Distributed consensus under ambiguous information

Gabriele Oliva*

Complex Systems and Security Laboratory,
University Campus Biomedico of Rome,
Via Álvaro del Portillo 21, 00128, Roma, Italy
E-mail: g.oliva@unicampus.it
*Corresponding author

Stefano Panzieri

Dipartimento di Informatica e Automazione,
University 'Roma TRE',
Via della Vasca Navale, 79, 00146, Roma, Italy
E-mail: panzieri@uniroma3.it

Roberto Setola

Complex Systems and Security Laboratory,
University Campus Biomedico of Rome,
Via Álvaro del Portillo 21, 00128, Roma, Italy
E-mail: r.setola@unicampus.it

Abstract: This paper addresses the problem of composing different ambiguous and vague pieces of information without a central authority through a set of distributed agents, each with limited perspective, converging to a shared point of view by exchanging information only with a reduced subset of nodes (i.e., their respective neighbourhood). To this end, the distributed consensus problem is extended in the fuzzy fashion. As a result, the framework allows to compose several heterogeneous and ambiguous/linguistically expressed opinions in a decentralised way, both in terms of value with higher belief and in terms of ambiguity associated to the final agreement value. The proposed framework is applied to a case study related to crisis management for critical infrastructures, where human operators, each able to observe directly the state of a given infrastructure (or of a given area considering the vast and geographically dispersed infrastructures), reach a distributed consensus on the overall criticality of a situation expressed in a linguistic, fuzzy way. Such a consensus is reached in terms of actual severity of the scenario (single integrators) or in terms of both severity and evolution tendency (double integrators).

Keywords: critical infrastructures; fuzzy systems; consensus; distributed systems.

Reference to this paper should be made as follows: Oliva, G., Panzieri, S. and Setola, R. (2013) 'Distributed consensus under ambiguous information', *Int. J. System of Systems Engineering*, Vol. 4, No. 1, pp.55–78.

Biographical notes: Gabriele Oliva received his Master's degree in Computer Science and Automation in 2008 and his PhD in Computer Science and Automation in 2012, both at University Roma Tre of Rome, Italy. His research interests include fuzzy systems, distributed systems and synchronisation, critical infrastructures and in general complex systems. He is a member of European Society for Fuzzy Logic and Technology, Italian Association of Critical Infrastructures' Experts; IEEE Control Systems Society and Robotics and Automation Society. He is involved in several national and European projects on critical infrastructures interdependency modelling and networked control.

Stefano Panzieri received his Laurea degree in Electronic Engineering in 1989 and his PhD in Systems Engineering in 1994, both from the University of Roma 'La Sapienza'. Since February 1996, he has been with the 'Dipartimento di Informatica e Automazione' (DIA) of the University of 'Roma Tre', where he is currently an Associate Professor. His research interests are in the field of industrial control systems, robotics and sensor fusion. In the field of critical infrastructures protection, he has contributed to the development of a simulation model, the CISIA project. He is also interested in the security-related problems of industrial controllers (SCADA), and more recently, in the application of complex networks theory.

Roberto Setola received his Laurea degree in Electronic Engineering (1992) and his PhD (1996) from the University of Naples 'Federico II'. From 1999 to 2004, he served at the Italian Prime Minister's Office. He is currently Associated Professor of Automatic Control at the University CAMPUS Bio-Medico and Head of the Complex System and Security Lab. He was responsible for the Italian Government Working Group on CIIP and a member of the G8 Senior Experts' group for CIP/CIIP. He wrote three books about the simulation of dynamic systems and more than 100 scientific papers related to the modelling and control of complex systems (electro-mechanical, biological and social) and critical infrastructures.

1 Introduction

In the literature, the distributed *consensus* of dynamic agents has been widely investigated, both in the continuous (Olfati-Saber and and Murray, 2004; Olfati-Saber et al., 2007; Ren, 2008; Ren and Beard, 2008) and discrete-time (Ren, 2007, 2008; Ren and Beard, 2008) cases. The behaviour of the agents is typically expressed as a single integrator, whereby the agents maintain their value if isolated (Olfati-Saber and Murray, 2004). However, more recently the consensus of double integrators has been inspected (Olfati-Saber et al., 2007; Ren, 2008), due to their capability to model the behaviour of agents or systems whose initial state grows with a constant ratio if isolated. According to the literature, the state of each agent is assumed as perfectly known, while in many cases this is not verified. This is especially true when the agents represent human operators, thus representing in a linguistic and vague way the value of the different state variables. Therefore, there exists a need to handle such an ambiguity and subjectivity in a proper way.

Consider the case of a failure or terroristic attack affecting a scenario composed of different, highly interconnected *critical infrastructures* (Haimes and Jiang, 2001; Haimes et al., 2005): in this case, each operator has the duty to determine the actual level of criticality in order to adopt the most suitable strategies. However, due to the geographical extension and dispersion of such systems, each operator may observe only a portion of the whole system.

Hence, in all those situations where there is not a central authority, there is the need to reach consensus on the overall level of criticality. From the operators with partial vision, resultant of limited and localised information, to the presence of ambiguous and clashing statistics, the need to make informed decisions based on compatible data is paramount.

This generally happens immediately after a negative event when the different operators have very limited and local information.

Since an attack may inhibit or destroy a central authority, it is important to note the value of a distributed approach. The collapse of a centralised coordination due to the lack of any recognised hierarchy could also benefit from such a method.

Another interesting feature is the ability to reach consensus on the expected evolution of the crisis thus providing a higher level of clarity to emergency response units and incident commanders.

Yet, another critical factor to consider is the need to provide standardisation for the consensus of interconnected systems characterised by vague information. This can be achieved within the framework of the fuzzy theory, assuming that the state of each agent and/or perception of the failure, is described by means of fuzzy variables which are derived from the linguistic expressions used by the operators.

To this end, in this paper, the consensus of arrays of *fuzzy difference systems* (FDS) (Oliva et al., 2011, 2012; Oliva, 2012) is addressed. Specifically, the fuzzy consensus problem is formally defined, considering the cases of single and double discrete-time integrators.

The paper is organised as follows: after some preliminary definitions, Section 2 is devoted to review the consensus problem for crisp systems; in Section 3, the theory of FDS is discussed; a framework for the consensus of fuzzy single and double discrete-time integrators is provided in Section 4; while the proposed case study, as well as some simulation results are provided in Section 5; finally, some conclusive remarks are collected in Section 6.

1.1 Preliminaries

Let us now provide some preliminary definitions and concepts that will be used in the following. Note that, in the following, boldface variables will indicate vectorial quantities.

In the following, to avoid confusions, $\mathbf{x}$ will denote a vector with fuzzy entries, while crisp (i.e., non-fuzzy) vectors will be denoted as $\mathbf{z}$. Let $\mathbb{R}$, $\mathbb{N}$ be the set of reals and integers, respectively and $\mathbb{R}^+$, $\mathbb{N}^+$ be the set of non-negative real and integer numbers, respectively.

In this paper, we will consider the following distance in $\mathbb{R}^N$, defined as

$$d_{\mathbb{R}^N}\left(\mathbf{z}_1, \mathbf{z}_2\right) = \sum_{i=1}^{N} d_{\mathbb{R}}\left(z_{1i}, z_{2i}\right) \tag{1}$$

where $d_{\mathbb{R}}(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}$ and z_{1i} is the i^{th} component of vector z_1.

Given two sets $A, B \in \mathbb{R}^N$, let $\rho^*_{\mathbb{R}^N}(A, B)$ and $\rho_{\mathbb{R}^N}(A, B)$ be the Hausdorff separation and the Hausdorff metric, respectively.

Let us now describe discrete-time systems and their stability definitions. Consider the following discrete-time system:

$$\mathbf{z}(k+1) = G\big(\mathbf{z}(k), k\big), \quad \mathbf{z}(0) = \mathbf{z}_0 \tag{2}$$

where $k \in \mathbb{N}^+$ represents the discrete time step, $G : \mathbb{R}^N \times \mathbb{N}^+ \rightarrow \mathbb{R}^N$ is continuous and $\mathbf{z}$, $\mathbf{z}_0 \in \mathbb{R}^N$.

Let $\mathbf{0}_N = [0, ..., 0]^T \in \mathbb{R}^N$; we have that $\mathbf{0}_N$ is a stable equilibrium point for system (2) if, for each $\varepsilon > 0$ there exists a positive function $\delta(\varepsilon)$ such that

$$d_{\mathbb{R}^N}\left[\mathbf{z}_0, \mathbf{0}_N\right] < \delta(\varepsilon) \text{ implies } d_{\mathbb{R}^N}\left[\mathbf{z}(k), \mathbf{0}_N\right] < \varepsilon, \forall k \geq 0 \tag{3}$$

Moreover, if further than (3), $\lim_{k\rightarrow+\infty} d_{\mathbb{R}^N}[\mathbf{z}(k), \mathbf{0}_N] \rightarrow 0$, then $\mathbf{0}_N$ is said to be an *asymptotically stable* solution of equation (2).

