Integrable Ermakov-Pinney equations with nonlinear Chiellini ‘damping’

Stefan C. Mancas∗

Department of Mathematics,
Embry-Riddle Aeronautical University,
Daytona Beach, FL. 32114-3900, U.S.A.

Haret C. Rosu†

IPICyT, Instituto Potosino de Investigacion Cientifica y Tecnologica,
Apdo. Postal 3-74 Tangamanga, 78231 San Luis Potosí, S.L.P., Mexico

For the constant frequency case, we introduce a special type of Ermakov-Pinney equations with nonlinear dissipation based on the corresponding Chiellini integrable Abel equation. General solutions of these equations are obtained following the Abel equation route. Based on particular solutions, we also provide general solutions containing a factor with the phase of the Milne type. In addition, the same kinds of general solutions are constructed for the cases of higher-order Reid nonlinearities. The Chiellini ‘dissipative’ function is actually a dissipation-gain function because it can be negative on some intervals. These are the first examples of integrable Ermakov-Pinney equations with nonlinear ‘damping’.

I. INTRODUCTION

The nonlinear Ermakov-Pinney (EP) equation has long been known to have profound connections with the linear equations of identical operatorial form without the inverse cubic nonlinearity and because of this it has been seen as an example of ‘nonlinearity from linearity’ [1]. A historical overview has been written by Leach and Andriopoulos [2] and the fundamental importance of the EP equation for parametric oscillators, both classical and quantum-mechanical, with their vast application reaches, is well established in the literature.

Already in 1880, Ermakov introduced the following pair of equations [3]

\[
\begin{align*}
\ddot{y} + \omega^2(t)y &= 0, \\
\ddot{x} + \omega^2(t)x - \kappa x^{-3} &= 0,
\end{align*}
\] (1)

for which the linear equation is endowed with the so-called Ermakov dynamic invariant

\[ I_{0-\kappa}(x, y, \dot{x}, \dot{y}) = \kappa \left( \frac{y}{x} \right)^2 + (x\dot{y} - y\dot{x})^2. \]

We label here this invariant by the tuning parameters of the nonlinear term of the two equations. When \( \kappa = 0 \) one gets \( I_{00} = W^2 \), where \( W \) is the Wronskian of two solutions of the linear equation. Thus, Ermakov’s invariant can be considered as a generalization of the Wronskian. Modern research on this invariant started in the 1960s when Lewis rediscovered it and also provided applications in quantum-mechanics [4]. In particular, Ray and Reid [5] and also Sarlet [6] introduced the following generalized Ermakov systems

\[
\begin{align*}
\ddot{y} + \omega^2(t)y - \frac{1}{y^3} f(x/y) &= 0, \\
\ddot{x} + \omega^2(t)x - \frac{1}{x^3} g(y/x) &= 0,
\end{align*}
\] (3)

and their associated invariants

\[ I_{-f-g} = 2 \int_{x/y}^{x} f(\eta)d\eta + 2 \int_{y/x}^{y} g(\eta)d\eta + (x\dot{y} - y\dot{x})^2. \] (4)

When \( f = 0 \) and \( g = \kappa(y/x) \) the invariant \( I_{0-\kappa} \) is recovered. Recent works on the connections of the Ermakov systems with the nonlinear superposition principle belong to Cariñena and collaborators [7, 8] where the reader can find more.

∗Electronic address: stefan.mancas@erau.edu
†Electronic address: hcr@ipicyt.edu.mx
references. The importance of Ermakov equations stems from the fact that they can be used to model the propagation of laser beams in nonlinear optics [9], magneto-gas dynamics [10], the mean field dynamics of pancake-shaped Bose-Einstein condensates [11], and cosmology [12, 13] and the same is expected from Ermakov systems.

In this paper, we determine the general solution of an Ermakov equation with an Abel-integrable nonlinear dissipation $g_{\lambda}(u)$ and constant ‘frequency’ function $\omega^2(\zeta) = \lambda^2$:

$$u_{\zeta\zeta} + g_{\lambda}(u)u_{\zeta} + \lambda^2 u + bu^{-3} = 0 .$$  \hspace{1cm} (5)

We get the solution using the corresponding integrable Abel equation and also we give a theorem for obtaining the general solution if a particular solution is known. In the latter case, the phase of the solution is of the Milne type [14] and the Ermakov invariant for a pair of nonlinear dissipative Ermakov equations of the type (5) with different nonlinearity parameters $b$ and $c$ is used in the derivation. We emphasize that Ermakov equations with nonlinear dissipation have never been considered before. For the general case of Ermakov equations with a linear dissipative term one has to resort to numerical methods because there are no Lie symmetries and reductions to simpler forms are useful only in particular cases [15].

