A monolithic finite element approach using high-order schemes in time and space applied to finite strain thermo-viscoelasticity

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Abstract

This article addresses a thermo-mechanically coupled problem of thermo-viscoelasticity at finite strains using a monolithic approach. The underlying equations are based on the non-linear transient heat equation, the local equilibrium conditions and the evolution equations of the internal variables. The latter describe the hardening behavior of the material. If the method of vertical lines is applied, its first step – namely the spatial discretization – yields a system of differential-algebraic equations (DAE-system). Here, we employ the p-version of the finite element method based on integrated Legendre polynomials. This can lead to very precise solutions in the spatial domain. In order to be accurate in the time-domain as well, stiffly accurate, diagonally-implicit Runge-Kutta methods are applied to solve the DAE-system yielding a coupled system of non-linear algebraic equations. In this article, the system is solved monolithically by employing the Multilevel-Newton algorithm. Accordingly, a high-order result is obtained in the space and the time domain. The numerical concept is applied to a constitutive model of finite strain thermo-viscoelasticity. Several examples are applied to demonstrate the efficiency and applicability of the numerical scheme. It is especially the transient problems that call for time-adaptive schemes which are naturally embedded in the concept.

Keywords: differential-algebraic equations, thermo-viscoelasticity, high-order time-integration, p-version finite element analysis

1. Introduction

Thermo-mechanically coupled computations are of great importance if either the heat equation is influenced by the deformation or if the material properties are temperature sensitive. The first case occurs either when internal mechanical dissipation generates heat – such as in metal forming or in cyclic processes, respectively – or in large deformation cases where the entire surface changes so significantly that the amount of heat of the material body is essentially influenced. In the second case, the material properties depend on temperatures during hot forming processes, see, for instance, (Quint, 2012; Hartmann et al., 2009; Quint et al.,

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and the literature cited therein, or are connected to polymers and rubber-like material that are highly temperature-sensitive. In the field of rubber-elasticity, first numerical attempts were made by Holzapfel (2008), Holzapfel and Simo (1996b), Lion (1997b), Lion (2000b), and Reese (2001). (Holzapfel, 2008) is an introductory textbook dealing with entropic elastic material behavior. While Lion (1997b, 2000b) addresses finite thermo-viscoplastic material behavior, the derivation of a thermodynamically consistent material model for finite thermo-viscoelastic material behavior is discussed in (Reese, 2001). For additional information on this topic see also (Holzapfel and Simo, 1996a). In (Reese, 2001), the spatial discretization is carried out with elements based on a stabilized reduced integration technique that enables large deformations. For reduced integration techniques in the field of finite thermo-viscoelastic material behavior see (Reese, 2003). The employed variational formulation contains a parameter which is, to a certain extent, artificial. Lion (1997a) addresses the investigation and modeling of a filler loaded tread compound. The material behavior is characterized by non-linear elasticity in combination with non-linear rate-dependence. Furthermore, a weak equilibrium hysteresis can be observed. Accordingly, it exhibits the characteristics of a thermo-viscoplastic material. In (Reese and Govindjee, 1998), a thermo-viscoelastic material model for large strain applications is discussed, drawing on a monolithic approach employing a Backward-Euler time integration scheme. Miehe (1996) considers problems of large strain rate-independent multi-surface thermo-plasticity. The governing equations are solved by means of an isothermal operator split – implying that, firstly, the isothermal sub-problem at frozen temperature is solved, and that, secondly, the heat conduction problem at frozen configuration is computed. The spatial discretization is carried out by means of Q1P0-elements, see (Simo and Miehe, 1992). The topic of large strain thermo-plasticity is also addressed in (Miehe, 1995b). (Miehe, 1995a) deals with entropic thermoelastic material behavior at finite strains. On the one hand an isothermal and on the other hand an isentropic deformation predictor is taken into account. In both cases a purely thermal heat conduction corrector phase is introduced after the predictor phase in order to solve the governing equations. Heimes (2005) treated the case of finite strain thermo-viscoelasticity experimentally, adapting a constitutive modeling concept and implementing it into a commercial finite element program. All investigations are related to a “linear” discretization scheme, both in space and in time. Now, the question is how to develop a program using high-order space and time-discretization.

