SOME REMARKS ON A NEW DQ–BASED
METHOD FOR SOLVING A CLASS OF VOLterra
INTEGRO–DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper a numerical Picard–like method, which combines successive approximations
with integral and differential quadrature, is considered to solve some Volterra integro–
differential equations where rational functions are involved. Under certain conditions, the
method provides solutions in explicit form, without any recursive computation, and for
some cases it is possible to give a formula to compute the related error. Several numerical
examples are proposed to illustrate the behaviour of the method, by comparing it, where
possible, with some existing methods.

KEYWORDS: integral and differential quadrature; integro–differential equations; grid
points; Lagrange polynomials
1. Introduction

In this paper the following class of linear integro–differential equations is considered

\[ \sum_{i=0}^{r} p_i(z) y^{(i)}(z) + \lambda \int_{0}^{z} k(z) y^{(s)}(z) dz = q(z), \quad z \in [0, 1] \]  

with \(0 \leq s \leq r\) and under the initial conditions

\[ y^{(i)}(0) = a_i, \quad i = 0, \ldots, r - 1\]

where \(y^{(i)}\) stands for the \(i\)th–order derivative of \(y(z)\), \(\lambda\) and \(a_i\) are real constants, \(p_i(z)\), \(k(z)\) and \(q(z)\) are real continuous functions in \([0, 1]\); in particular, \(p_i(z)\) and \(k(z)\) may be rational functions. The choice of the interval \([0, 1]\) can be considered of general interest because of dimensionless equations or through a reference change.

Many problems in science and engineering involve integro–differential equations, e.g. viscoelasticity [1–3], heat transfer [4], economics [5], chemostat [6], HIV models [7], biotissues [8], static analysis of wind towers or chimneys [9].

Several numerical methods have been proposed to solve Volterra integro–differential equations: spline collocation methods [10–11], finite element method [12], sinc collocation method [13], Taylor expansion [14–15], Bessel polynomials [16], Legendre polynomials [17], Bernstein polynomials [18], Lagrange polynomials [19].

The method herein discussed is a numerical adaptation of the Picard method, by combining successive approximations with integral and differential quadrature. Compared to methods where Lagrange polynomials are used to reduce the problem to the solution of a discretized equations system ([19], [9]), which often fails to give the desired accuracy when there are rational functions [9], this approach has the main advantage, under certain conditions, to lead to an explicit solution without solving any equations system, in addition to exactly reproduce the initial conditions.

Although the method of successive approximations can be used as a semi–analytical approach by computing the integrals exactly, some complications may arise when there are rational functions, even for the linear case, since computations become very time–expensive. To overcome this drawback the \(r\)-fold inverse operator applied to the \(r\)th–order differential eq. (1) is replaced by a weighted sum of the functional values at \(N\) grid points, where the weights are obtained by integrating \(r\) times the Lagrange polynomials. Similarly, the derivatives are replaced by a (different) weighted sum of the functional values at \(N\) grid
points, according to the so-called differential quadrature (DQ) rules. DQ rules were introduced by Bellman and Casti [20] and attracted a lot of interest in the last decades (a recent review of DQ–based methods can be retrieved in [21]).

DQ rules are based on Lagrange polynomials. The great attractiveness of Lagrange polynomials is in their simplicity and the aim of this paper is to discuss another possible application, since the single DQ method often fails for the problem here considered [9], whereas it works for different problems (e.g. [22–25]).

The paper is structured in two theoretical sections in order to give an overview of the method and a final section devoted to numerical results and discussion.

2. INTEGRAL AND DIFFERENTIAL QUADRATURE

Integral and differential quadrature are very similar in their idea. Integral quadrature consists in replacing the integral of a function $u(z)$ over an interval, e.g. $[0, 1]$, by a weighted sum of the functional values $u_1, \ldots, u_N$ at the discrete points $0 = z_1 < z_2 < \ldots < z_N = 1$:

$$\int_0^1 u(z)dz = \sum_{i=1}^N \overline{C}_i u_i$$  \hspace{1cm} (2)

The differential quadrature (DQ) approximates the $r$th–order derivative of the function $u(z)$ at a point $z = z_j$ by the weighted sum:

$$\left[ \frac{d^r u}{dz^r} \right]_{z=z_j} = \sum_{l=1}^N A_{jl}^{(r)} u_l$$  \hspace{1cm} (3)

where $A_{jl}^{(r)}$ are the weighting coefficients computed in $N$ grid (or sampling) points.

