Loops, regular permutation sets and graph colourings

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\textbf{Abstract}
We establish a correspondence among loops, regular permutation sets and directed graphs with a suitable edge colouring and relate some algebraic properties of the loop to configurations of the associated graph.

\textit{Keywords:} Loops, regular permutation sets, colouring of directed graphs

\section{Introduction}
Let $P$ be a non-empty set, $\Gamma$ a set of permutations acting regularly on $P$ and $o$ a distinguished point of $P$, then it is possible to equip $P$ with a loop operation with neutral element $o$ (see e.g. [2]), conversely if $(P, +)$ is a loop, it is possible to associate to $P$ a regular permutation set in several ways. Here, motivated by our works on slid product of loops [5] and [4], we deepen the investigation

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of the relationships between regular permutation sets and loops. Since many different regular permutation sets can be associated to a loop, we characterize those giving rise to the same loop, to isomorphic loops and to isotopic loops.

A correspondence between loops and directed graphs with a suitable edge colouring is introduced and some of the algebraic properties of the loop \((P,+)\) are related to configurations of the associated graph. All the results presented here have been proved in [3]. For the notation on loops we shall refer to [1].

2 Loops and regular permutation sets

Let \(P\) be a non-empty set (whose elements are called points), \(\Gamma \subseteq \text{Sym} P\) a regular permutation set, \(J := \{ \gamma \in \text{Sym} P \mid \gamma^2 = \text{id}\}\) and \(o \in P\) be a distinguished point. The triple \((P,\Gamma,o)\) will denote a regular permutation set acting on \(P\) together with a fixed element \(o \in P\); for all \(a,b \in P\) the unique element of \(\Gamma\) mapping \(a\) to \(b\) is denoted by \(\tilde{\gamma}ab\) (when \(a = b\) we simply write \(\tilde{\gamma}a\)).

Starting from \((P,\Gamma,o)\) we can define the following loop operation on \(P\) (see e.g. [2]): \(\forall a,b \in P:\ a + b := \tilde{\gamma}oa \circ \tilde{\gamma}o^{-1}(b)\).

It is easy to see that \(o\) is the neutral element and that if we denote by \(\nu : P \to P; a \mapsto \tilde{\gamma}oa \circ \tilde{\gamma}o^{-1}(o)\), then \(a + \nu(a) = o\). Moreover for all \(a \in P\) let \(a^+ : P \to P; x \mapsto a + x\) and \(P^+ := \{a^+ \mid a \in P\}\).

**Definition 2.1** The loop \((P,+,o)\) defined as above from \((P,\Gamma,o)\) is called loop derivation in the point \(o\). We will denote such loop by \(L(P,\Gamma,o)\).

**Definition 2.2** Let \((P,+,o)\) be a loop with neutral element \(o\) and let \(\Gamma := \nu \circ \omega\); \(\nu \circ \omega\) is a regular permutation set called permutation derivation of the loop \((P,+,o)\), denoted by \(P(P,+,o)\).

**Proposition 2.3** Let \((P,+,o) = L(P,\Gamma,o)\) and \((P,+_1,o_1) = L(P,\Gamma,o_1)\). Then there exist \(\alpha,\beta \in \text{Sym} P\) such that \((\alpha,\beta,\text{id})\) is a principal isotopism of \((P,+_1)\) upon \((P,+)\) (i.e. for all \(a,b \in P\): \(\alpha(a) +_1 \beta(b) = a + b\)).

**Proposition 2.4** Let \((P,+,o) = L(P,\Gamma,o)\) and \(\Gamma = P^+ \circ \nu\). Then

(i) \(\Gamma = \Gamma \circ \omega\) where \(\omega : P \to P; x \mapsto \tilde{\omega}x^{-1}(o)\);
(ii) \(P^+ = \Gamma \circ \tilde{\omega}^{-1}\);
(iii) \(\Gamma = \Gamma\) if and only if for all \(a \in P\): \(\tilde{\omega}a(o) = o\);
(iv) \(\Gamma = P^+\) if and only if \(\text{id} \in \Gamma\);
(v) \(L(P,\Gamma,o) = L(P,\Gamma,o) = L(P,P^+,o)\).