Let $\mathbf{z}_a$, $\mathbf{z}_b \in \mathbb{R}^N$; define the partial orderings $\geq$ as follows

$$\mathbf{z}_a \geq \mathbf{z}_b \Leftrightarrow z_{ai} \geq z_{bi}, \forall i = 1, ..., N \tag{4}$$

The definition of $\leq$ is analogous.

A continuous function $G(\mathbf{z}, k) : \mathbb{R}^N \times \mathbb{N}^+ \rightarrow \mathbb{R}^N$ is said to be *monotone non-decreasing* in $\mathbf{z}$ if for each $k \in \mathbb{N}^+$ and for each $\mathbf{z}_a$, $\mathbf{z}_b \in \mathbb{R}^N$ it preserves the partial ordering $\leq$ ($\geq$), i.e., $\forall k \in \mathbb{N}^+$ if $\mathbf{z}_a \geq \mathbf{z}_b$ then $G(\mathbf{z}_a, k) \geq G(\mathbf{z}_b, k)$.

In the following, we will refer to non-fuzzy systems/numbers as *crisp* systems/numbers.

2 Consensus problem

Let $\Gamma = \{\mathscr{V}, \mathscr{E}, A\}$ be a graph with p nodes, where set $\mathscr{V}$ denotes the nodes $\{v_i\}$, $\mathscr{E}$ is the set of edges (v_i, v_j).

Matrix $A = \{a_{ij}\}$ is the *adjacency matrix* describing the network topology; elements a_{ij} of A are such that $a_{ij} = 1$ if $(v_i, v_j) \in \mathscr{E}$, otherwise $a_{ij} = 0$. The graph is said to be *undirected* if, for each edge $(v_i, v_j) \in \mathscr{E}$ also $(v_j, v_i) \in \mathscr{E}$. Let $d_i^{out} = \sum_{j=1}^{n} a_{ij}$ be the *out-degree* of node i (i.e., the number of outgoing edges), and let $d_i^{in} = \sum_{j=1}^{n} a_{ji}$ be the *in-degree* of node i (i.e., the number of incoming edges). An undirected graph is said to be *balanced* if for each node $v_i \in \mathscr{V}$, $d_i^{out} = d_i^{in}$. An undirected graph is said to be *connected* if it contains at least a *spanning tree*; a directed graph is said to be *simply connected* if it contains at least a directed spanning tree rooted in a given node v_i, while it is *strongly connected* if for each couple of nodes v_i, v_j there exists a path that connects the nodes respecting the orientation of edges.

The set of *neighbours* of a node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$.

Let us consider the scalar consensus, and let $z_i \in \mathbb{R}$ be the state variable associated to the i^{th} node.

Nodes i and j are said to agree if $z_i = z_j$ and consequently the graph Γ *agrees* if each couple of nodes v_i and v_j agrees, for all $v_i,\ v_j \in \mathcal{V}$. Whenever all the nodes of a network are in agreement, the common value of all nodes is called the *group decision value*.

Let each node in the network be modelled as a discrete-time *dynamic agent*, whose dynamics is in the form:

$$z_i(k+1) = g\left(z_i(k), e_i(k)\right), \quad z_i(0) = z_i^0 \tag{5}$$

where $e_i(k) \in \mathbb{R}$ represents the input for i^{th} agent and z_i^0 is the initial condition vector for i^{th} agent. The stacked dynamics for all the p agents is given by:

$$\mathbf{z}(k+1) = G\left(\mathbf{z}(k), \mathbf{e}(k)\right) \quad \mathbf{z}(0) = \left[z_1^0, \cdots, z_p^0\right]^T \tag{6}$$

where $\mathbf{z},\ \mathbf{e} \in \mathbb{R}^p$ are stack vectors composed of the state variables and inputs of each agent, respectively, i.e., $\mathbf{z} = [z_1 \cdots z_p]^T$ and $\mathbf{e} = [e_1 \cdots e_p]^T$; $G(\mathbf{z}(k),\ \mathbf{e}(k))$ is the column-wise concatenation of the elements $G_i(\mathbf{z}(k), \mathbf{e}(k)) = g(z_i(k), e_i(k))$.

Let a function $\chi : \mathbb{R}^p \to \mathbb{R}$; the *χ-consensus problem* in a dynamic graph can be interpreted as a distributed way to calculate $\chi(\mathbf{z}(0))$ by using as inputs for each node only information depending on the values of its neighbours $\mathcal{N}_i$.

Define a *protocol*, i.e.,

$$e_i(k) = f_i\left(z_{j1}(k), \ldots, z_{jm_i}(k)\right) \tag{7}$$

with $j_1, \ldots, j_{m_i} \in \mathcal{N}_i \cup \{i\}$ and, obviously, $m_i < p$. A protocol asymptotically solves the χ-consensus problem if and only if there exists an asymptotically stable equilibrium z^* of (6) such that for each node $z_i^* = \chi(\mathbf{z}(0))$ for all $i \in [1, p]$.

In the literature different typologies of consensus have been addressed; in the following, we will discuss the average consensus problem for networks of discrete-time single and double integrators.

2.1 Single integrators

Consider a network composed of p dynamic agents, each one described by an integrator (Olfati-Saber and Murray, 2004):

$$\dot{z}_i(t) = e_i(t), \quad z_i(0) = z_{i0} \quad \forall v_i \in \mathcal{V} \tag{8}$$

where $z_i \in \mathbb{R}, e_i \in \mathbb{R}$. In the discrete-time fashion the system becomes:

$$z_i(k+1) = z_i(k) + \tau e_i(k), \quad z_i(0) = z_i^0 \quad \forall v_i \in \mathcal{V} \tag{9}$$

where $\tau > 0$ represents the sampling time. In Olfati-Saber and Murray (2004), the following protocol is used to solve the continuous-time average consensus problem:

$$e_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \left[z_j(t) - z_i(t) \right] \tag{10}$$

where a_{ij} are the coefficients of the adjacency matrix of the considered graph. The resulting stacked dynamic system for the p agents is given by

$$\dot{\mathbf{z}}(t) = -L\mathbf{z}(t), \quad \mathbf{z}(0) = \mathbf{z}^0 \tag{11}$$

where $\mathbf{z} \in \mathbb{R}^p$ and L is the *graph Laplacian* induced by Γ, whose elements $\{l_{ij}\}$ are in the form:

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{p} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \tag{12}$$

In Ren and Beard (2008), it is proved that, if the graph is (simply) connected a consensus equal to a linear combination of the initial conditions of the agents is reached, while the consensus coincides with the actual average of the initial conditions if the graph is balanced and strongly connected.

When the protocol is applied in the discrete-time fashion, the resulting stacked dynamics is in the form:

$$\mathbf{z}(k+1) = P_\tau \mathbf{z}(k) \tag{13}$$

where $P_\tau = I_p - \tau L$ is called the *Perron matrix* (Olfati-Saber and Murray, 2004).

The following lemma (Olfati-Saber and Murray, 2004; Ren and Beard, 2008) provides a stability condition for the discrete time first order average consensus problem.

Lemma 2.1: Let $\tau_1^* = 1/l^*$; then choosing $\tau < \tau_1^*$, protocol (10) solves the consensus problem for a network of discrete-time single integrator agents if the graph Γ contains at least a directed spanning tree. If the graph Γ is also strongly connected and balanced, then the average consensus is achieved and $(I - \tau L)^k$ tends to $\frac{1}{p}\mathbf{1}\mathbf{1}^T$ (i.e., a matrix with all entries equal to $\frac{1}{p}$).

Notice that, as illustrated in Section 5.1, the condition on the sampling rate $\tau < \tau_1^*$ is a sufficient and conservative estimation of the maximum sampling rate that guarantees the convergence of the consensus problem.

2.2 Double integrators

Consider a network composed of p dynamic agents, each one described by a double integrator (Ren, 2007, 2008 Olfati-Saber, 2006; Ren and Beard, 2008):

$$\ddot{z}_i(t) = e_i(t), \quad \dot{z}_i(0) = \dot{z}_i^0, \quad z_i(0) = z_i^0, \quad \forall v_i \in \mathcal{V} \tag{14}$$

where $z_i \in \mathbb{R}$ and $e_i \in \mathbb{R}$.

In Olfati-Saber (2006), the protocol used to solve the continuous-time average consensus problem is:

$$e_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \left[\dot{z}_j(t) - \dot{z}_i(t) \right] + \sum_{j \in \mathcal{N}_i} a_{ij} \left[z_j(t) - z_i(t) \right] \tag{15}$$

Defining $\mathbf{z}_a = \mathbf{z} \in \mathbb{R}^p$ and $\mathbf{z}_a = \dot{\mathbf{z}} \in \mathbb{R}^p$, the resulting dynamics for the p agents can be posed in the form

$$\begin{cases} \mathbf{z}_a(t) = \mathbf{z}_b(t) \\ \dot{\mathbf{z}}_b(t) = -L\mathbf{z}_a(t) - L\mathbf{z}_b(t) + \mathbf{e}(t) \end{cases} \tag{16}$$

where $\mathbf{z}_a(0) = \mathbf{z}_0 = [z_1^0, \cdots, z_p^0]^T$ and $\mathbf{z}_b(0) = \dot{\mathbf{z}}_0 = [\dot{z}_1^0, \cdots, \dot{z}_p^0]^T$.