Since structures are influenced by mechanical and thermal loads, we are interested in the numerical solution of the aforementioned problems. In finite element computations the numerical treatment can be divided into partitioned and monolithic approaches. In a partitioned approach both fields are solved by different programs. The advantage is that each program can be handled separately, even with different discretization schemes. The coupling has to be treated very carefully – both for the time- and the space-discretization as well as for the coupling itself – because no heat should be generated or lost due to the numerical algorithm, as this can lead to unstable computations. Monolithic approaches have the advantage to be completely independent of such problems and no coupling program has to be developed. One common disadvantage is that the same discretization is used for each field (temperature and displacements), and that the resulting non-linear system is larger. However, it is – even in one program – not necessary to have the same discretization schemes, meaning the same mesh and/or the same polynomial
degree. If different spatial integration points occur, the mapping of the internal variables must be discussed in more detail, see (Rothe et al., 2014).

It is very common to apply numerical discretization methods of low order, both for the space and for the time-discretization. To overcome locking phenomena in linear finite element approximations, various mixed element formulations or stabilization techniques have been developed, but they do not really solve the problem that the convergence to the exact solution is not sufficient for decreasing the element-size. The element increase using linear (or quadratic elements) is called “h-version of finite elements”, whereas “h-extension” defines the mesh refinement. Caused by these limited convergence properties, see (Düster et al., 2003; Netz et al., 2013a), we make use of hierarchical shape function based on integrated Legendre-polynomials. In this respect, we keep the mesh fixed and increase the polynomial degree. This is called the “p-extension”. Naturally, decreasing the element-size and increasing the polynomial order can be called “hp-extension”. In the context of high-order elements, two main branches have been followed. The first is based on the use of integrated Legendre-polynomials, see (Szabo and Babuska, 1991; Düster, 2001; Demkowicz, 2007; Demkowicz et al., 2008) and (Szabo et al., 2004). This approach is associated with the p-version of the finite element method which is generalized to finite strains in (Düster et al., 2003; Heisserer et al., 2008b; Yosibash et al., 2007; Heisserer et al., 2008a). In the second branch, iso-geometric element formulations are proposed, see e.g. (Cottrell et al., 2009). In CAD-systems, which provide the polynomial description of the shape function, however, it is common to apply polynomial degrees of third order. Even here, an hp-extension can introduced, which, however, violates the original idea of an isogeometric formulation.

Within the last three decades the p-extension has been shown to be a very competitive approach to classical h-extension finite elements. Low order elements have the disadvantage of their convergence behavior, and of showing volumetric locking phenomena for nearly incompressible materials. In the case of the p-extension, it is shown that above a certain polynomial degree, locking phenomena decrease, see (Heisserer et al., 2008b). In (Szabo and Babuska, 1991) it is demonstrated numerically that the error measured by means of the energy norm approaches zero at a certain rate, independent of the Poisson ratio. This was considered for problems of small strain elasticity. Theoretical considerations are discussed in (Vogelius, 1983). For further readings concerning the p-extension in the context of small strain elasticity see e.g. (Suri et al., 1995) and (Babuska and Suri, 1992). Problems of large strain (isotropic) hyperelasticity in conjunction with the p-extension are addressed in (Düster et al., 2003; Noel and Szabo, 1997) or (Netz et al., 2013a). The superior characteristics of the p-extension in the context of the locking phenomenon also hold true in the large strain regime, see (Yosibash et al., 2007) and (Heisserer et al., 2008b). For an extension to problems of anisotropic hyper-elasticity, see (Yosibash and Priel, 2012).

The topic of small strain plasticity is treated in (Nübel, 2005), whereas problems of large strain powder plasticity are discussed in (Heisserer, 2007; Heisserer et al., 2008a). For the application of the p-extension for the spatial discretization to problems of finite strain viscoelasticity see (Netz et al., 2013a), where diagonally-implicit Runge-Kutta schemes (DIRK-schemes) as well as Rosenbrock-type schemes are used, both employing an embedded time step-size control for the temporal discretization. An extension of the p-version FEM to coupled problems of fi-
nite strain thermo-elasticity, or electro-thermoelasticity is given in (Erbts and Düster, 2012) and
(Erbts et al., 2014), where order one methods in the time domain are applied. For stationary,
one-dimensional considerations, see (Yosibash et al., 2014) as well.