By using the Lagrange interpolation polynomial, i.e. by using the approximation

$$u(z) = \sum_{l=1}^N l_l(z)u_l$$  \hspace{1cm} (4)

where $l_l(z)$ are the Lagrange polynomials at abscissae $z_l$, one has for the integral quadrature:

$$\overline{C}_i(z) = \int_0^z l_i(z)dz$$  \hspace{1cm} (5)

and for the differential quadrature:
\[ A_{ij}^{(r)} = \sum_{l=1}^{N} \frac{d^r l_l(z)}{dz^r} u_l \]  

(6)

For a symmetric distribution of sampling points, the matrix of the weighting coefficients of the \(r\)th derivative \(A^{(r)}\) turns out to be skew centrosymmetric [26], i.e.:

\[ A_{ij}^{(r)} = -A_{(N+1-i)(N+1-j)}^{(r)} \]  

(7)

Besides, the matrix \(A^{(r)}\) can be obtained by multiplying \(r\) times the matrix \(A^{(1)}\).

The DQ method (DQM) is substantially a polynomial approximation. After the application of DQ rules to a differential equation, the approximate solution can be written as [21]

\[ u(z) = \sum_{j=1}^{N} \tilde{V}_j(z) u_j \]  

(8)

where

\[ \tilde{V}_j(z) = \delta_{1j} + \sum_{r=1}^{N-1} \frac{A_{1j}^{(r)} z^r}{\Delta z^r r!} \]  

(9)

being \(\delta_{1j}\) the Kronecker delta, \(\Delta z\) the length of the interval where the solution is sought and \(A_{1j}^{(r)}\) computed in an interval with unit length.

Eq. (8) is equivalent to a Taylor expansion around \(z = 0\), truncated at the \((N-1)\)th power. In fact, eq. (8) can be written as follows (being \(\Delta z = 1\))

\[ u(z) = u_1 + \sum_{k=1}^{N-1} \left( \sum_{j=1}^{N} A_{1j}^{(k)} u_j \right) \frac{z^k}{k!} = u_1 + \sum_{k=1}^{N-1} u^{(k)}(0) \frac{z^k}{k!} \]  

(10)

where \(u_1 = u(0)\).

Assuming the Lagrange polynomials as test functions, there is no restriction in the choice of the grid coordinates. Grid points can be equally spaced or not. In this second case, a general way to generate the grid points can be computing the zeros of the first order derivative of Gegenbauer polynomials [21]. An usual choice is the Gauss–Chebyshev–Lobatto (GCL) distribution.
3. Fundamental relations and theorems

Let us introduce the operator

\[ L = \frac{d^r}{dz^r} \]  

(11)

whose inverse is the \( r \)-fold operator

\[ L^{-1}(\cdot) = \int_0^z \ldots r \text{ fold } \int_0^z (\cdot) dz \ldots dz \]  

(12)

and applying \( L^{-1} \) on both sides of eq. (1) one has

\[ y(z) = \sum_{i=0}^{r-1} a_i \frac{z^i}{i!} + L^{-1} \left( \frac{p_i(z)}{p_r(z)} \right) - L^{-1} \left( \sum_{i=0}^{r-1} \frac{p_i(z)}{p_r(z)} y^{(i)}(z) \right) - \lambda L^{-1} \left( \frac{1}{p_r(z)} \int_0^z k(z)y^{(s)}(z)dz \right) \]  

(13)

In accordance with the idea of the successive approximations, the solution \( \overline{y}(z) \) can be written by a series of unknown functions \( y_i(z) \) which can be determined recursively

\[ \overline{y}(z) = \sum_{k=0}^{\infty} y_k(z) \]  

(14)

\[ y_0(z) = \sum_{i=0}^{r-1} a_i \frac{z^i}{i!} + L^{-1} \left( \frac{q(z)}{p_r(z)} \right) \]  