In order to obtain a discrete-time representation of an agent described by a double integrator, in Ren and Cao (2008), Cao and Ren (2009), and Ren and Beard (2008) the above model is sampled with sample time τ, and the following protocol is adopted

$$e_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij} \left[\left(z_{aj}(k) - z_{ai}(k) \right) \right] + \sum_{j \in \mathcal{N}_i} a_{ij} \left[z_{bj}(k) - z_{bi}(k) \right] \tag{17}$$

providing a condition for the stability of the resulting system.

Utilising a unique approach, this paper will adopt a different formulation, implementing feedback also for $\mathbf{z}_a$, and considering two inputs e_{ai} and e_{bi}. This is done since the resulting model will be simpler to adopt in a fuzzy fashion, as will be explained in the next sections. The dynamics of the i^{th} agent, in the proposed formulation, becomes:

$$\begin{cases} \dot{z}_{ai}(t) = z_{bi}(t) + e_{ai}(t) \\ \dot{z}_{bi}(t) = e_{bi}(t) \end{cases} \tag{18}$$

and consequently its discretisation with uniform sampling time τ is

$$\begin{cases} z_{ai}(k+1) = z_{ai}(k) + \tau z_{bi}(k) + \tau e_{ai}(k) \\ z_{bi}(k+1) = z_{bi}(k) + \tau e_{bi}(k) \end{cases} \tag{19}$$

Theorem 2.2: Let p systems in the form of equation (19), such that their graph Γ is strongly connected and balanced. Then if $\tau < \frac{1}{l^*}$ the agents reach the average consensus using the protocol

$$\begin{cases} e_{ai}(k) = \sum_{j \in \mathcal{N}_i} a_{ij} \left[\left(z_{aj} - z_{ai} \right) \right] \\ e_{bi}(k) = \sum_{j \in \mathcal{N}_i} a_{ij} \left[\left(z_{bj} - z_{bi} \right) \right] \end{cases} \tag{20}$$

Proof: The overall dynamics for the p systems, considering the protocol (20) becomes

$$\begin{bmatrix} \mathbf{z}_a(k+1) \\ \mathbf{z}_b(k+1) \end{bmatrix} = \begin{bmatrix} I - \tau L & \tau I \\ 0 & I - \tau L \end{bmatrix} \begin{bmatrix} \mathbf{z}_a(k) \\ \mathbf{z}_b(k) \end{bmatrix} \tag{21}$$

Note that, in this formulation, the dynamics of $\mathbf{z}_b$ is decoupled from $\mathbf{z}_a$, and has the standard form of a single integrator system. Therefore, an agreement on $\mathbf{z}_b$ is reached if $t < 1/l^*$.

From simple calculations, it is easy to prove the following equality:

$$\begin{bmatrix} I - \tau L & \tau I \\ 0 & I - \tau L \end{bmatrix}^k = \begin{bmatrix} (I - \tau L)^k & (k-1)\tau(I - \tau L)^{k-1} \\ 0 & (I - \tau L)^k \end{bmatrix}$$

therefore the solution of the double integrator systems is given by:

$$\begin{bmatrix} \mathbf{z}_a(k) \\ \mathbf{z}_b(k) \end{bmatrix} = \begin{bmatrix} (I-\tau L)^k & (k-1)\tau(I-\tau L)^{k-1} \\ 0 & (I-\tau L)^k \end{bmatrix} \begin{bmatrix} \mathbf{z}_a(0) \\ \mathbf{z}_b(0) \end{bmatrix}$$

From Lemma 2.1, if Γ is strongly connected and balanced, then $(I - \tau L)^k$ tends to $\frac{1}{n}\mathbf{11}^T$ (i.e., a matrix with all entries equal to $\frac{1}{n}$) as $k \to \infty$. Hence, each component of $\mathbf{z}_b(k)$ tends to the average of the entries of $\mathbf{z}_b(0)$.

Similarly, the components of $\mathbf{z}_a(k)$ asymptotically have the same value

$$z_a^*(k) = \frac{1}{n}\sum_{j=1}^{p} z_{aj}(0) + \tau(k-1)\frac{1}{n}\sum_{j=1}^{p} z_{bj}(0)$$

although the evolution is divergent. □

Note that, in this case, the average consensus is an agreement on the 'velocity' which becomes constant, while the 'position', is asymptotically the same for all the agents, although being diverging.

It is also important to highlight in this case the condition on the maximum value of the sampling rate has to be considered as a conservative estimation.

3 Discrete-time fuzzy systems

Let a fuzzy subset of $\mathbb{R}$ be defined in terms of a membership function $\mu : \mathbb{R} \to (0, 1]$ which assigns to each point $p \in \mathbb{R}$ a grade of membership in the fuzzy set; such function is used to denote the corresponding fuzzy set. Let the p-membership $\mu(p)$ be defined as the grade of membership of $p \in \mathbb{R}$ in the set μ.

For each $\alpha \in [0, 1]$, the α-level set $[\mu]^a$ of a fuzzy set is the subset of points $p \in \mathbb{R}$ with membership grade $\mu(p) \geq \alpha$. The support $[\mu]^0$ of a fuzzy set is defined as the closure of the union of all its α-level sets (see Figure 1).

Let $\mathbb{E}$ be the space of all fuzzy subsets μ of $\mathbb{R}$ such that:

1 μ maps $\mathbb{R}$ onto [0, 1]

2 $[\mu]^0$ is a bounded subset of $\mathbb{R}$

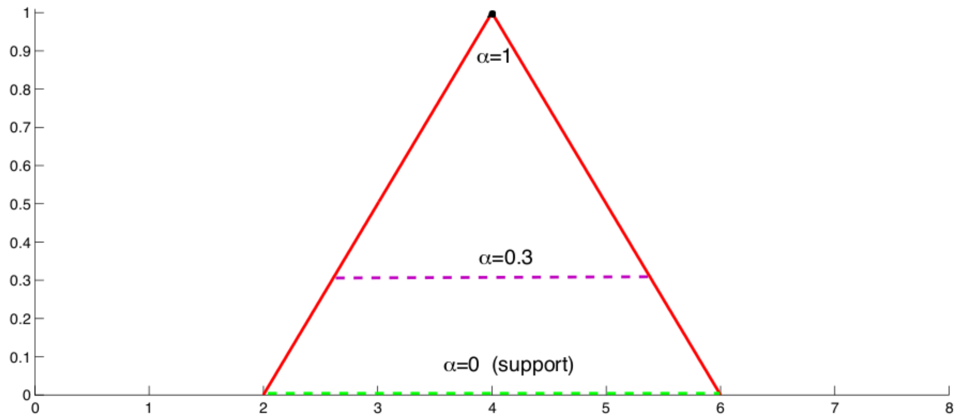
3 $[\mu]^\alpha$ is a compact subset of $\mathbb{R}$ for all $\alpha \in (0, 1]$

4 μ is *fuzzy convex*, that is: $\mu(\phi p + (1 - \phi)q) \geq \min[\mu(p), \mu(q)]$ for all $p, q \in 2\ \mathbb{R}$ the fuzzy sets of $\mathbb{E}$ are often called *fuzzy numbers* (FNs).

Such a space is closed with respect to addition and scalar multiplication (Lakshmikantham and Mohapatra, 2003).

The use of α-levels, as shown in Figure 1, allows to address FNs as a set of real intervals. As will be explained later, the evolution of a fuzzy dynamic system will be evaluated *level-wise* by considering for each α, the evolution of a system whose state is described by an interval (i.e., the corresponding α-level).

Figure 1 Different α-levels of a triangular-shaped fuzzy membership function (see online version for colours)



Notes: In particular the support ($\alpha = 0$) coincides with the base of the triangle (i.e., the interval of real numbers on the abscyssae between the two endpoints), while for $\alpha = 1$, due to the particular shape considered, a single (i.e., crisp) value is obtained.

A *triangular* fuzzy number (TFN) $\mu \in \mathbb{E}$, in particular, is described by an ordered triple $\{\mu_l, \mu_c, \mu_r\} \in \mathbb{R}^3$ with $\mu_l \leq \mu_c \leq \mu_r$ and such that $[\mu]^0 = [\mu_l, \mu_r]$ and $[\mu]^1 = \{\mu_c\}$, while in general the α-level set is given, for any $\alpha \in [0, 1]$ by:

$$[\mu]^\alpha = \left[\mu_c - (1-\alpha)\left(\mu_c - \mu_l\right), \mu_c + (1-\alpha)\left(\mu_r - \mu_c\right)\right] \tag{22}$$

Figure 2 shows two TFNs that represent the fuzzification of number 4 with different ambiguities; notice that the width of the base of the triangle is a measure of the uncertainty associated to the TFN. Triangular representation is not the sole available alternative; as depicted in Figure 3 many other shapes are possible, and the more complex is the shape, the more descriptive is the resulting FN (i.e., the vagueness is better codified). For instance, the existence of a plateau for a given interval represents complete indeterminacy for that interval, or, in the case of risk impact analysis, an asymmetry with respect to the peak may represents different beliefs for the best and worst cases.