Here, we follow a decisive concept of a discretization in space and time. According to (Wittekindt, 1991) and (Fritzen, 1997) the classical finite element approach can be related to the
method of vertical lines. First of all, the spatial discretization is carried out by means of finite
elements yielding a system of differential algebraic equations (DAE-system). The algebraic part
of the DAE-systems consists of the spatially discretized weak form of the equilibrium condition,
which constitutes a system of non-linear equations in the coefficients contained in the ansatz for
the displacement and temperature field, here in the sense of p-version finite elements. In the
isothermal case, the differential part contains the evolution equations describing the material
behavior using internal variables. Since the internal variables, which describe the non-linear
hardening behavior of the material, are independent from Gauss-point to Gauss-point, they can
be compiled in a global vector containing all internal variables of the structure, see (Ellsiepen
and Hartmann, 2001) for small strain applications and (Hartmann, 2002) for problems of large
strain viscoelasticity. In the case of transient thermo-mechanical coupling, a further set of ordi-
mary differential equations occur, i.e. the heat equation extends the DAE-system by an additional
differential part.

For isothermal applications the application of the Backward-Euler time integration scheme
to the DAE-system in conjunction with the Multilevel-Newton algorithm (MLNA) is equivalent
to the classical finite element approach employing a Backward-Euler time integration scheme on
Gauss-point level, see (Elsiepen and Hartmann, 2001). It is demonstrated that high-order time
integration schemes with an embedded step-size control can be applied to the DAE-system,
which results in very accurate and efficient computations. Quint (2012) addresses thermo-
mechanically coupled problems, pointing out that the structure of the MLNA does not change
when thermo-mechanics are taken into consideration.

Further publications treating the DIRK/MLNA approach are (Hartmann, 2006; Hartmann
et al., 2008; Hartmann and Bier, 2008). In (Hartmann, 2006) small strain problems are taken
into consideration, while (Hartmann et al., 2008) addresses problems of large strain von Mises-
type viscoplasticity, – and compressible plasticity (metal powder compaction) is discussed in
(Hartmann and Bier, 2008). In the field of damage mechanics at finite strain plasticity, this
concept is applied by Borgqvist and Wallin (2013) as well.

In contrast to this approach, which employs the classical structure known from implicit fi-
nite elements, Rosenbrock-type methods, see (Hairer and Wanner, 1996), constitute another
approach in order to solve the DAE-systems. They allow a high-order time integration proce-
dure although they are totally iteration-free. For their successful application in the small strain
regime see (Hartmann and Wensch, 2007). In (Hartmann and Hamkar, 2010) and (Hamkar
et al., 2012) Rosenbrock-type time integration schemes are applied to problems of large strain
viscoelasticity. There, h-version finite elements are employed for the spatial discretization. In
(Hamkar et al., 2012) a new Rosenbrock-type method is proposed, specially designed for the
DAE-approach. Its application to problems of thermo-hyperelasticity results in particularly ef-
ficient computations, see (Hamkar and Hartmann, 2012).

Apart from DIRK-type time integration schemes, Netz et al. (2013a) also propose Rosenbrock-
type time integration schemes for the temporal discretization in combination with the p-version of the finite element for the spatial discretization. Here, this concept is extended by thermo-mechanical coupling in a monolithic sense.

In (Rothe et al., 2012) half-explicit Runge-Kutta methods (HERK-methods) employed for the temporal discretization are discussed in the context of small and finite strain viscoelasticity. The idea of HERK-methods is to apply an explicit time integration scheme to the differential part of the DAE-system and to make use of a Newton-Raphson-method to solve the algebraic part of the spatially discretized weak form of the equilibrium condition. However, it turned out that the method is not always a good choice.