(15)

\[ y_{k+1}(z) = -L^{-1} \left( \sum_{i=0}^{r-1} \frac{p_i(z)}{p_r(z)} y^{(i)}_k(z) \right) - \lambda L^{-1} \left( \frac{1}{p_r(z)} \int_0^z k(z)y^{(s)}_k(z)dz \right) \]  

(16)

By introducing numerical integration, one can write, with regard to a function \( \overline{T}(z) \):

\[ L^{-1} \left( \overline{T}(z) \right) = \sum_{i=1}^{N} C_i(z) \overline{T}(z_i) \]  

(17)

where

\[ C_i(z) = \int_0^z \ldots \int_0^z l_i(z)dz \ldots dz \]  

(18)

being \( l_i(z) \) the Lagrange polynomials and \( N \) the number of grid points at abscissae \( 0 = z_1 < z_2 < \ldots < z_{N-1} < z_N = 1 \).

So one has:
\[
y_0(z) = C_0(z)q_0
\]  
(19)  
where
\[
C_0(z) = (1, z, \ldots, z^{r-1}, C_1(z), \ldots, C_N(z))
\]  
(20)  
and
\[
q_0^T = \left( a_0, a_1, \ldots, \frac{a_{r-1}}{(r-1)!}, \frac{q(z_1)}{p_r(z_1)}, \ldots, \frac{q(z_N)}{p_r(z_N)} \right)
\]  
(21)  
\[
y_{k+1}(z) = C(z) \left[ \overline{P}_0 y_k + \ldots + (\overline{P} + \lambda \overline{B}) y_k^{(s)} + \ldots + \overline{P}_{r-1} y_k^{(r-1)} \right]
\]  
(22)  
where
\[
C(z) = (C_1(z), \ldots, C_N(z))
\]  
(23)  
\[
y_k^{(i)} = \left( y_k^{(i)}(z_1), \ldots, y_k^{(i)}(z_N) \right), \quad i = 0, \ldots, m
\]  
(24)  
with \( m = r - 1 \) if \( s < r \) or \( m = r \) if \( s = r \), \( \overline{P}_i \) diagonal matrices whose non-null elements are \( p_i(z_j)/p_r(z_j) \), with \( i = 1, \ldots, r-1, j = 1, \ldots, N \), \( \overline{B} \) a \( N \times N \) matrix whose elements are \( \overline{C}_j(z_i) \cdot \frac{k(z_i)}{p_r(z)} \) with \( i, j = 1, \ldots, N \) and \( \overline{C}_i(z) = \int_0^z l_i(z)dz \).  
Eq. (22) can also be written as
\[
y_{k+1}(z) = \sum_{i=0}^{m} B_i(z)y_k^{(i)}
\]  
(25)  
where \( B_i \) are \( N \times N \) matrices opportunely written according to eq. (22).  
By introducing DQ rules:
\[
y^{(i)}(z_j) = \sum_{s=1}^{N} A_{js}^{(i)} y_s, \quad j = 1, \ldots, N
\]  
(26)  
with \( i = 1, \ldots, m \), or in compact form
\[
y^{(i)} = A^{(i)} y
\]  
(27)  
eq \text{eq. (25) can be rewritten as}
\[ y_{k+1}(z) = \sum_{i=0}^{m} B_i(z) A^{(i)} y_k = D(z) y_k \]  

(28)

where \( y_k^T = (y_k(z_1), \ldots, y_k(z_N)) \) and \( D(z) = \sum_{i=0}^{m} B_i(z) A^{(i)} \) is a row vector. Since it is

\[ y_k = \overline{D} y_k = \overline{D}^k y_0 \]  

(29)

where

\[ \overline{D}^T = (D(z_1), \ldots, D(z_N)) \]  

(30)

then eq. (28) can be written as follows

\[ y_{k+1}(z) = D(z) \overline{D}^k y_0 \]  

(31)

so the solution given by the first \( k \) series terms (besides \( y_0 \)), \( y^{[k]}(z) = \sum_{i=0}^{k} y_i(z) \), can be written as

\[ y^{[k]}(z) = y_0 + D(z) \sum_{i=0}^{k-1} \overline{D}^i y_0 \]  

(32)

The spectral radius of the matrix \( \overline{D} \), \( \rho(\overline{D}) \), influences the solution, as will be proved.

