mass formula depends on the classification of binary Type IV-residue codes. The mass formula for the classification of such codes is not still known.

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References