Further approaches, where high-order finite elements in space are combined with high-order time integration methods, can be found, for example, in (Schötzau and Schwab, 2001), where a discontinuous Galerkin method is applied to parabolic systems. Various time-integrators are discussed in (Carstens and Kuhl, 2012) as well, where discontinuous Galerkin methods in combination with Lagrange functions are applied. One disadvantage is related to the increase of the size of the linear equation systems if higher order time-integration is applied. Karniadakis and Sherwin (1999) investigate various time integrators in high-order spectral method in fluid mechanics, whereas in (Joulaian et al., 2014) a second-order time scheme is applied to a hyperbolic system. In all these schemes, however, no internal variable computations are treated, because the point-wise definition of the material properties have an essential influence on the applicability of the methods.

The paper is structured as follows: first, the constitutive model of thermo-viscoelasticity is recapped, which has a direct influence on the heat equation due to the heat production. Second, the spatial and temporal discretization is recalled, and, finally, several examples are provided to investigate the entire scheme.

In the following sections, bold letters, such as $\mathbf{A}$, define second-order tensors. Geometrical vectors are denoted by a superscript arrow, $\vec{a}$, and fourth-order tensors are symbolized by calligraphic letters $\mathcal{A}$. In addition, we introduce matrices at global level symbolized by bold-type italic letters $\mathbf{A}$ and matrices on the local level using bold-type Roman letters $\mathbf{A}$.

2. Constitutive modeling in thermo-viscoelasticity

In the following a model of finite strain thermo-viscoelasticity is summarized, see for details (Heimes, 2005; Netz, 2013). In this model the deformation gradient $F(\vec{X}, t) = \text{Grad} \chi_R(\vec{X}, t)$ is decomposed multiplicatively into a thermal and mechanical part,

$$F = F_\Theta F_M.$$  

($\vec{x} = \chi_R(\vec{X}, t)$ is the motion of the material point $\vec{X}$ occupying the place $\vec{x}$ at time $t$. The decomposition (1) is the opposite of the decomposition of Lu and Pister (1975); for a discussion see (Hartmann, 2012) and for their application (Holzapfel and Simo, 1996a; Miehe, 1995a; Lion, 2000a; Heimes, 2005). As a first approach the thermal part of the deformation gradient is assumed to be purely volumetric

$$F_\Theta = \varphi^{1/3}\mathbf{I} \quad \text{with} \quad \varphi := \dot{\varphi}(\Theta - \Theta_0) = 1 + \alpha_\Theta(\Theta - \Theta_0),$$  

(2)
where the heat expansion is assumed to be linear with respect to the temperature, $\vartheta = \Theta - \Theta_0$. $\Theta$ defines the absolute and $\Theta_0$ a reference temperature. The mechanical part of the deformation gradient is split into a volume preserving, $\hat{\mathbf{F}}_M$, and a volume changing part, $\mathbf{F}_M$,

$$\mathbf{F}_M = \hat{\mathbf{F}}_M \mathbf{F}_M$$  \hspace{1cm} (3)

with

$$\mathbf{F}_M = J^{-1/3}_M \mathbf{F}_M, \quad \det \mathbf{F}_M = 1, \quad \hat{\mathbf{F}}_M = J^{1/3}_M \mathbf{I} \quad \text{and} \quad J_M := \det \hat{\mathbf{F}}_M = \det \mathbf{F}_M, \hspace{1cm} (4)$$

which goes back to Flory (1961). Furthermore, the unimodular mechanical part of the deformation gradient is decomposed into an elastic and a viscous part

$$\mathbf{F}_M = \mathbf{F}_e \mathbf{F}_v$$  \hspace{1cm} (5)

(in the elastic and viscous parts $\mathbf{F}_e = (\det \mathbf{F}_e)^{-1/3} \mathbf{F}_e$ and $\mathbf{F}_v = (\det \mathbf{F}_v)^{-1/3} \mathbf{F}_v$, respectively, the index M is omitted for brevity) so that Eq.(1) reads

$$\mathbf{F} = \mathbf{F}_\Theta \hat{\mathbf{F}}_M \mathbf{F}_e \mathbf{F}_v = \mathbf{F}_\Theta \hat{\mathbf{F}}_e \mathbf{F}_v$$  \hspace{1cm} (6)

with the abbreviation

$$\hat{\mathbf{F}}_e = \hat{\mathbf{F}}_M \mathbf{F}_e.$$  \hspace{1cm} (7)