The paper is structured as follows. We first discuss briefly the basic properties of the simplest nondissipative Ermakov equation corresponding to the constant ‘frequency’ case. We next discuss the special EP equation with dissipation determined by Chiellini’s integrability condition for the corresponding Abel equation of the first kind [16]. Its general solution is obtained through a method that makes usage of the Ermakov invariant of a pair of such dissipative Ermakov equations. Finally, the method is also applied to Chiellini-dissipative Ermakov equations with higher order Ermakov nonlinearities for which we also provide the general solutions. The paper ends up with some conclusions and three appendices in which several related mathematical issues are included for self-consistency reasons.

II. THE SIMPLEST EP (SEP) EQUATION

A. Solution of SEP

As is well known, if we have two linear independent solutions $u_1$, and $u_2$ to

$$u_{\zeta\zeta} + h(\zeta)u_{\zeta} = 0 ,$$  \hspace{1cm} (6)

then a particular solution of the corresponding EP equation

$$v_{\zeta\zeta} + h(\zeta)v + cv^{-3} = 0$$  \hspace{1cm} (7)

is given by [17]

$$v(\zeta) = \sqrt{u_2^2 - cW^2} ,$$  \hspace{1cm} (8)

where $W$ is the Wronskian of the two linearly independent solutions, while the general solution can be written

$$v_\psi(\zeta) = (\alpha_1 u_1^2 + \alpha_2 u_2^2 + 2\alpha_3 u_1 u_2)^{1/2} ,$$

with the three constants constrained by the condition $\alpha_1\alpha_2 - \alpha_3^2 = c$.

Let’s take the simplest case, i.e., $h(\zeta) = \lambda^2$, a constant.

$$u_{\zeta\zeta} + \lambda^2 u_{\zeta} = 0 .$$  \hspace{1cm} (9)

We call the EP equation for this case as SEP

$$v_{\zeta\zeta} + \lambda^2 v + cv^{-3} = 0 ,$$  \hspace{1cm} (10)

which we also write in the form

$$v_{\zeta\zeta} + h_\lambda(v) = 0 , \hspace{1cm} h_\lambda(v) = \lambda^2 v + cv^{-3} .$$  \hspace{1cm} (11)

From [8], its solution can be written in the form

$$\begin{cases}
\tilde{v}_-(\zeta) = \sqrt{1 + \left(1 - \frac{c\lambda^2}{\lambda^2}\right)\sinh^2\lambda\zeta} , \hspace{0.5cm} \lambda^2 < 0 , \\
\tilde{v}_0(\zeta) = \sqrt{1 - c\lambda^2} , \hspace{0.5cm} \lambda^2 = 0 , \\
\tilde{v}_+(\zeta) = \sqrt{1 - \left(1 + \frac{c\lambda^2}{\lambda^2}\right)\sin^2\lambda\zeta} , \hspace{0.5cm} \lambda^2 > 0 .
\end{cases}$$  \hspace{1cm} (12)

The case $\lambda^2 > 0$ corresponds to the simple harmonic oscillator.
B. SEP equation with Chiellini dissipation

We build now the Ermakov-Pinney equation with Chiellini-type dissipation as an equation of the following format

\[ v'' + g_\lambda(v)v' + h_\lambda(v) = 0 , \]  

where \( h_\lambda(v) \) is as given in [11]. The dissipation term \( g_\lambda(v) \) is obtained from \( h_\lambda(v) \) using Chiellini’s integrability condition [10]

\[ \frac{d}{dv} \left( \frac{b(v)}{g(v)} \right) = kg(v), \quad k, \text{ a real constant} \]  

for Abel’s equation of the first kind \( dy/dv = h_\lambda(v)y^3 + g_\lambda(v)y^2 \) corresponding to (13). From (14) one easily gets

\[ g_\lambda(v) = \frac{\lambda^2 v^2 + cv^{-2}}{\sqrt{k\lambda^2 v^4 + c_1 v^2 - k}}. \]  

Before proceeding further, we want to mention that one can not get rid of the Chiellini dissipation term \( g_\lambda(v) \) because this term corresponds to the quadratic nonlinearity of the Abel’s equation

\[ \frac{dy}{dv} = g_\lambda(v)y^2 + h_\lambda(v)y^3 \],

see Appendix A.