Figure 2 The two TFNs depicted represent the same value 'about 4'; however the smaller one (dotted line) is characterised by less uncertainty (see online version for colours)

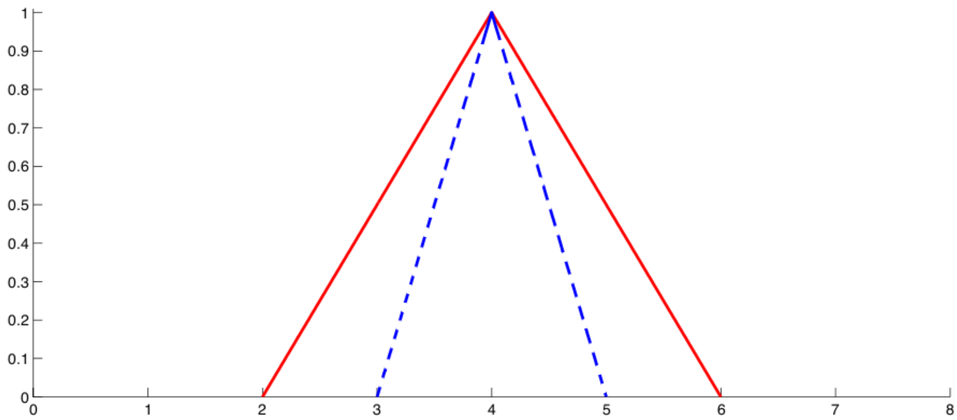
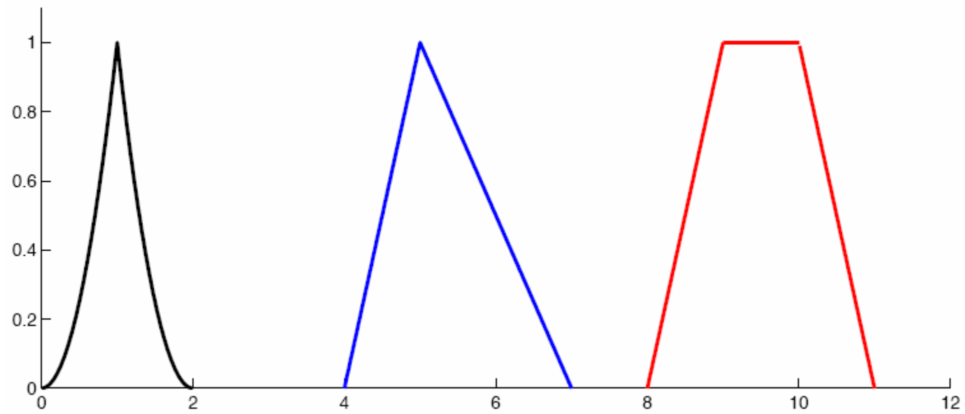


Figure 3 Examples of FNs $\mu \in \mathbb{E}$; many shapes are possible, although the TFNs (in the middle) are the most used, because they can be described by the triple of the abscissae of their vertices ({4, 5, 7} in this case) (see online version for colours)



Notes: More complex shapes, however, allow to characterise better the vagueness of the data; the leftmost FN (in black) represents a case where vagueness rapidly decreases while approaching the central value; the rightmost FN, due to its trapezoidal shape, models the case where a single value with maximum belief can not be found. Notice further that the shape of a FN needs not to be symmetric, thus allowing to represents different beliefs on the left and right spread of uncertainty with respect to the value associated with the maximum belief.

The space $\mathbb{E}$ can be equipped with the following metric, which correlates the distance with the α-sets (Lakshmikantham and Mohapatra, 2003):

$$d_{\mathbb{E}}(\mu, v) = \sup_{\alpha > 0} \left\{ \rho_{\mathbb{R}} \left([\mu]^{\alpha}, [v]^{\alpha} \right) \right\} \; \mu, v \in \mathbb{E} \tag{23}$$

The *level-wise convergence* (i.e., the convergence of the α-levels of a sequence of FNs) is defined as follows. Let $\{\mu_n\}$ be a sequence on $\mathbb{E}$, then $\{\mu_n\}$ converges level-wise to $\mu \in \mathbb{E}$ if, for all $\alpha \in (0, 1]$:

$$\rho_{\mathbb{R}}\left([\mu_n]^\alpha, [\mu]^\alpha\right) \to 0 \text{ as } n \to \infty \tag{24}$$

Define

$$\Psi = \left\{\mu \in \mathbb{E} : \mu\left(\phi p + (1-\phi)q\right) \geq \phi\mu(p) + (1-\phi)\mu(q)\right\} \tag{25}$$

for each $p, q \in [\mu]^0$, $\phi \in [0, 1]$. In Lakshmikantham and Mohapatra (2003), it is proved that convergence in $(\mathbb{E}, d_{\mathbb{E}})$ a implies level-wise convergence; moreover limiting to sequences in Ψ, the implication among convergence in $(\mathbb{E}, d_{\mathbb{E}})$ and level-wise convergence can be revised. The following theorem extends the above result to TFNs.

Theorem 3.1: Limiting to sequences in the set of TFNs, level-wise convergence implies convergence in $(\mathbb{E}, d_{\mathbb{E}})$.

Proof: See Oliva et al. (2012) □

In order to consider vectors of N components, each being a FN, the space $\mathbb{E}^N$ has to be characterised; to this end we choose to equip $\mathbb{E}^N$ with the following metric:

$$d_{\mathbb{E}^N}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{N} d_{\mathbb{E}}\left(x_i, y_i\right) \tag{26}$$

where $\mathbf{x} = [x_1, \ldots, x_N]^T$ and $\mathbf{y} = [y_1, \ldots, y_N]^T$, $\mathbf{x}, \mathbf{y} \in \mathbb{E}^N$.

Define the α-level of a vector of FNs $\mathbf{x} \in \mathbb{E}^N$ as the set of vectors $\mathbf{z} \in \mathbb{R}^N$ such that, $\forall i = 1, \ldots, N$ z_i belongs to the α-level of i^{th} component x_i.

Let $\{x_n\}$ be a sequence on $\mathbb{E}^N$, then $\{x^n\}$ converges level-wise to $\mathbf{x} \in \mathbb{E}^N$ if, for all $\alpha \in (0, 1]$:

$$\rho_{\mathbb{R}^N}\left([\mathbf{x}_n]^\alpha, [\mathbf{x}_n]^\alpha\right) \to 0 \text{ as } i \to \infty \tag{27}$$

Define

$$\Psi^N = \left\{\mathbf{x} \in \mathbb{E}^N : x_i\left(\phi p + (1-\phi)q\right) \geq \phi x_i(p) + (1-\phi)x_i(q);\ p, q \in \left[x_i\right]^0;\ \forall i = 1, \ldots, N\right\} \tag{28}$$

for each $\phi \in [0, 1]$; therefore $\Psi^N \subset \mathbb{E}^N$.

Notice that, with the above definition of α-level of a vector of FNs, we have that

$$\lim_{n\to\infty} \rho_{\mathbb{R}^N}\left([\mathbf{x}_n]^\alpha, [\mathbf{x}]^\alpha\right) = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{\mathbb{R}}\left([x_{ni}]^\alpha, [x_i]^\alpha\right) = 0 \tag{29}$$

$\forall i = 1, \ldots, N$, therefore the scalar results on level-wise convergence are extended to the vectorial case.

Let a *discrete-time fuzzy system* (DFS) be defined as follows:

$$\mathbf{x}(k+1) = F\left(\mathbf{x}(k), k\right); \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{30}$$

where $\mathbf{x},\ \mathbf{x}_0 \in \mathbb{E}^N$ are vectors composed by N FNs and $F : \mathbb{E}^N \times \mathbb{N}^+ \rightarrow \mathbb{E}^N$ a is continuous.

Such a system extends the discrete-time system (2) allowing each state variable to be a fuzzy variable, i.e., a set of values with different levels of belief. In this way, system (30) is able to model the dynamic evolution of data with ambiguity and vagueness.

In the following section, the stability of linear and stationary DFS will be discussed.

3.1 Stability of linear and stationary DFS

Let a linear and stationary DFS system in the form

$$\mathbf{x}(k+1) = F\mathbf{x}(k); \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{31}$$

where $\mathbf{x},\ \mathbf{x}_0 \in \mathbb{E}^N$ and F is a $N \times N$ matrix, and let a linear and stationary crisp system of the same dimension in the form

$$\mathbf{z}(k+1) = G\mathbf{z}(k), \quad \mathbf{z}(0) = \mathbf{z}_0 \tag{32}$$

where $\mathbf{z},\ \mathbf{z}_0 \in \mathbb{R}^N$ and G is a $N \times N$ matrix of real numbers. Note that since $\mathbb{E}^N$ is closed with respect to sum and scalar multiplication, there are no constrains on the structure of F.