The application of the multiplicative decomposition motivates different intermediate configurations, and, related to each configuration, different strain measures (as well as stress measures later on). For a very short description, we introduce the viscous strain tensor

$$\mathbf{E}_v = \frac{1}{2}(\mathbf{C}_v - \mathbf{I})$$  \hspace{1cm} (8)

of Green-Lagrange type, which is related to the reference configuration. $\mathbf{C}_v = \mathbf{F}_v^T \mathbf{F}_v$, $\det \mathbf{C}_v = 1$, is the unimodular, viscous, right Cauchy-Green tensor. Relative to the elastic intermediate configuration, we have

$$\hat{\mathbf{C}}_e = \frac{1}{2}(\hat{\mathbf{F}}_e^T \hat{\mathbf{F}}_e - \mathbf{I}) = \frac{1}{2}(\hat{\mathbf{C}}_e - \mathbf{I}),$$  \hspace{1cm} (9)

with $\hat{\mathbf{C}}_e := \hat{\mathbf{F}}_e^T \hat{\mathbf{F}}_e$, which yields by a pull-back operation to

$$\mathbf{E}_e = \mathbf{F}_v^T \hat{\mathbf{G}}_e \mathbf{F}_v = \frac{1}{2}(\mathbf{F}_M^T \mathbf{F}_M - \mathbf{F}_v^T \mathbf{F}_v).$$  \hspace{1cm} (10)

The mechanical right Cauchy-Green tensor $\mathbf{C}_M := \mathbf{F}_M^T \mathbf{F}_M$ is introduced for later purposes as well. The third strain measure stems from the temperature-induced deformation

$$\mathbf{\Gamma}_\Theta = \frac{1}{2}(\mathbf{F}_\Theta^T \mathbf{F}_\Theta - \mathbf{I}) = \frac{1}{2}(\varphi^{2/3} - 1) \mathbf{I}$$  \hspace{1cm} (11)

yielding

$$\mathbf{E}_\Theta = \mathbf{F}_v^T \hat{\mathbf{G}}_e \mathbf{F}_e \mathbf{F}_\Theta = \frac{1}{2}(\mathbf{F}_M^T \mathbf{F}_M - \mathbf{F}_v^T \mathbf{F}_v) = \mathbf{E} - \mathbf{E}_M$$  \hspace{1cm} (12)
with
\[ \mathbf{E}_M := \mathbf{E}_e + \mathbf{E}_v = \lim_{\Theta \to \Theta_0} \mathbf{E} = \frac{1}{2}(\mathbf{F}_M^T \mathbf{F}_M - \mathbf{I}). \] (13)
In total, the Green strain decomposes additively into three parts,
\[ \mathbf{E} = \mathbf{E}_e + \mathbf{E}_v + \mathbf{E}_\Theta, \] (14)
namely an elastic, a viscous and a thermal part.

According to Lion (2000a) and Heimes (2005), the free-energy
\[ \psi(\mathbf{C}_M, \tilde{\mathbf{C}}_e, \Theta) = \frac{\Theta}{\Theta_0} \tilde{\psi}_M^e(J_M, \mathbf{C}_M) + \tilde{\psi}_M^v(\tilde{\mathbf{C}}_e, \Theta) + \psi_\Theta(\Theta) \] (15)
is chosen, while here the following parts are proposed: for the equilibrium part \( \tilde{\psi}_M^e(J_M, \mathbf{C}_M) \) the decomposition into a volume-changing and volume-preserving part is applied
\[ \tilde{\psi}_M^e = U(J_M) + \tilde{v}(\mathbf{C}_M) \] (16)
with
\[ U(J_M) = \frac{K}{50\varrho_R} (J_M^5 + J_M^{-5} - 2) \] (17)
and
\[ \tilde{v}(\mathbf{C}_M) = \tilde{v}(\mathbf{C}) = w(I_{\mathbf{C}}, II_{\mathbf{C}}) = \frac{1}{\varrho_R} \left( c_{10}(I_{\mathbf{C}} - 3) + c_{01} \left( I_{\mathbf{C}}^{3/2} - 3\sqrt{3} \right) + \alpha (I_{\mathbf{C}}^3 - 27) \right). \] (18)