The inverse of the Abel solution, \( \frac{1}{\eta} = \frac{g_\lambda(v)}{h_\lambda(v)} |_{k=-2} \), is obtained from

\[ \frac{1}{\eta} = \frac{v}{\sqrt{-2\lambda^2 v^4 + c_1 v^2 + 2c}} \],

which gives

\[ \zeta - \zeta_0 = \int \frac{vdv}{\sqrt{-2\lambda^2 v^4 + c_1 v^2 + 2c}}. \]  

Depending on the sign of \( \lambda^2 \) we have the following cases which give the general solution of (13)

\[ \begin{align*}
  v_-(\zeta) &= \sqrt{-c_1 + \sqrt{16\lambda^2 c - c_1^2 \sinh (2\sqrt{2\lambda}(\zeta - \zeta_0)}}) \div 4\lambda^2, & \lambda^2 &= -\tilde{\lambda}^2 < 0 \\
  v_0(\zeta) &= \sqrt{c_1 (\zeta - \zeta_0)^2 - \frac{2c_1}{c_1}}, & \lambda^2 &= 0, \\
  v_+(\zeta) &= \sqrt{c_1 + \sqrt{16\lambda^2 c + c_1^2 \sinh (2\sqrt{2\lambda}(\zeta - \zeta_0)}}) \div 4\lambda^2, & \lambda^2 &= > 0. 
\end{align*} \]

It is worth noting that if we take \( c = 0 \), the reduced equation

\[ v'' + g_\lambda(v)v' + \lambda^2 v = 0 \]  

has the harmonic solutions \( v_{1r} = \sqrt{\frac{2c}{\sqrt{2}\lambda}} \sin \sqrt{2\lambda} \zeta \) and \( v_{2r} = \sqrt{\frac{2c}{\sqrt{2}\lambda}} \cos \sqrt{2\lambda} \zeta \) as if the nonlinear dissipation is not acting at all and the equation (21) is linear. Thus, one can also construct solutions of (13) in the standard Pinney form.

The only feature is that the amplitudes of the harmonic modes are inverse proportional to the frequency.

It is also possible to construct a different form of the general solution for a SEP with Chiellini dissipation in terms of a particular solution as follows. We first get a particular solution from (19) by fixing \( \zeta_0 = 0 \) and \( c_1 = \gamma \)

\[ \begin{align*}
  v_\gamma-:\gamma(\zeta) &= \sqrt{-\gamma^2 + \sqrt{16\lambda^2 c - \gamma^2 \sinh (2\sqrt{2\lambda}\zeta)}} \div 4\lambda^2, & \lambda^2 &= -\tilde{\lambda}^2 < 0 \\
  v_\gamma0(\zeta) &= \sqrt{\gamma^2 \zeta^2 - \frac{2c}{\gamma}}, & \lambda^2 &= 0, \\
  v_\gamma+:\gamma(\zeta) &= \sqrt{\gamma^2 + \sqrt{16\lambda^2 c + \gamma^2 \sinh (2\sqrt{2\lambda}\zeta)}} \div 4\lambda^2, & \lambda^2 &= > 0. 
\end{align*} \]
Then, the solution of the equation
\[ u_{\zeta\zeta} + g_\lambda(u)u_\zeta + \lambda^2 u + bu^{-3} = 0, \] (22)
with \( g_\lambda \) as given in (15) is provided by the following theorem.

**Theorem.** For the Ermakov equation (22), with \( v_\gamma(\zeta) \) the particular solution as given in (21), and \( g_\lambda(v) \) given by (15), the general solution \( u(\zeta) \) is obtained by

\[
\begin{align*}
\left\{
\begin{array}{ll}
  u_-(\zeta) &= \sqrt{-I_{bc} + \sqrt{-I_{bc}^2 + 4bc \sinh(2\sqrt{c}(\Theta - \Theta_0))}} v_\gamma(\zeta), & \lambda^2 = -\tilde{\lambda}^2 < 0 \\
  u_0(\zeta) &= \sqrt{I_{bc}(\Theta - \Theta_0)^2 - \frac{1}{I_{bc}}v_\gamma(\zeta)}, & \lambda^2 = 0 \\
  u_+(\zeta) &= \sqrt{I_{bc} + \sqrt{I_{bc}^2 + 4bc \sin(2\sqrt{c}(\Theta - \Theta_0))}} v_\gamma(\zeta), & \lambda^2 > 0
\end{array}
\right.
\tag{23}
\end{align*}
\]