Analogously to the crisp case, let $\hat{\mathbf{0}} \in \mathbb{E}^N$ denote the trivial solution of equation (31), which we assume to exist. The trivial solution $\hat{\mathbf{0}}$ of system (31) is stable if, for each $\varepsilon > 0$ there exists a positive function $\delta(\varepsilon)$ such that

$$d_{\mathbb{E}^N}\left[\mathbf{x}_0, \hat{\mathbf{0}}\right] < \delta(\varepsilon) \text{ implies } d_{\mathbb{E}^N}\left[\mathbf{x}(k), \hat{\mathbf{0}}\right] < \varepsilon, \forall k \geq 0 \tag{33}$$

Moreover, if $d_{\mathbb{E}^N}[\mathbf{x}(k), \hat{\mathbf{0}}] \rightarrow 0$, as $k \rightarrow +\infty$, then the trivial solution $\hat{\mathbf{0}}$ of system (31) is said to be asymptotically stable.

Let us now report a theorem that characterises the solution of (31) in terms of solution of the crisp system (32) (Lakshmikantham and Mohapatra, 2003); the proof is given in Oliva et al. (2012) and Oliva (2012).

Theorem 3.2: Let a DFS (31) and let a crisp system (32) where $G\mathbf{z}(k)$ is a continuous linear mapping, monotone-non-decreasing with respect to $\mathbf{z}(k)$. Suppose that exists a continuous and positive valued *defuzzyfication function* $V(\mathbf{x}(k)) : \mathbb{E}^N \rightarrow \mathbb{R}_+^N$ such that, posing $\mathbf{z}(0) = V(\mathbf{x}(0))$, for each $k \geq 0$

$$V\left(\mathbf{x}(k+1)\right) \leq GV\left(\mathbf{x}(k)\right) \tag{34}$$

Suppose further that there exists a continuous, monotone non-decreasing function $a(\cdot)$ defined in $\mathbb{R} \rightarrow \mathbb{R}_+$ such that

$$a\left(d_{\mathbb{E}^N}\left[\mathbf{x}(k), \hat{0}\right]\right) \leq V_0\left(\mathbf{x}(k)\right) \tag{35}$$

where $d_{\mathbb{E}^N}$ is the distance in $\mathbb{E}^N$ defined in (26) and $V_0(\mathbf{x}(k))$ is defined as

$$V_0\left(\mathbf{x}(k)\right)=\sum_{j=1}^{N}V_i\left(\mathbf{x}(k)\right),\quad \forall k\geq 0 \tag{36}$$

Then the stability properties of the trivial solution of equation (32) imply the corresponding stability properties of the trivial solution of equation (31).

Proof: See Oliva et al. (2012) □

The above comparison theorem is very useful, since the stability of fuzzy systems can be derived from the stability of a non-fuzzy system. The following corollary (Oliva et al., 2012) provides a useful parallelism between the stability of a fuzzy system and the stability of a crisp system obtained by defuzzyfying the system itself (i.e., the case in which $G = F$).

Corollary 3.3: Let a DFS in the form of equation (31) and let the following crisp systems

$$\mathbf{z}(k+1)=F\mathbf{z}(k)\quad \mathbf{z}(0)=V\left(\mathbf{x}(0)\right) \tag{37}$$

such that $F\mathbf{z}(k)$ is monotone non-decreasing in $\mathbf{z}(k)$; then the stability properties of crisp system (37) imply the corresponding stability properties of the DFS (31).

Proof: See Oliva et al. (2012). □

Each state variable x_i of a DFS (30) is such that, at any time k its α-level is given by

$$\left[x_i(k)\right]^{\alpha}=\left[\underline{x}_i^{\alpha}(k),\overline{x}_i^{\alpha}(k)\right],\quad \forall i=1,\dots,N \tag{38}$$

The set (38) is generally referred to as the *level-wise* representation of the fuzzy variable $x_i(k)$, because it represents the evolution of the fuzzy variable for the different membership grades.

In Pearson (1997), Seikkala (1987), and Kay and Kuipers (1993), it is shown that, for each time k and for each α-level, the evolution of a DFS can be described by $2N$ crisp difference equations for the endpoints of the intervals of equation (38). Such equations are in general complicated; however in the linear case these equations become

$$\begin{bmatrix}\underline{x}^{\alpha}(k+1)\\ \overline{x}^{\alpha}(k+1)\end{bmatrix}=\begin{bmatrix}F^{+} & F^{-}\\ F^{-} & F^{+}\end{bmatrix}\begin{bmatrix}\underline{x}^{\alpha}(k)\\ \overline{x}^{\alpha}(k)\end{bmatrix} \tag{39}$$

where F^+ and F^- are $N \times N$ matrices such that F^+ contains only the positive coefficients of F (and is zero otherwise) and F^- contains only the negative coefficients (and is zero otherwise).

4 Consensus of fuzzy agents

In this section, a framework for the discrete-time first order and second order average consensus problem with fuzzy initial condition is provided extending the results for crisp systems (Olfati-Saber and Murray, 2004). Let a *dynamic graph* Γ for p agents, where the dynamics of each agent is described by the DFS

$$\mathbf{x}_i(k+1)=F\left(\mathbf{x}_i(k),\mathbf{e}_i(k)\right),\quad \mathbf{x}_i(0)=\mathbf{x}_{i0} \tag{40}$$

where $\mathbf{x}_i$, $\mathbf{x}_{i0} \in \mathbb{E}^N$ and $\mathbf{e}_i \in \mathbb{E}^q$ depends only on the state of agent i and his neighbours N_i, according to the topology described by the matrix A. The array of fuzzy agents reaches consensus if

$$\lim_{k\to\infty} d_{\mathbb{E}^N}\left[\mathbf{x}_i(k), \mathbf{x}_j(k)\right] = 0, \quad \forall i, j = 1, ..., p \tag{41}$$

where $d_{\mathbb{E}^N}$ is the metric defined in equation (26).

Conversely, the array of fuzzy agents reaches consensus *level-wise* if, for all $\alpha \in [0, 1]$ and $\forall i = 1, \ldots, p$:

$$\lim_{k\to\infty} \rho_{\mathbb{R}^N}\left[\mathbf{x}_i^\alpha(k), \mathbf{x}_j^\alpha(k)\right] = \mathbf{0} \tag{42}$$

The following theorem correlates consensus and level-wise consensus of fuzzy systems:

Theorem 4.1: The consensus of fuzzy agents in the sense of (41) implies level-wise consensus (42).

Proof: Substituting equation (23) for each component of the summation that define $d_{\mathbb{E}^N}$ inside equation (41), the statement is verified. □

Note that, limiting each state variable to the set Ψ, defined in equation (28), the implication can be reversed; hence the theorem is true for systems with initial conditions described by TFNs.

Theorem 4.2: Consider a fuzzy consensus problem with p interconnected discrete time agents with fuzzy dynamics described by equation (40), where the graph $\mathcal{G}$ is strongly connected and balanced. Then both the following statements hold true:

1 if the dynamics of each agent is a discrete-time single integrator in the form of equation (9) then protocol (10) solves the problem for $\tau < \frac{1}{l^*}$ and the agents reach an agreement

2 if the dynamics of each agent is a discrete-time double integrator in the form of equation (19) then protocol (20) solves the problem for $\tau < \frac{1}{l^*}$ and the agents reach an agreement.

Proof: From Lemma 2.1 system (9) reaches consensus for $\tau < 1/l^*$; moreover the overall dynamic matrix has only-negative entries and is monotone non-decreasing. Hence, by Theorem 3.3, it follows that system (9) reaches consensus also in the fuzzy fashion. Analogously, by Theorem 2.2, system (21) reaches consensus in the fuzzy fashion for $\tau < \frac{1}{l^*}$. □

Corollary 4.3: Under the hypotheses of Theorem 4.2 and assuming that Γ is strongly connected and balanced, then the following statements holds true for each α-level:

1 in the case of discrete-time single integrators

$$\begin{cases} \underline{\mathbf{z}}(k) - (I - \tau L)^k \, \underline{\mathbf{z}}(0) \\ \overline{\mathbf{z}}(k) = (I - \tau L)^k \, \overline{\mathbf{z}}(0) \end{cases} \tag{43}$$

2 in the case of discrete-time double integrators

$$\begin{cases} \underline{\mathbf{z}}_b(k) = (I - \tau L)^k \underline{\mathbf{z}}_b(0) \\ \overline{\mathbf{z}}_b(k) = (I - \tau L)^k \overline{\mathbf{z}}_b(0) \\ \underline{\mathbf{z}}_a(k) = (I - \tau L)^k \underline{\mathbf{z}}_a(0) + \tau(k-1)(I - \tau L)^k \underline{\mathbf{z}}_b(0) \\ \overline{\mathbf{z}}_a(k) = (I - \tau L)^k \overline{\mathbf{z}}_a(0) + \tau(k-1)(I - \tau L)^k \overline{\mathbf{z}}_b(0) \end{cases} \tag{44}$$

Proof: From Theorem 4.2 it follows that, assuming a connected and balanced topology, the array of single or double integrators reach consensus. Since, in both cases, the dynamical matrix is composed only of non-negative elements, for each α-level the consensus reached assumes the structure of equation (39), where $F^+ = (I - \tau L)$ and $F^- = 0$, in the case of single integrators, while

$$F^+ = \begin{bmatrix} (I - \tau L) & \tau I \\ 0 & (I - \tau L) \end{bmatrix}; \quad F^- = 0$$

in the case of double integrators, proving the statement. □

Corollary 4.4: For $\alpha = 1$ fuzzy consensus coincides with crisp consensus for both single and double integrators.