We aim now at investigating the relationship between regular permutation sets giving rise to isotopic loops, to isomorphic loops and to the same loop.
Note that for a fixed $\Gamma \subseteq \text{Sym} P$ and for any $\alpha, \beta \in \text{Sym} P$ the set $\Gamma_1 = \alpha \Gamma \beta$ acts regularly on $P$ whenever $\Gamma$ acts regularly on $P$, in fact for all $x, y \in P$ if $\gamma$ is the unique permutation of $\Gamma$ mapping $\beta(x)$ to $\alpha^{-1}(y)$, then $\alpha \gamma \beta(x) = y$, with $\alpha \gamma \beta \in \Gamma_1$. We have the following results:

**Proposition 2.5** Let $(P, \Gamma, o)$ and $(P, \Gamma_1, o_1)$ be regular permutation sets.

(i) If $\mathcal{L}(P, \Gamma, o)$ and $\mathcal{L}(P, \Gamma_1, o_1)$ are isotopic, then there exist $\varphi, \psi \in \text{Sym} P$ such that $\Gamma_1 = \varphi \Gamma \psi$;

(ii) if $\Gamma_1 = \varphi \Gamma \psi$ with $\varphi, \psi \in \text{Sym} P$, then $\mathcal{L}(P, \Gamma, o)$ and $\mathcal{L}(P, \Gamma_1, o_1)$ are isotopic loops.

**Proposition 2.6** (see also [2, 3.1 and 3.2]) Let $(P, \Gamma, o)$ and $(P, \Gamma_1, o_1)$ be regular permutation sets.

(i) If there exists an isomorphism $\alpha \in \text{Sym} P$ between $\mathcal{L}(P, \Gamma_1, o_1)$ and $\mathcal{L}(P, \Gamma, o)$, then there exists $\beta \in \text{Sym} P$ with $\beta(o_1) = o$ such that $\Gamma_1 = \alpha^{-1} \Gamma \beta$;

(ii) if there exist $\alpha, \beta \in \text{Sym} P$ with $\alpha(o_1) = \beta(o_1) = o$ such that $\Gamma_1 = \alpha^{-1} \Gamma \beta$, then $\alpha$ is an isomorphism between the loops $\mathcal{L}(P, \Gamma_1, o_1)$ and $\mathcal{L}(P, \Gamma, o)$.

As a corollary of (2.6) we have:

**Proposition 2.7** Let $(P, \Gamma, o)$ and $(P, \Gamma_1, o)$ be regular permutation sets, $(P, +, o) = \mathcal{L}(P, \Gamma, o)$ and $(P, +_1, o) = \mathcal{L}(P, \Gamma_1, o)$. If there exists $\alpha \in \text{Sym} P$ such that $\alpha(o) = o$ and $\Gamma_1 = \Gamma \alpha$ then “+” = “+1”.

Conversely if “+” = “+1” then $\Gamma_1 = \Gamma \circ \alpha$ where $\alpha \in \text{Sym} P$ and $\alpha(o) = o$.

The results above motivate the following definition:

**Definition 2.8** The regular permutation sets $(P, \Gamma_1, o)$ and $(P, \Gamma_2, o)$ are called l-equivalent if there exists $\alpha \in \text{Sym} P$ such that $\Gamma_2 = \Gamma_1 \circ \alpha$ and $\alpha(o) = o$.

3 **Loops and edge colourings of directed graphs**

Let $P$ be a non-empty set and let $\mathcal{O} = \{(x, y) \mid x, y \in P\}$, then $(P, \mathcal{O})$ is a complete directed graph. Given a non-empty set $K$ whose elements will be called colours, a map $k : \mathcal{O} \to K$ is called an edge colouring of the graph $(P, \mathcal{O})$. Assume now the following axiom:

(G0) For all $c \in K$, for all $p \in P$ there exists exactly a pair $(x, y) \in P^2$ such that $k(p, x) = c$ and $k(y, p) = c$.

(G0) implies that $k$ is a surjective map and that, if $o$ is a distinguished point, the map $f_o : P \to K$, $x \mapsto k(o, x)$ is a bijection, thus we can identify every colour $c \in K$ with the unique point $p \in P$ such that $f_o(p) = c$, hence
for all \( p \in P \): \( k(o, p) = p \).

From now on the quadruple \( (P, O, k, o) \) will denote a complete directed graph \( (P, O) \) together with the edge colouring \( k \) fulfilling (G0), a fixed element \( o \in P \) and the identification \( K = P \).