The identity \( \mathbf{C}_M = \mathbf{C} \) is obvious
\[ \mathbf{C}_M = J_M^{-2/3} \mathbf{C} = \left( \frac{J}{\varphi} \right)^{-2/3} \varphi^{-2/3} \mathbf{C} = J^{-2/3} \mathbf{C} = \mathbf{C} \] (19)
and the invariants are defined by
\[ I_{\mathbf{C}} = \text{tr} \mathbf{C} \quad \text{and} \quad II_{\mathbf{C}} = \frac{1}{2}(\text{tr} \mathbf{C}_e^2 - \text{tr} \mathbf{C}_e^2) = \text{tr} \mathbf{C}_e^{-1}, \] (20)
where \( \text{tr} \mathbf{A} = a_k^k \) defines the trace of a second-order tensor. The choice of the free-energy (17) is related to the fact that – in uniaxial tensile problems – unphysical results occur for the classical approach \( U(J_M) = K(J_M - 1) \), see (Ehlers and Eipper, 1998) and (Hartmann and Neff, 2003). The ansatz (18) is proposed in (Hartmann and Neff, 2003) showing an S-shaped curve in a stress-stretch diagram. Additionally, the thermal part of the free-energy is assumed to be
\[ \psi_\Theta = \varrho_R c_{d0} \left( \left( \Theta - \Theta_0 - \Theta \ln \frac{\Theta}{\Theta_0} \right) (1 - c_{dk}(\Theta_0) - \frac{1}{2} c_{dk}(\Theta - \Theta_0)^2) \right), \] (21)
see (Heimes, 2005) for a detailed discussion as well. For the overstress part, a Neo-Hookean type strain-energy is chosen
\[ \tilde{\psi}_M^v(\tilde{\mathbf{C}}_e, \Theta) = \frac{\mu}{\varrho_R} (I_{\mathbf{C}_e} - 3) \] (22)
with $I_{\overline{C}_e} = \text{tr} \overline{C}_e$ and $\overline{C}_e = (\det \overline{\mathbf{C}}_e)^{-1/3} \overline{\mathbf{C}}_e$.

The free-energy (15) has to be inserted into the Clausius-Duhem inequality
\[ \delta = -\dot{\psi} - s\dot{\Theta} + \frac{1}{\Theta_{\dot{g}_R}} \mathbf{T} \cdot \dot{\mathbf{E}} - \frac{1}{\Theta_{\dot{g}_R}} \text{Grad} \Theta \cdot \mathbf{g}_R \geq 0, \] (23)
where $\mathbf{T}$ represents the 2nd Piola-Kirchhoff tensor, $\mathbf{T} = (\det \mathbf{F})^{-1} \mathbf{F}^T$ ($\mathbf{T}$ denotes the Cauchy stress tensor). If we introduce the decomposition
\[ \mathbf{T} = \mathbf{T}_{eq} + \mathbf{T}_{ov}, \] (24)
with the equilibrium stresses $\mathbf{T}_{eq} = \mathbf{T}_{eq}^{vol} + \mathbf{T}_{eq}^{iso}$, we obtain by inequality (23)
\[ \mathbf{T}_{eq}^{vol} = \frac{\Theta}{\Theta_0} \frac{J}{\varphi} U'(J/\varphi) \overline{\mathbf{C}}^{-1}, \quad U'(J/\varphi) = \frac{K}{10 \Theta_{\dot{g}_R}} \left( \left( \frac{J}{\varphi} \right)^4 - \left( \frac{J}{\varphi} \right)^6 \right) \] (25)
\[ \mathbf{T}_{eq}^{iso} = 2 \left( \frac{\Theta}{\Theta_0} J^{-2/3} \left( (w_1 + w_2 II_{\overline{\mathbf{C}}}) \mathbf{I} - w_2 \overline{\mathbf{C}} - \frac{1}{3} (w_1 \overline{\mathbf{C}} + 2 w_2 II_{\overline{\mathbf{C}}}) \overline{\mathbf{C}}^{-1} \right) \right) \] (26)
with
\[ w_1 = \frac{1}{\Theta_{\dot{g}_R}} \left( c_{10} + 3 \alpha \mathbf{I} \right), \quad w_2 = \frac{3}{2 \Theta_{\dot{g}_R}} c_{01} \mathbf{I}^{1/2} \] (27)
and the overstresses
\[ \mathbf{T}_{ov} = 2 \mu J^{-2/3} \left( \mathbf{C}^{-1} - \frac{1}{3} (\mathbf{C} \cdot \mathbf{C}^{-1}) \mathbf{C}^{-1} \right). \] (28)