where the phase \( \Theta = \int \frac{1}{v_\gamma^2} d\zeta \) is of the Milne type (14), and \( I_{bc} \) is the following quantity
\[ I_{bc} = -b\left(\frac{u_1}{u_2}\right)^2 - c\left(\frac{u_1}{u_2}\right)^2 + (u_1v - uv_\zeta)^2, \] (24)
which is the Ermakov invariant for an Ermakov pair of equations of type (22) of nonlinear parameters \( b \) and \( c \), respectively.

A version of this theorem for the non-dissipative case can be found in a paper by Qin and Davidson (18).

**Proof:**

First, we construct the (invariant) quantity \( I_{bc} \), using (24). Then, using \( \Theta = \int \frac{1}{v_\gamma^2} d\zeta \), and \( \theta = \frac{\Theta}{b} \) in (24) we obtain the following separable equation
\[ \frac{\theta d\theta}{\sqrt{b + I_{bc}\theta^2 + 4\theta^4}} = d\Theta. \] (25)

(25) has the same format as (18) with \( c = -2\lambda^2 \), \( I_{bc} = c_1, \) \( b = 2c \).
But from \( \theta = \frac{\Theta}{b} \), one gets \( u = \theta v \) which leads to (23).

Two more lemmas on \( I_{bc} \) are given in the Appendix B, whereas the factorization of the dissipative Ermakov equations on which we focus here can be found in Appendix C.

**C. Abel-dissipative SEP equations with \( 2m - 1 \) negative power nonlinearity**

In this subsection, we show that the method of obtaining the general solution of the previous section can be also applied to equations with higher order Ermakov nonlinearities and associated Abel dissipation.

Reid has shown in (19) that the general solution to
\[ v_{\zeta\zeta} + h(\zeta)v = q_m(\zeta)v^{-(2m-1)} \] (26)
is
\[ v(\zeta) = \left(u_1^m + \frac{\tilde{c}}{(m-1)W^2u_2^m}\right)^{\frac{1}{m}} \] (27)
provided that \( u_1 \) and \( u_2 \) are two independent solutions of (10), and
\[ q_m(\zeta) = \tilde{c}(u_1u_2)^{m-2}. \] (28)
Notice that (27) is a direct generalization of (8). We are interested in a general solution to
\[ u_{\zeta\zeta} + g_\lambda(u)u_\zeta + h(\zeta)u = q_m(\zeta)u^{1-2m}, \] (29)
via the machinery of invariants. 
First, let us choose \( h(\zeta) = +\lambda^2, 0, -\lambda^2 \), then (20) has solutions
\[
\begin{cases}
v_-(\zeta) = (a^m e^{m\lambda \zeta} + \tilde{c}_m b^m e^{-m\lambda \zeta}) \frac{\ddot{\theta}}{\dot{\theta}}, & h(\zeta) = -\lambda^2 \\
v_0(\zeta) = \left(1 + \frac{\ddot{\theta}}{m-1} \zeta^m \right) \frac{\ddot{\theta}}{\dot{\theta}}, & h(\zeta) = 0 \\
v_+(\zeta) = (a^m \cos m\lambda \zeta + \tilde{c}_m b^m \sin m\lambda \zeta) \frac{\ddot{\theta}}{\dot{\theta}}, & h(\zeta) = \lambda^2,
\end{cases}
\]
where
\[
\tilde{c}_m = \frac{\ddot{\theta}}{4\lambda^2(ab)^m (m-1)}.
\]
For simplification, let us write (30) as
\[
\begin{cases}
v_-(\zeta) = \left(A e^{m\lambda \zeta} + B e^{-m\lambda \zeta}\right) \frac{\ddot{\theta}}{\dot{\theta}} \\
v_0(\zeta) = \left(1 + B_0 \zeta^m \right) \frac{\ddot{\theta}}{\dot{\theta}} \\
v_+(\zeta) = \left(A \cos m\lambda \zeta + B \sin m\lambda \zeta\right) \frac{\ddot{\theta}}{\dot{\theta}},
\end{cases}
\]
where \( A = a^m, B = \frac{\ddot{\theta}}{2\lambda a^m (m-1)^2}, B_0 = \frac{\ddot{\theta}}{m-1} \) and by integrating \( \Theta_+ = \int \frac{1}{v_+} d\zeta \) and \( \Theta_0 = \int \frac{1}{v_0} d\zeta \) leads to
\[
\begin{cases}
\Theta_-(\zeta) = \frac{A e^{2m\lambda \zeta} + B}{2\lambda B \left(A e^{m\lambda \zeta} + B e^{-m\lambda \zeta}\right)} \frac{\ddot{\theta}}{\dot{\theta}} \frac{2F_1\left(1, \frac{m-1}{m}, \frac{m+1}{m}; -\frac{4}{B} e^{2m\lambda \zeta}\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m\lambda \zeta + \arctan \frac{A}{B}\right)\right)}. \\
\Theta_0(\zeta) = \frac{2 \cos \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}}{m \left(\cos \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}\right)} \frac{\ddot{\theta}}{\dot{\theta}} \frac{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}.
\end{cases}
\]
Finally, the solution to (29) is \( u = u_0 \), with \( \theta \) obtained by integrating (25).