**Theorem 1.**

If \( \rho(\overline{D}) \leq 1 \), then the solution \( \overline{y}(z) \) (eq. (14)) is

\[ \overline{y}(z) = y_0(z) + D(z)(I - \overline{D})^{-1} y_0 \]  

(33)

with \( I \) the identity matrix of order \( N \).

**Proof.**

The solution \( y^{[k]}(z) \) given by eq. (32) can be written as follows

\[ y^{[k]}(z) = y_0 + D(z) \left[ (I - \overline{D})^{-1}(I - \overline{D}^k) \right] y_0 \]  

(34)

For \( k \) which tends to infinity, if \( \rho(\overline{D}) \leq 1 \) one has eq. (33). \( \square \)

As a consequence of Theorem 1, the error \( e(z) = y(z) - \overline{y}(z) \) can be written as follows

\[ e(z) = -D(z)(I - \overline{D})^{-1} y_0 + e_0(z) \]  

(35)
where \( e_0(z) = y(z) - y_0(z) \). The values of the error at grid points \( e(z_i) \) can be collected in a vector \( e \) and predicted according to the following Theorem 2.

By deriving one time eq. (1), one can write

\[
\sum_{i=0}^{r} \left( p'_i(z)e_0^{(i)} + p_i(z)e_0^{(i+1)} \right) + k(z)e_0 = - \sum_{i=0}^{r} \left( p'_i(z)y_0^{(i)} + p_i(z)y_0^{(i+1)} \right) - k(z)y_0 + q'(z) \tag{36}
\]

By applying the quadrature rules

\[
Me_0 + R_N = -My_0 + \overline{R}_N + Q \tag{37}
\]

where \( e^T_0 = (e_0(z_1), \ldots, e_0(z_N)) \),

\[
M = \sum_{i=0}^{r} \left( P^{(1)}_iA^{(i)} + P_iA^{(i+1)} + K \right) \tag{38}
\]

with \( K, P^{(1)}_i, P_i \) diagonal matrices where \( k_{jj} = k(z_j), p^{(1)}_{i,jj} = p'_i(z_j), p_{i,jj} = p_i(z_j) \) respectively and \( R_N, \overline{R}_N \) the vectors of the residuals referred to the functions \( e_0 \) and \( -y_0 \) after the application of the DQ rules.

**Theorem 2.**

Let \( M \) (eq. (38)) be an invertible matrix; if the derivative \( y^{(N-1)} \) is a constant for any \( z \in [0, 1] \) then

\[
e = - \left[ D(I - D)^{-1} + I \right] y_0 + M^{-1}Q \tag{39}
\]

**Proof.**

By rewriting eq. (10) for \( e_0 \), one has

\[
e_0 = (y^{(r)}(0) - y_0^{(r)}(0)) \frac{z^r}{r!} + \ldots + (y^{(N)}(\zeta) - y_0^{(N)}(\zeta)) \frac{z^N}{N!}
\]

By rewriting eq. (10) also for \( -y_0 \), it is simple to verify that the residuals of \( e_0 \) and \( -y_0 \) are equal.\( \square \)

It should be pointed out that, even if \( y(z) \) is unknown, Theorem 2 has practical application, since \( e_1 \) should be null.

In fact, if eq. (39) is applicable, it is simple to check that \( e_1 = 0 \). However, ongoing research is attempting to generalize these results.
4. NUMERICAL EXAMPLES AND DISCUSSION

In this section, several numerical examples are given to illustrate the properties of the method.

For all the examples, good results can be achieved by a relatively small number of grid points.

For all the cases considered, the spectral radius of the matrix $\mathbf{D}$ turns out to be less than 1.

In order to compare the present method with existing ones, the Example 6 and 7 deal with integro–differential equations where there are no rational functions. In particular, Example 6 compares the results obtained by the present method with the ones obtained in [19], where Lagrange polynomials are used to obtain a discretized equations system. Example 7 is referred to an integro–differential equation with weakly singular kernel solved in [18] by means of Bernstein polynomials and in [14] by means of the Taylor expansion.

For all the cases, the method behaves quite satisfactorily.

Example 1.