The overstresses depend on the evolution of the viscous right Cauchy-Green tensor $\mathbf{C}_v = \mathbf{F}_v^T \mathbf{F}_v$,
\[ \dot{\mathbf{C}}_v = 4 \mu \eta \left( \varphi, J \right)^{-2/3} \left( \mathbf{C} - \frac{1}{3} (\mathbf{C} \cdot \mathbf{C}^{-1}) \mathbf{C} \right) \] (29)
see for a detailed discussion (Heimes, 2005; Netz, 2013; Hamkar, 2013). We choose a temperature and process-dependent viscosity according to Lion (2000b), which reads
\[ \eta(\mathbf{C}, \mathbf{C}_v, \Theta) = \eta_0(\Theta) \exp \left( -\frac{\|\varphi^{-2/3} \mathbf{C} \mathbf{T}_{ov}\|}{m_0(\Theta) \left( \frac{1}{\sqrt{3}} \|\mathbf{C}_v^{-1}\| \right)^{\nu_0}} \right) \] (30)
with the abbreviations
\[ \eta_0(\Theta) = \bar{\eta}_0 \exp \left( \frac{\beta}{\Theta} - \frac{\beta}{\Theta_0} \right), \] (31)
\[ m_0(\Theta) = m_{\infty} + (\bar{m}_0 - m_{\infty}) \exp(-\omega_m (\Theta - \Theta_0)). \] (32)
Moreover, Fourier’s heat conductivity is assumed, $\mathbf{g}_R = -\kappa \text{Grad} \Theta$, which reads, relative to the reference configuration,
\[ \mathbf{g}_R = -\kappa R \text{Grad} \Theta \quad \text{with} \quad \kappa_R = \kappa J \mathbf{C}^{-1}. \] (33)
defines the heat conductivity. Finally, the specific entropy $s$ results in

$$s = -\frac{\partial \psi}{\partial \Theta} + \frac{1}{\rho R} \frac{\varphi'(\Theta)}{3\varphi^{1/3}} \text{tr} \left( C_M \tilde{T} \right).$$  

(34)

$\rho_R$ is the density in the reference configuration.

The constitutive equations have to be inserted into the balance of linear momentum

$$\text{Div}(\mathbf{F} \tilde{T}) + \rho_R \vec{k} = \vec{0}$$  

(35)

and the balance of energy

$$\dot{e} = -\text{Div} \tilde{q} + \frac{1}{\rho R} \tilde{T} \cdot \dot{\mathbf{E}}$$  

(36)

(a volume-distributed heat source is not assumed), where the internal energy $e$ is expressed by the free-energy $\psi$, absolute temperature $\Theta$ and the specific entropy $s$

$$e = \psi - \Theta s.$$  

(37)

Inserting the time-derivative of the internal energy into the balance of internal energy yields the heat equation, expressed by quantities relative to the reference configuration,

$$c_\Theta \dot{\Theta} = \frac{1}{\rho R} \text{Div}(\kappa_R \text{Grad} \Theta) + d + p,$$  