Proof: For $\alpha = 1$ it follows that $\underline{\mathbf{z}}^1(0) = \overline{\mathbf{z}}^1(0)$ (i.e., the interval collapses into a single point). Substituting inside system (43), it follows that $\underline{\mathbf{z}}^1(k) = \overline{\mathbf{z}}^1(k)$, for all $k \geq 0$. The proof is analogous in the case of double integrators. □

5 Case study

In this section, a case study is outlined, in order to illustrate a concrete application of the proposed approach.

Consider the following scenario: composed of p highly interconnected infrastructures or lands with a respective operator or a team responsible for determining the local effects of an adverse event (i.e., terrorist attack, natural disaster or a distributed technological failure) in the absence of a central coordination authority.

Within the scenario, the following two cases will be analysed. In the first case, one the operators has to reach a consensus on the actual severity of the failure affecting the whole scenario based solely on their partial observations. In the second case, they must determine the expected evolution of the phenomena.

Each expert expresses a linguistic measurement of the perceived severity of the failure (and of its expected growth ratio) affecting his/her infrastructure or land using the expressions reported in Table 1 (and Table 2) providing, also, an estimate about his/her confidence on the provided data, in accordance with the confidence scale of Table 3. The values are then encoded into TFNs according to the last column of Tables 1, 2 and 3. Specifically, Table 1 encodes the actual level of failure perceived by the operator, and Table 2 the expected growth/reduction rate (which can as well be negative); these numbers can be regarded as the central values of the TFNs, while the left and right endpoints are obtained by applying the confidence scale reported in Table 3.

Hence, each operator being aware only of their own domain, they need an instrument akin to the one outlined in this paper in order to reach a distributed consensus.

Table 1 Perceived severity estimation table

Perceived severity	*Description*	*Value*
Nothing	The event does not induce any effect on the infrastructure/land	0
Negligible	The event induces some very limited and geo-graphically bounded consequences that have no direct impact on the infrastructure's or land's operativeness	0.025
Very limited	The event induces some geographically bounded consequences that have no direct impact on the infrastructure's or land's operativeness	0.05
Limited	The event induces consequences only on subsystems/zones that have no direct impact on the infrastructure's or land's operativeness	0.1
Circumscribed degradation	The event induces geographically bounded consequences	0.2
Significant degradation	The event significantly degrades the operativeness of the infrastructure/land	0.30
Severe degradation	The impact on the infrastructure/land is severe	0.500
Quite complete stop	The impact is quite catastrophic	0.700
Stop	Total disruption	1

Table 2 Expected growth estimation table

Expected growth/reduction	*Description*	*Value*
Steady	The severity of the event is expected to remain constant.	0
Negligible	The severity of the event is expected to have a very limited growth/reduction.	± 0.0001
Very slow	The severity of the event is expected to grow/reduce only in the long-term.	± 0.001
Slow	The severity of the event is expected to grow/reduce in the long-term and eventually in the mid-term.	± 0.03
Quite slow	The severity of the event is expected to grow/reduce in the mid-term	± 0.005
Not so slow	The severity of the event is expected to grow/reduce in the mid-term and eventually in the short-term	± 0.010
Quite fast	The severity of the event is expected to grow/reduce in the short-term	± 0.05
Fast	The severity of the event is expected to grow/reduce significantly in the short-term	± 0.07
Very Fast	The severity of the event is expected to grow/reduce dramatically in the short-term	± 0.1

Table 3 Confidence estimation scale

Confidence	*Description*	*Value (severity)*	*Value (growth)*
*	Perfect knowledge (no uncertainty)	0	0
**	Excellent confidence	± 0.005	± 0.0005
***	Good confidence	± 0.050	± 0.0050
****	Relative confidence	± 0.100	± 0.0100
*****	Uncertain	± 0.200	± 0.0200

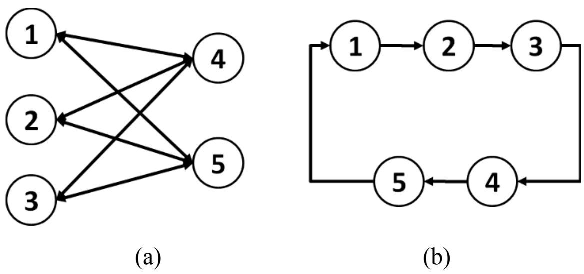
5.1 Simulation results

Let us consider a scenario composed of five infrastructures and assume that their topology is a bipartite graph [see Figure 4(a)]. Such a topology may represent a scenario where some infrastructures are not able to communicate directly (e.g., due to physical or commercial constrains) Assuming unitary weight the corresponding laplacian L_a is

$$L_a = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 \\ -1 & -1 & -1 & 0 & 3 \end{bmatrix} \tag{45}$$

Since $l_a^* = 3$, in order to respect the condition required by Theorem 4.2, we have that for bipartite topology $\tau_1^* = \frac{1}{3}[s]$ and $\tau_2^* = \frac{1}{4}[s]$; however for the sake of uniformity, we chose $\tau = \frac{1}{4}[s]$, for both single and double integrator cases.

Figure 4 Topologies chosen for simulations, (a) bipartite graph and (b) chain



Note: All the edges have unitary weight.

It is immediate to recognise that, due to the choice of τ, the dynamic matrices of both single and double integrator case are composed by non-negative entries.

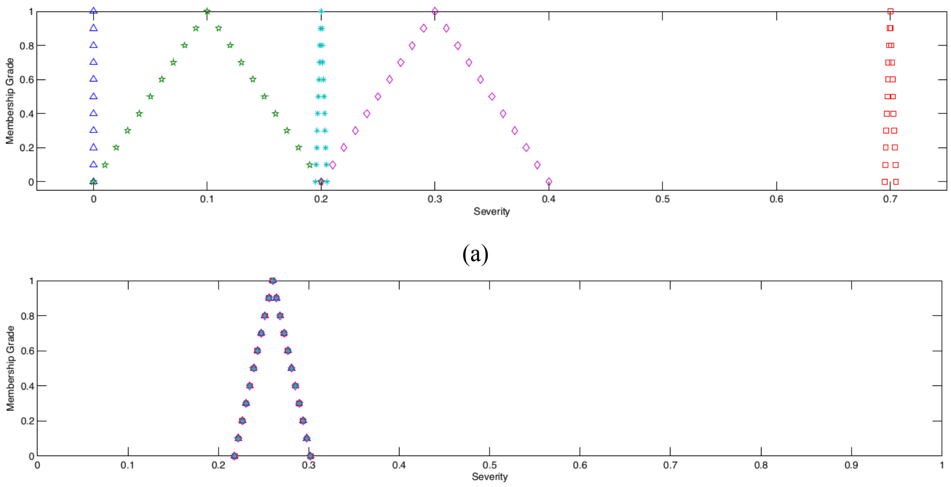
Table 4 shows the initial conditions for both perceived severity and expected growth, each with the associated confidence, as well as the corresponding TFN.

Table 4 Initial conditions for the case study

n.	*Severity*		*Confidence*	*TFN*
1	Nothing		*	[0, 0, 0]
2	Limited		****	[0, 0.1, 0.2]
3	Quite Complete stop		**	[0.695, 0.7, 0.705]
4	Circumscribed degradation		**	[0.195, 0.2, 0.205]
5	Significant degradations		****	[0.2, 0.3, 0.4]
n.	*Expected growth*	*Growth/reduction*	*Confidence*	*TFN*
1	Steady	Growth	*****	[–0.2, 0, 0.2]
2	Quite fast	Growth	**	[0.0495, 0.05, 0.0505]
3	Slow	Reduction	***	[–0.035, –0.03, –0.025]
4	Very fast	Reduction	***	[–0.105, –0.1, –0.095]
5	Fast	Growth	*	[0.07, 0.07, 0.07]

Figure 5 shows the initial conditions and final synchronised state in the case of single integrators. More specifically in a situation where only one operator observes a very bad situation (i.e., operator n. 3 sees a 'quite complete stop') while all the others have no direct perception of the crisis (i.e., they estimate the event ranging from 'nothing' to 'circumscribed'), they distributedly agree on a circumscribed degradation crisis condition with a good confidence. Hence, the agents are able to share vague and ambiguous information in a distributed way and they reach a consensus obtaining a consistent qualification of the actual crisis. Note that the consensus is obtained both in terms of expected severity (e.g., the central endpoint of the triangle) and in terms of confidence on the estimation (i.e., the width of the base of the triangle).