Let us consider the Volterra integro–differential equation of the second kind

$$y''(z) + y(z) + \int_0^z \frac{y(z)}{1 + z + z^2} dz = q(z)$$

$$y(0) = y'(0) = 0$$

$$q(z) = 2 + z + z^2 + \frac{1}{\sqrt{3}} \left[ \frac{\pi}{6} - \arctan \left( \frac{1 + 2z}{\sqrt{3}} \right) \right] - \frac{1}{2} \ln (1 + z + z^2)$$

with exact solution $y(z) = z^2$.

The norm $\| e \|_{\infty} = \max_i |e_i|$ for several distributions of grid points is tabulated in Table 1. The best approximation seems to be achieved by means of the GCL distribution.

For $N = 9$ and the GCL distribution one finds

$$\bar{y} = z^2 + 2.21179 \times 10^{-7} z^3 - 3.90186 \times 10^{-6} z^4 + 2.31740 \times 10^{-5} z^5 - 6.74075 \times 10^{-5} z^6 +$$

$$+ 1.08363 \times 10^{-4} z^7 - 9.85227 \times 10^{-5} z^8 + 4.75724 \times 10^{-5} z^9 - 9.49954 \times 10^{-6} z^{10}.$$
Example 2.
Consider the linear Volterra integro–differential equation

\[ y'''(z) + y(z) + \int_0^z \frac{y(z)}{(1 + z)^4} dz = q(z) \]

\[ y(0) = y'(0) = y''(0) = 0 \]

\[ q(z) = 6 + z^3 + \frac{1}{4} \ln (1 + z^4) \]

with exact solution \( y(z) = z^3 \).

For this example, the norm \( \| e \|_\infty \) for several distributions of grid points is tabulated in Table 2. The best results are obtained again by means of the GCL distribution.

Example 3.
Consider the following Volterra integro–differential equation

\[ y'''(z) + \frac{y(z)}{1 + z^2} + \int_0^z \frac{y(z)}{1 + z^4} dz = q(z) \]

\[ y(0) = y''(0) = 0, \quad y'(0) = 1 \]

\[ q(z) = 6 + z + \frac{1}{4} \ln (1 + z^4) + \frac{1}{2} \arctan (z^2) \]

with exact solution \( y(z) = z + z^3 \).

The error norm \( \| e \|_\infty \) for several distributions of grid points is tabulated in Table 3. Once again the best results are achieved by means of the GCL distribution. Besides, for \( N = 9 \) and equally spaced points a worst approximation is obtained.

Example 4.
Consider the linear Volterra integro–differential equation

\[ y'''(z) + y(z) + \int_0^z \frac{y(z)}{(1 + z^2)^2} dz = q(z) \]

\[ y(0) = y'(0) = 1, \quad y''(0) = -2 \]

\[ q(z) = 7 + z - z^2 + z^3 + \frac{z}{1 + z^2} + \frac{1}{2} \ln (1 + z^2) \]
with exact solution \( y(z) = 1 + z - z^2 + z^3 \).

The behaviour of the error norm \( \|e\|_\infty \) is shown in Table 4.

The best approximation is referred to GCL points (e.g. see the values obtained for \( N = 5 \)).

**Example 5.**

Consider the linear Volterra integro–differential equation

\[
y^{(4)}(z) + y(z) + \frac{1}{12} \int_0^z \frac{y''(z)}{a^4 + z^4} dz = q(z)
\]

\[
y(0) = -a, \quad y'(0) = y''(0) = y'''(0) = 0
\]

\[
q(z) = 24 - a + z^3 + \frac{1}{4a\sqrt{2}} \left\{ \ln \left( \frac{a^2 - a\sqrt{2} + z^2}{a^2 + a\sqrt{2} + z^2} \right) - 2\arctan \left( \frac{az\sqrt{2}}{z^2 - a^2} \right) \right\}
\]

with exact solution \( y(z) = -a + z^4 \), being \( a \) an integer.

In Table 5 one can find the values of the error norm \( \|e\|_\infty \) for two values of the parameter \( a \) and for different distributions of grid points. For the higher value of \( a \) and \( N = 7 \) the best results seem to be referred to equally spaced points.