Using the same \( \lambda = \frac{1}{2} \), together with \( a = b = \ddot{\theta} = c_1 = 1 \), then the solution is the following
\[
\begin{cases}
u_-(\zeta, m) = \left(e^{\frac{m \zeta}{2}} + \frac{1}{m-1} e^{-\frac{m \zeta}{2}}\right) \frac{\ddot{\theta}}{\dot{\theta}} (-1 \mp \sqrt{3} \sinh(\sqrt{2}\Theta_-)) \frac{\ddot{\theta}}{\dot{\theta}}, & h(\zeta) = -\lambda^2 \\
u_0(\zeta, m) = \left(1 + \frac{m \zeta}{2} \zeta^m \right) \frac{\ddot{\theta}}{\dot{\theta}} (\Theta_0 - 1) \frac{\ddot{\theta}}{\dot{\theta}}, & h(\zeta) = 0 \\
u_+(\zeta, m) = \left(\cos \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}\right) \frac{\ddot{\theta}}{\dot{\theta}} (1 \pm \sqrt{3} \sinh(\sqrt{2}\Theta_+)) \frac{\ddot{\theta}}{\dot{\theta}}, & h(\zeta) = \lambda^2,
\end{cases}
\]
where
\[
\begin{cases}
\Theta_-(\zeta) = \frac{(m-1)e^{m \zeta} + 1}{m \left(e^{m \zeta} + \frac{1}{m-1} e^{-m \zeta}\right)} \frac{\ddot{\theta}}{\dot{\theta}} \frac{2F_1\left(1, \frac{m-1}{m}, \frac{m+1}{m}; - \left(\sqrt{m-1} e^{\frac{m \zeta}{2}}\right)^2\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}.
\\
\Theta_0(\zeta) = \frac{2 \sin \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}}{m \left(\sin \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}\right)} \frac{\ddot{\theta}}{\dot{\theta}} \frac{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}.
\\
\Theta_+(\zeta) = \frac{2 \sin \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}}{m \left(\sin \frac{m \zeta}{2} + \frac{1}{m-1} \sin \frac{m \zeta}{2}\right)} \frac{\ddot{\theta}}{\dot{\theta}} \frac{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}{2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{m}; \frac{3}{2}; \cos^2 \left(m \lambda \zeta + \arctan \frac{A}{B}\right)\right)}.
\end{cases}
\]
Note that for the case \( m = 2 \) which is the standard EP case the solution simplifies to
\[
\begin{cases}
u_-(\zeta, 2) = \sqrt{(e^\epsilon + e^{-\epsilon}) \left(-1 \mp \sqrt{3} \sinh \left(\sqrt{2} \arctan \cos \epsilon\right)\right)} \\
u_0(\zeta, 2) = \sqrt{(\epsilon^2 + 1)(\arctan^2 \epsilon - 1)} \\
u_+(\zeta, 2) = \sqrt{\left(\cos \zeta + \sin \zeta\right) \left(1 \pm \sqrt{\sinh(\arctanh \frac{\cos \zeta - \sin \zeta}{\sqrt{2}})}\right)}.
\end{cases}
\]