Figure 5 (a) Initial conditions and (b) synchronised state for five discrete time single integrators (see online version for colours)



(a)

(b)

Note: The result is the same for both topologies; however, the consensus is reached after 10 steps for bipartite topology, while for chain topology 32 steps are required.

In the case of double integrator models, Figure 8 shows the initial conditions for the expected growth and the consensus reached. In this case, it is more evident that even in the presence of very different local perception of which should be the evolution of the phenomena (most of the operators express no overlapping estimations, both in terms of magnitude and sign, and two operators even have strong credibility) they reach a distributed consensus or a common understanding of the effective growth of the evolution of the phenomenon, again, both in terms of magnitude and confidence. Note that the criticality in this latter framework is assumed to be growing with a constant rate, and the agents reach an agreement also on this varying quantity, both in terms of magnitude and confidence.

Notice that the reached consensus does not depend on the peculiar topology adopted. Any strongly connected and balanced topology with the proposed protocol allows to reach the same consensus. For example, let us consider the ring topology of Figure 4(b), where each agent is able to communicate only with its nearest neighbours. In this case, the Laplacian L_b is

$$L_b = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{46}$$

and we have that $\tau_1^* = 1[s]$ and $\tau_2^* = \frac{1}{2}[s]$. Hence, also in this case $\tau = \frac{1}{4}$ satisfies the conditions of Theorem (4.2).

Figure 6 Synchronisation of (a) left and (b) right extrema of an α-level of for five discrete time single integrators connected by the bipartite topology, for $\alpha = 0.3$ (see online version for colours)

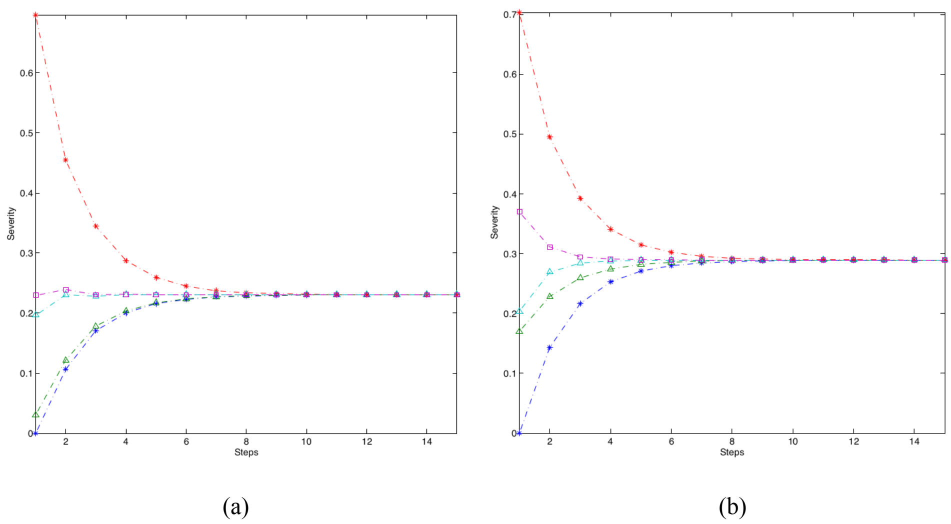
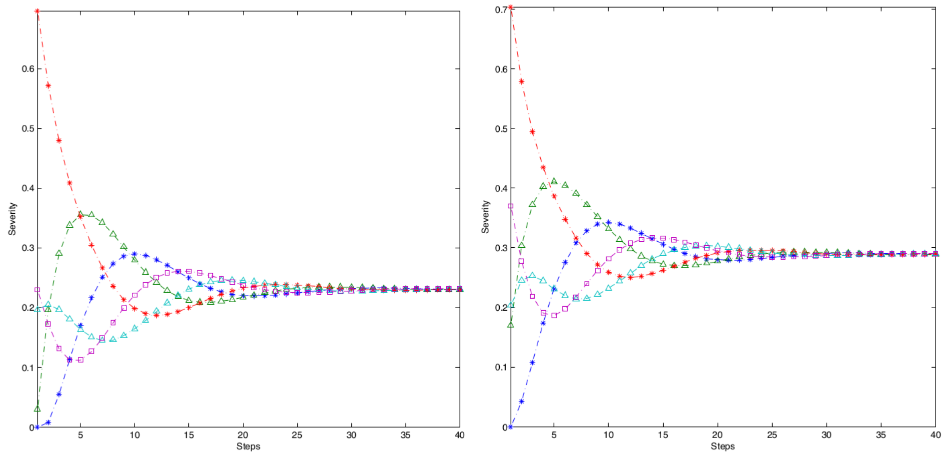
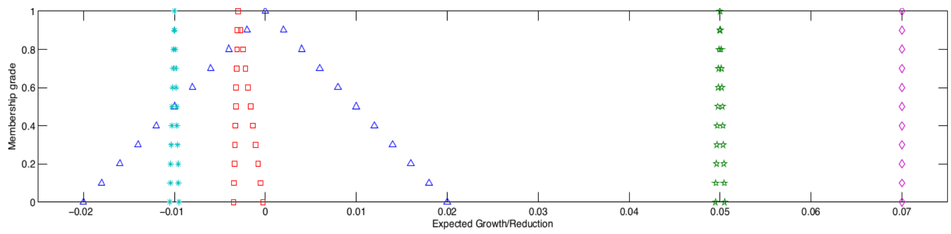


Figure 7 Synchronisation of (a) left and (b) right extrema of an α-level of for five discrete time single integrators connected by the chain topology, for $\alpha = 0.3$ (see online version for colours)

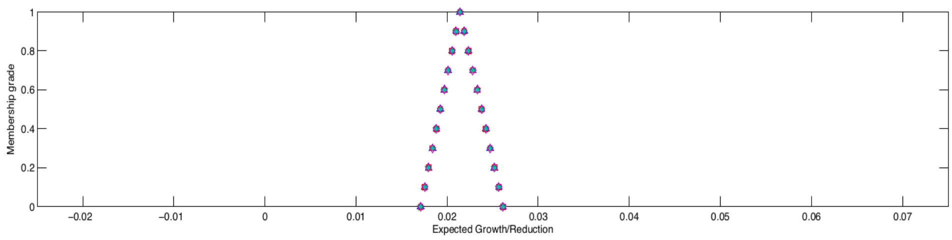


(a) (b)

Figure 8 Initial conditions for (a) expected growth and (b) consensus reached for five discrete time double-integrators; the consensus is reached after 8 steps for bipartite topology, and after 29 steps for chain topology (see online version for colours)

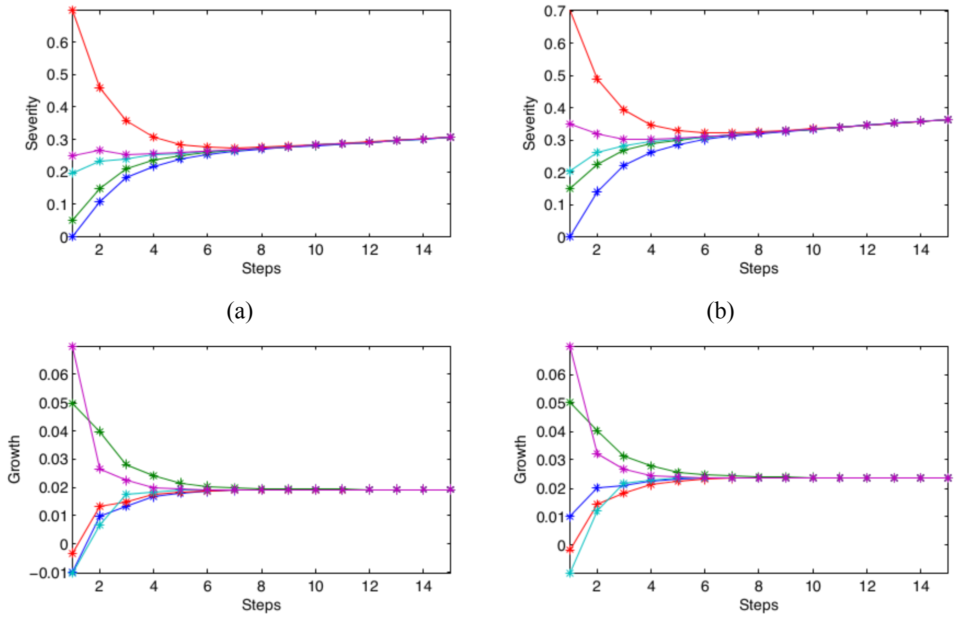


(a)



(b)

Figure 9 Synchronisation of an α-level of five discrete time double integrators connected by the topology of Figure 4(a) for $\alpha = 0.5$, (a) left extrema of severity; (b) right extrema of severity; (c) left extrema of growth; (d) right extrema of growth (see online version for colours)



(a) (b) (c) (d)

Let us initialise both the single and double integrator models with the same initial conditions used for the bipartite topology (Table 4).