**Example 6.** [19]

In order to compare the present method with the one proposed in [19], based on the simple application of Lagrange polynomials, consider the following

\[
y''(z) + z \exp(z)y'(z) + y(z) + \int_0^z \exp(-z + t) dt = q(z)
\]

\[
y(0) = 1, \quad y'(0) = -1
\]

\[
q(z) = 2 \exp(-z) - z + z \exp(-z)
\]

with exact solution \( y(z) = \exp(-z) \)

The norm \( \|e\|_2 = \sqrt{\sum_{i=1}^N e_i^2/N} \) for several distributions of grid points is tabulated in Table 6 and compared with the results proposed in [19].

**Example 7.** [18, 14]

Let us consider the following integro–differential equation with a weakly singular kernel

\[
y''(z) + y(z) + \frac{1}{\sqrt{\pi}} \int_0^z \frac{y''(t)}{\sqrt{z-t}} dt = q(z)
\]
\[ y'(0) = y(0) = 1 \]

\[ q(z) = 2 \exp(z) + \exp(z) \text{erf}(\sqrt{z}) \]

with exact solution \( y(z) = \exp(z) \).

Following [18,14], the following result is used:

\[ \int_0^z \frac{t^i}{\sqrt{z-t}} \, dt = \sqrt{\pi} \frac{z^{\frac{3}{2} + i} \Gamma(i+1)}{\Gamma(\frac{3}{2} + i)} \]

where \( i \) is an integer, \( i \geq 0 \).

Since

\[ y''(z) = \sum_{j=1}^{N} l_{ij}''(z) y_j = \sum_{j=1}^{N} y_j \sum_{k=0}^{N-3} b_{kj} z^k \]

one has

\[ \int_0^z \frac{y''(t)}{\sqrt{z-t}} \, dt = \sum_{j=1}^{N} y_j \sum_{k=0}^{N-3} \sqrt{\pi} b_{kj} \frac{z^{\frac{3}{2} + k} \Gamma(i+k)}{\Gamma(\frac{3}{2} + k)} \]

where \( b_{kj} \) are real parameters referred to the Lagrange polynomial; so the matrix \( \overline{B} \) in this case becomes

\[ \overline{B} = RT \]

where \( R \) is a \( N \times (N-3) \) matrix with \( R_{ij} = z_i^{\frac{3}{2} + j} \) and \( T \) is a \( (N-3) \times N \) matrix with \( T_{ij} = b_{ij} \gamma_i \), being \( \gamma_i = \sqrt{\pi} \frac{\Gamma(i+1)}{\Gamma(\frac{3}{2} + i)} \).

The norm \( \|e\|_2 \) is shown in Table 7. As one can see, for \( N = 4 \) the present method (both for equally spaced and GCL points) gives an error norm of the same order of the one obtained by the method proposed in [18]; this value decreases by adding one or two grid points.

The value obtained for \( N = 4 \) by the method proposed in [14] is instead similar to the one obtained by the present method for \( N = 3 \) and the second grid coordinate \( z_2 = 0.2 \).
5. Conclusions

In this paper a numerical Picard–like method, which combines successive approximations with integral and differential quadrature, has been discussed. Integral and differential quadrature use Lagrange polynomials. Compared to methods where Lagrange polynomials are used to reduce the problem to the solution of a discretized equations system, the present approach has the main advantage, under certain conditions, to lead to an explicit solution without solving any equations system, or recursive computations, in addition to exactly reproduce the initial conditions. The method has been applied to a class of integro–differential equations, where rational functions are involved. Several numerical examples show that the method behaves quite satisfactorily, even when applied to different cases (e.g. weakly singular kernel) solved by other methods.
REFERENCES


Table captions

Table 1 – Example 1: the error norm for several $N$ and distributions of grid points (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)

Table 2 - Example 2: the error norm for several $N$ and distributions of grid points (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)

Table 3 - Example 3: the error norm for several $N$ and distributions of grid points (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)

Table 4 - Example 4: the error norm for several $N$ and distributions of grid points (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)

Table 5 - Example 5: the error norm for two values of the parameter $a$ and several $N$ and distributions of grid points (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)

Table 6 - Example 6: a comparison of the error norm (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)