Plots of the \( u \) solutions for different values of \( m \) and all the other parameters set to the value of unity are given in Figs. (11)-(13). Solutions \( u_- \) are real, while \( u_+ \) are periodically pure real and pure imaginary. As for the solution \( u_0 \) displayed for the case \( m = 2 \) in Fig. (10), it is pure imaginary on a symmetric interval around the origin and real in the rest. Notice also the diminishing of the amplitudes with increased order of the negative-power nonlinearity. But the most interesting feature of the ‘dissipative’ solutions is that they may have larger amplitudes than the nondissipative ones on some time intervals. This reveals the presence of gain effects. Indeed, from the plots of the function \( g(\zeta) \) in Figs. (17)-(19), one can infer that this function is not always dissipative but it also has gain intervals.
III. CONCLUSION

A class of dissipative Ermakov-Pinney equations, either with standard \( m = 2 \) or higher-order \( m > 2 \) nonlinearities, and with nonlinear dissipation of the Abel-Chiellini-integrable type has been introduced. The general solutions are obtained directly by the Abel equation route and also using the dynamic invariant of Ermakov pairs of equations of this type and the particular solution of one member of the pair, both in the standard case and in the case of any odd Ermakov nonlinearity. The technique based on Abel’s equation we used here is unique to the constant frequency systems and cannot be directly generalized to the case of time-dependent oscillators. This is due to the fact that the Chiellini integrability condition which plays a central role in obtaining the results do not apply to the cases when there exist an explicit dependence on the independent variable. Another remarkable aspect is that the Chiellini nonlinear dissipative function is in fact a dissipation-gain function which could have interesting applications in the propagation of pulses.
Appendix A: The Abel equation

The importance of Abel’s equation in its canonical forms stems from the fact that its integrability cases lead to closed form solutions to nonlinear ODEs of the form

\[ u_{\zeta\zeta} + f_2(u)u_\zeta + f_3(u) + f_1(u)u_\zeta^2 + f_0(u)u_\zeta^3 = 0. \]  

(37)

This can be expressed by the following Lemma.

**Lemma 1**: Solutions to a general second order ODE of type (37) may be obtained via the solutions to Abel’s equation (42), and vice versa using the following relationship

\[ \frac{du}{d\zeta} = \eta(u(\zeta)). \]  

(38)

**Proof**: To show the equivalence, one just need the simple chain rule

\[ \frac{d^2 u}{d\zeta^2} = \frac{d\eta}{du} \frac{du}{d\zeta} = \eta \frac{du}{d\zeta} \]  

(39)

which turns (37) into the second order Abel equation in canonical form

\[ \frac{d\eta}{du} + f_2(u)\eta + f_3(u) + f_1(u)\eta^2 + f_0(u)\eta^3 = 0. \]  

(40)

Moreover, via transformation of the dependent variable

\[ \eta(u) = \frac{1}{y(u)} \]  

(41)

(40) becomes

\[ \frac{dy}{du} = f_0(u) + f_1(u)y + f_2(u)y^2 + f_3(u)y^3 \]  

(42)

and letting \( \hat{y} = ye^{\int f_1(u)du} \), one gets

\[ \frac{d\hat{y}}{du} = f_0(u)e^{-\int f_1(u)du} + f_2(u)e\int f_1(u)du \hat{y} + f_3(u)e^{2\int f_1(u)du} \hat{y}, \]  

(43)

which is an Abel equation without the linear term.

Appendix B: Two lemmas for \( I_{bc} \)

Here we formulate two lemmas related to \( I_{bc} \).

**Lemma 1**: The invariant \( I_{bc} \) is related to the factoring functions of the Ermakov equations of the corresponding system.

Let \( \Phi_1 \), and \( \Psi_1 \) be the first factoring functions of two nonlinear Ermakov ODEs as in the previous appendix. Then, according to (52), \( u = c_\alpha e^{\int \Phi_1 d\zeta} \), and \( v = c_\epsilon e^{\int \Psi_1 d\zeta} \). Then, using (24) we have

\[ I_{bc} = \theta^2 v^4(\Phi_1 - \Psi_1)^2 - m \cosh \left( \int (\Phi_1 - \Psi_1) d\zeta \right), \]  

(44)

where \( m = 2 \left( b c_\alpha^2 + c_\epsilon^2 \right) \).