Obviously the two topologies are not completely equivalent, because greater is the communication capability of the agents, faster is the consensus is reached.

This can be immediately recognised looking at the time evolution of the state variables of the different agents, as reported in Figures 6 and 7 for the single integrator model with reference to the level-wise representation for $\alpha = 0.3$ (i.e., the left and right extrema).

While in the bipartite graph topology the consensus is achieved after nine iterations, with the ring topology we need more than 30 iterations.

Analogously, in the case of double integrator model, Figures 9 and 10 report the state variables of all the agents for the crisis severity estimation and expected growth with reference to the level-wise representation for $\alpha = 0.5$.

In order to better understand the influence of the sampling rate τ, Figure 11 shows the time required for the consensus for a given topology (i.e., tk^*, where k^* is the number of steps required for consensus) for $\tau \in (0, 1]$. Note that such a time is almost constant for small values of τ, while it diverges for τ that reaches the stability boundaries. Finally, note that since Theorem 4.2 only provides a sufficient condition, the system may reaches consensus even for values of τ greater than τ_1^*.

Figure 10 Synchronisation of an α-level of 5 discrete time double integrators connected by the topology of Figure 4(b) for $\alpha = 0.5$, (a) left extrema of severity; (b) right extrema of severity; (c) left extrema of growth; (d) right extrema of growth (see online version for colours)

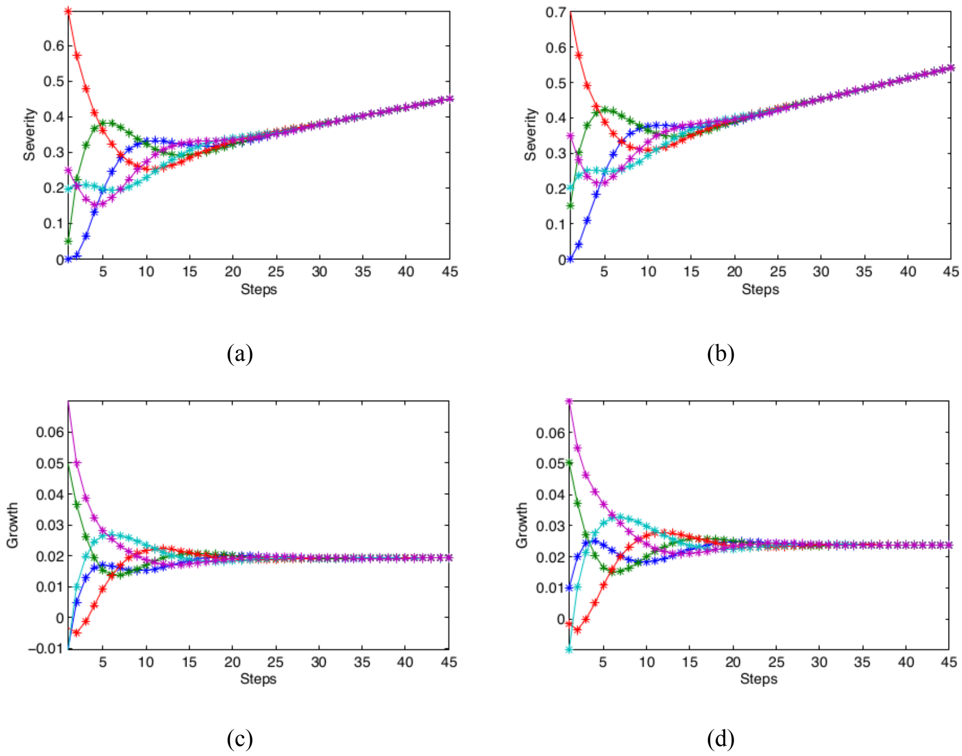
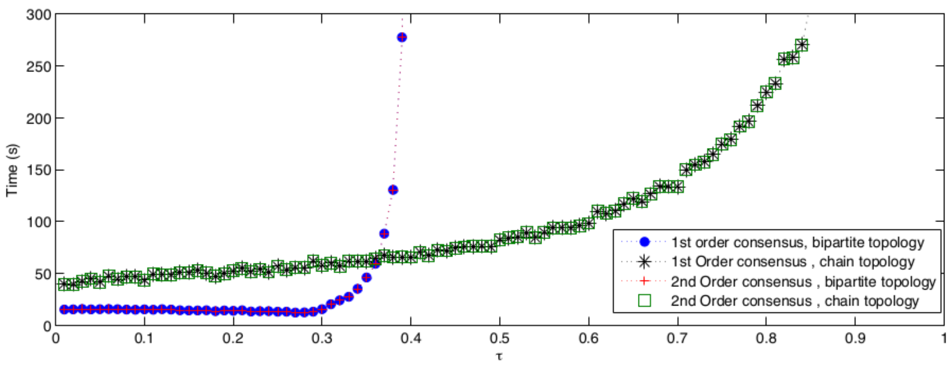


Figure 11 Time required for consensus depending on the choice of τ in the interval (0, 1] (see online version for colours)



6 Conclusions

In this paper, a methodology for the composition of different ambiguous or linguistically expressed opinions has been provided, resorting to the theory of fuzzy systems. The framework has then been applied to the distributed estimation of the severity of failure in scenarios composed of interdependent critical infrastructures, where the distributed agreement on both the actual severity (single order consensus) and the expected growth of the severity (second order consensus) have been performed in the absence of a central authority considering ambiguous linguistic values.

The proposed methodology represents an effective way to cope with ambiguity while merging subjective opinions in a distributed manner. In fact the different agents, each with its independent point of view, are able to reach an agreement both in terms of final agreement value and degree of ambiguity.

Further works will be devoted to find less conservative conditions for the convergence and to consider time-varying topologies.

References

Cao, Y. and Ren, W. (2009) 'Sampled-data discrete-time coordination algorithms for double-integrator dynamics under dynamic directed interaction', *Int. Journal of Control*, Vol. 83, No. 3, pp.506–515.

Haimes, Y. and Jiang, P. (2001) 'Leontief-based model of risk in complex interconnected infrastructures', *Journal of Infrastructure Systems*, Vol. 7, No. 1, pp.1–12.

Haimes, Y., Horowitz, B., Lambert, J., Santos, J., Lian, C. and Crowther, K. (2005) 'Inoperability input-output model for interdependent infrastructure sectors. I: theory and methodology', *Journal of Infrastructure Systems*, June, Vol. 11, No. 2, pp.67–79.

Kay, H. and Kuipers, B. (1993) 'Numerical behavior envelopes for qualitative models', *Proc. of 11th National Conf. on Artificial Intelligence*, MIT Press, pp.606–613.

Lakshmikantham, V. and Mohapatra, R.N. (2003) *Theory of Fuzzy Differential Equations and Inclusions*, CRC Press, New York.

Olfati-Saber, R. (2006) 'Flocking for multi-agent dynamic systems: algorithms and theory', *IEEE Trans. Autom. Control*, Vol. 51, No. 3, pp.401–420.

Olfati-Saber, R. and Murray, R.M. (2004) 'Consensus problems in networks of agents with switching topology and time-delays', *IEEE Trans. on Autom. Control*, Vol. 49, No. 9, pp.1520–1533.

Olfati-Saber, R., Fax, J.A. and Murray, R.M. (2007) 'Consensus and cooperation in networked multi-agent systems', *Proceedings of the IEEE*, Vol. 95, No. 1, pp.215–233.

Oliva, G. (2012) 'Stability and level-wise representation of discrete-time fuzzy systems', *International Journal of Fuzzy Systems*, Vol. 14, No. 2, pp.185–192.

Oliva, G., Panzieri, S. and Setola, R. (2011) 'Fuzzy dynamic input-output inoperability model', *International Journal on Critical Infrastructure Protection*, December, Vol. 4, Nos. 3–4, pp.165–175, doi:10.1016/j.ijcip.2011.09.003.

Oliva, G., Panzieri, S. and Setola, R. (2012) 'Distributed synchronization under uncertainty: a fuzzy approach', *Fuzzy Sets and Systems*, Elsevier, DOI: 10.1016/j.fss.2012.02.003.

Pearson, D. (1997) 'A property of linear fuzzy differential equations', *Appl. Math. Lett.*, Vol. 10, No. 3, pp.99–103.

Ren, W. (2007) 'Consensus strategies for cooperative control of vehicle formations', *IET Control Theory & Applications*, March, Vol. 1, No. 2, pp.505–512.

Ren, W. (2008) 'On consensus algorithms for double-integrator dynamics', *IEEE Trans. Autom. Control*, Vol. 53, No. 6, pp.15603–15609.

Ren, W. and Beard, R.W. (2008) *Distributed Consensus in Multi-vehicle Cooperative Control*, Springer, London.

Ren, W. and Cao, Y. (2008) 'Convergence of sampled-data consensus algorithms for double-integrator dynamics', *Proc. of 47th IEEE Conf. on Decision and Control*, Cancun, Mexico, 9–11 December.

Seikkala, S. (1987) 'On the fuzzy initial value problem', *Fuzzy Sets and Systems*, Vol. 24, No. 3, pp.319–330.