Table 7 - Example 7: a comparison of the error norm (ES: equally spaced points; GCL: Gauss–Chebyshev–Lobatto points)
<table>
<thead>
<tr>
<th>N</th>
<th>distribution</th>
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<th>$|e|_\infty$ (eq. (39))</th>
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<td>2.70E-05</td>
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<td>GCL</td>
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<td>GCL</td>
<td>7.66E-10</td>
<td>7.66E-10</td>
</tr>
</tbody>
</table>

Table 1
5 ES 2.55E-05 2.55E-05
5 GCL 6.13E-06 6.13E-06
7 ES 9.05E-06 9.05E-06
7 GCL 7.07E-07 7.07E-07
9 ES 1.59E-06 1.59E-06
9 GCL 2.21E-08 2.21E-08

Table 2
\[ ||e||_\infty = \infty \quad \text{or} \quad ||e||_\infty = \infty \ (\text{eq. (39)}) \]

| N  | distribution | \( ||e||_\infty \) | \( ||e||_\infty \) (eq. (39)) |
|----|--------------|---------------------|---------------------------------|
| 5  | ES           | 8.00E-05            | 8.00E-05                        |
| 5  | GCL          | 4.49E-05            | 4.49E-05                        |
| 7  | ES           | 5.56E-07            | 5.56E-07                        |
| 7  | GCL          | 3.87E-07            | 3.87E-07                        |
| 9  | ES           | 1.25E-06            | 1.25E-06                        |
| 9  | GCL          | 1.41E-08            | 1.41E-08                        |

Table 3
| N  | distribution | $||e||_{\infty}$ | $||e||_{\infty}$ (eq. (39)) |
|----|--------------|-----------------|-----------------------------|
| 5  | ES           | 1.12E-04        | 1.12E-04                    |
| 5  | GCL          | 8.46E-06        | 8.46E-06                    |
| 7  | ES           | 1.97E-06        | 1.97E-06                    |
| 7  | GCL          | 1.43E-07        | 1.43E-07                    |
| 9  | ES           | 8.14E-07        | 8.14E-07                    |
| 9  | GCL          | 1.06E-08        | 1.06E-08                    |

Table 4
| N  | distribution | $||e||_\infty$ | $||e||_\infty$ (eq. (39)) | $||e||_\infty$ | $||e||_\infty$ (eq. (39)) |
|----|--------------|-----------------|---------------------------|-----------------|---------------------------|
| 5  | ES           | 1.46E-07        | 4.65E-13                  | 1.46E-07        | 4.65E-13                  |
| 5  | GCL          | 1.99E-08        | 7.28E-14                  | 1.99E-08        | 7.28E-14                  |
| 7  | ES           | 3.23E-09        | 1.01E-16                  | 3.23E-09        | 1.01E-16                  |
| 7  | GCL          | 3.25E-10        | 1.06E-14                  | 3.25E-10        | 1.06E-14                  |

Table 5
<table>
<thead>
<tr>
<th>N</th>
<th>$|e|_2$ [19]</th>
<th>$|e|_2$ (ES)</th>
<th>$|e|_2$ (GCL)</th>
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</thead>
<tbody>
<tr>
<td>5</td>
<td>3.70E-07</td>
<td>1.06E-06</td>
<td>5.24E-07</td>
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<td>7</td>
<td>6.10E-10</td>
<td>2.20E-09</td>
<td>2.59E-10</td>
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<td>9</td>
<td>7.20E-13</td>
<td>4.81E-12</td>
<td>1.01E-13</td>
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</tbody>
</table>

Table 6
<table>
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<th>N</th>
<th>4 [18]</th>
<th>4 [14]</th>
<th>4 (ES)</th>
<th>4 (GCL)</th>
<th>3 ( (z_2 = 0.2) )</th>
<th>5 (ES)</th>
<th>5 (GCL)</th>
<th>6 (ES)</th>
<th>6 (GCL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>|(e|_2|\</td>
<td>2.13E-03</td>
<td>4.12E-02</td>
<td>5.55E-03</td>
<td>5.72E-03</td>
<td>5.35E-02</td>
<td>4.73E-04</td>
<td>3.82E-04</td>
<td>3.10E-05</td>
<td>2.03E-05</td>
</tr>
</tbody>
</table>

Table 7