**Lemma 2**: \( I_{bc} = c_1 \).

If one writes (24) as

\[ I_{bc} = -b \frac{1}{\theta^2} - c\theta^2 + \left( \frac{d\theta}{d\Theta} \right)^2, \]  

(45)
using chain rule and (25) and by substituting $\Theta = \int \frac{1}{u} d\zeta$ leads to $I_{bc} \equiv c_1$.

**Appendix C: Factorization of dissipative Ermakov ODEs**

The Ermakov ODEs with nonlinear dissipation-like coefficients, i.e.,

$$u_{\zeta\zeta} + g(u)u_{\zeta} + h(u) = 0, \quad h(u) \equiv h_{\lambda}(u) = \lambda^2 u + q_m(\zeta)u^{-(2m-1)}$$

(46)
can be factored in the form

$$\left( \frac{d}{d\zeta} - \Phi_2(u) \right) \left( \frac{d}{d\zeta} - \Phi_1(u) \right) u = 0.$$  

(47)

This gives

$$u_{\zeta\zeta} - \frac{d}{d\zeta} (\Phi_1(u)u) - \Phi_2 u_{\zeta} + \Phi_1 \Phi_2 u = 0.$$  

(48)

Furthermore, employing the grouping of terms used by Rosu and Cornejo-Pérez [20–22], one gets

$$u_{\zeta\zeta} - (\Phi_1 + \Phi_2) u_{\zeta} + (\Phi_1 \Phi_2) u = 0.$$  

(49)

Then, we have

$$\begin{cases} 
  g(u) = -(\Phi_1 + \Phi_2 + \frac{d\Phi_1}{du} u) \\
  h(u) = (\Phi_1 \Phi_2) u 
\end{cases}.$$  

(50)

From (17), we also have

$$u_{\zeta} = \Phi_1 u$$  

(51)

which leads to

$$u = c_1 e^{\int \Phi_1(u(x)) dx}.$$  

(52)

This is identical to a well-known result in the case of linear equations, especially in supersymmetric quantum mechanics, where it gives the ground state wavefunction in terms of the superpotential. Since $\Phi_1 = \frac{\lambda u}{v}$ and $\Phi_1 \Phi_2 = \frac{h}{v}$, and additionally we know that $v_{\zeta} = \eta = \frac{d}{g}$, then we get

$$\Phi_1 = \frac{\eta}{v} = \sqrt{-2\lambda^2 v^4 + c_1 v^2 + 2c},$$

$$\Phi_2 = g = \frac{\lambda^2 v^2 + cv^{-2}}{\sqrt{-2\lambda^2 v^4 + c_1 v^2 + 2c}},$$

which is exactly the dissipative term for which the equation is integrable.

---

FIG. 1: (Color online). The dissipative solution $u_-(\zeta)$ (blue) and the non-dissipative solution $v_-(\zeta)$ (red) for the case $m = 2$. All the parameters are set to unity in all of the plots.
FIG. 2: (Color online). The modulus square of the same solutions for the case $m = 3$ with the same code of colors.
FIG. 3: (Color online) The dissipative $u_+(\zeta)$ solution and the non-dissipative $v_+(\zeta)$ solution for the case $m = 2$ and the same color code.
Fig. 4: (Color online) The modulus square of $u_\pm(\zeta)$ and $v_\pm(\zeta)$ solutions for the case $m = 3$ and the same color code.
FIG. 5: (Color online) The modulus square of $u_m(\zeta)$ and $v_m(\zeta)$ solutions for the case $m = 4$ and the same color code.
FIG. 6: (Color online). The dissipative solution $u_0(\zeta)$ (blue) and the non-dissipative solution $v_0(\zeta)$ (red) for the case $m = 2$. 
FIG. 7: (Color online) Chiellini dissipation function $g(\zeta)$ for $v_-$ and $\lambda^2 = -\frac{1}{4}$. 

\[ g(\zeta, \lambda^2) = \frac{1}{4} \]
FIG. 8: (Color online) Chiellini dissipation function $g(\zeta)$ for $v_+ \text{ and } \lambda^2 = \frac{1}{4}$. 

$$g(v_+), \lambda^2 = \frac{1}{4}$$
FIG. 9: (Color online) Chiellini dissipation function $g(\zeta)$ for $v_0$ and $\lambda^2 = 0$. 