Every \( c \in P \) induces in a natural way the following map:

\[ \tilde{o}c : P \to P, \ x \mapsto y \quad \text{with} \quad k(x, y) = c. \]

Let \( \Gamma_k := \{ \tilde{o}c \mid c \in P \} \), then by (G0) we have:

**Proposition 3.1** \((P, \Gamma_k, o)\) is a regular permutation set.

By Proposition 3.1 and following Section 2, from the graph \((P, O, k, o)\) we can derive the loop \( \mathcal{L}(P, \Gamma_k, o) \) which will be called again the loop derivation of the graph \((P, O, k, o)\) and denoted by \( \mathcal{L}(P, O, k, o) \).

Note that if we define \( \tilde{o}c^1 : P \to P, \ x \mapsto y \quad \text{with} \quad k(y, x) = c \), then the set \( \Gamma_{k}^1 := \{ \tilde{o}c^1 \mid c \in P \} \) is a regular permutation set as well and \( \Gamma_{k}^1 = \Gamma_k^{-1} \). In general the algebraic structure induced by \( \Gamma_k \) and by \( \Gamma_{k}^1 \) are pretty different, in fact if we consider the loops \( \mathcal{L}(P, \Gamma_k, o) \) and \( \mathcal{L}(P, \Gamma_{k}^1, o) \) these two loops could even not be isotopic (see [3] for an example).

Conversely if \((P, \Gamma, o)\) is a regular permutation set we can define an edge colouring of the complete directed graph \((P, O)\) in the following way:

**Proposition 3.2** Let \((P, \Gamma, o)\) be a regular permutation set and let \( k : O = P^2 \to P ; (x, y) \mapsto \tilde{xy}(o) \), then \((P, O, k, o)\) is a coloured complete directed graph fulfilling (G0).

Starting from a loop \((P, +, o)\) and considering the permutation derivation \( \mathcal{P}(P, +, o) = (P, \bar{\Gamma}, o) \) (see def. 2.2) we can now associate to \((P, \bar{\Gamma}, o)\) the graph \((P, P^2, k, o)\), where \( k \) is defined by Proposition 3.2 starting from \( \bar{\Gamma} \), which is called the graph derivation of the loop \((P, +, o)\) and denoted by \( \mathcal{G}(P, +, o) \).

In [4] the following axioms for \((P, \Gamma, o)\) are considered:

(L1) For all \( a \in P \): \( \tilde{\bar{a}}(a) = o \)

(L2) \( \bar{o} \in J \)

(L3) \( \Gamma \subseteq J \)

(L4) for all \( a, b \in P \): \( \tilde{a}a^{-1}b \in \Gamma \)

For a given coloured complete directed graph \((P, O, k, o)\) we introduce the following conditions:

(G1) for all \( x \in P \): \( k(o, x) = k(x, o) \)

(G2) for all \( x \in P \): \( k(x, \tilde{o}(x)) = k(\tilde{o}(x), x) \)

(G3) for all \( x, y \in P \): \( k(x, y) = k(y, x) \)
(G4) for all \((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in P^4\): \(k(x_1, x_2) = k(y_1, y_2), k(x_3, x_4) = k(y_3, y_4), k(x_3, x_2) = k(y_3, y_2) = o \implies k(x_1, x_4) = k(y_1, y_2)\)

The following holds true:

**Proposition 3.3** Let \((P, \mathcal{O}, k, o)\) be a coloured complete directed graph and \((P, \Gamma_k, o)\) the corresponding regular permutation set, then \((P, \Gamma_k, o)\) satisfies (Li) if and only if \((P, \mathcal{O}, k, o)\) satisfies (Gi) for \(i = 1, 2, 3, 4\).

**Definition 3.4** Two coloured directed complete graphs \((P, \mathcal{O}, k, o)\) and \((P, \mathcal{O}, k_1, o)\) are said to be l-equivalent if there exists \(\alpha \in \text{Sym } P\) such that \(\alpha(o) = o\) and for all \(x, y \in P\): \(k_1(x, y) = k(\alpha(x), y)\).

**Proposition 3.5** The following statements are equivalent:

(i) \((P, \mathcal{O}, k, o)\) and \((P, \mathcal{O}, k_1, o)\) are l-equivalent graphs;

(ii) the corresponding regular permutation sets \((P, \Gamma_k, o)\) and \((P, \Gamma_{k_1}, o)\) are l-equivalent;

(iii) \(L(P, \mathcal{O}, k, o) = L(P, \mathcal{O}, k_1, o)\).

In [3] some properties of the loop \((P, +, o)\) (such as the presence of elements of exponent 2, or having two sided inverses, the left inverse property...) are translated in terms of configurations of the corresponding coloured complete directed graph \(G(P, +, o)\).

**References**


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