(38)

with the internal dissipation produced by the inelastic stress power

$$d(C, \dot{C}, C_v, \Theta) = \frac{\mu R}{\rho R} J^{-2/3} C_v^{-1} C C_v^{-1} \cdot \dot{C}$$  

(39)

and the thermo-elastic coupling term

$$p(C, \dot{C}, \Theta) = \frac{1}{2} \left( \Theta \beta J C^{-1} + \frac{1}{\rho R} T_{\text{iso}} \right) \cdot \dot{C}$$  

(40)

with

$$\beta = \frac{\Theta}{\Theta_0} \frac{\varphi'}{\varphi^2} \left( \frac{\varphi'}{\Theta_0 \varphi^2} - 1 \right) U'(J/\varphi) - \frac{J}{\varphi} U''(J/\varphi).$$  

(41)

Commonly, the highly non-linear heat capacity

$$c_\Theta(C, \Theta) = \frac{\Theta}{\rho R} \left( -\rho_R \frac{\partial^2 \psi_0}{\partial \Theta^2} + \left( \frac{\varphi'}{\Theta} \right) a + \varphi b \right),$$  

(42)

with $\bar{\varphi} = \varphi'/(3\varphi^{1/3})$ and

$$a = 3\rho_R \varphi^{-2/3} \frac{J}{\varphi} \frac{\Theta}{\Theta_0} U'(J/\varphi)$$  

(43)

$$b = 3\rho_R J \varphi^{-5/3} \left( \frac{1}{\Theta_0} - \frac{5}{3} \frac{\varphi}{\Theta_0} \right) U'(J/\varphi) - \frac{\Theta}{\Theta_0} \frac{\varphi'}{\varphi} U''(J/\varphi),$$  

(44)

is approximated by the linear function

$$c_\Theta = c_{\Theta 0}(1 + c_{\Theta k}(\Theta - \Theta_0)),$$  

(45)

which is assumed to be valid for moderate temperature changes $\vartheta = \Theta - \Theta_0$. 

9
3. Space-time discretization

In the following, the space-time discretization using the method of vertical lines is applied, see, for example, (Großmann and Roos, 1994). In the first step, we draw on the spatial discretization of Eqns.(35) and (38) using the finite element method. Accordingly, the weak forms are derived by multiplying the PDEs with the virtual displacements $\delta \bar{u}(\bar{X})$ and virtual temperatures $\delta \Theta(\bar{X})$, respectively, integrating over the volume in the reference configuration $V_R$ and applying the divergence theorem. This leads to

$$\pi_u(\bar{u}, \Theta, q, \delta \bar{u}) = 0,$$

$$\pi_\Theta(\bar{u}, \dot{\bar{u}}, \Theta, \dot{\Theta}, q, \delta \Theta) = 0,$$ (46) (47)

with

$$\pi_u(\bar{u}, \Theta, q, \delta \bar{u}) := \int_{V_R} \bar{T} \cdot \delta \bar{E} \ dV_R - \int_{A_R} \bar{\tau}_R \cdot \delta \bar{u} \ dA_R - \int_{V_R} \rho \bar{k} \cdot \delta \bar{u} \ dV_R$$ (48)

$$\pi_\Theta(\bar{u}, \dot{\bar{u}}, \Theta, \dot{\Theta}, q, \delta \Theta) := \int_{V_R} \rho \bar{c} \Theta \dot{\Theta} - p - d) \delta \Theta \ dV_R + \int_{V_R} \kappa \bar{R} \bar{e} T \cdot \bar{R} \bar{e} T \ dV_R$$

$$- \int_{A_R} g \delta \Theta \ dA_R.$$ (49)

$\delta \bar{E} = (\bar{F}^T \bar{R} e T \delta \bar{u} + (\bar{R} e T \delta \bar{u}) \bar{F})/2$ represents the virtual displacements, $\bar{\tau}_R = \bar{T}_R \bar{n}_R$ the surface traction relative to the reference configuration, where $\bar{T}_R = \bar{F} \bar{R} e T$ is the 1st Piola-Kirchhoff tensor and $\bar{n}_R$ the normal vector on the surface in the reference configuration. $\bar{q}_R$ represents the heat flux at the surface, which can be sub-divided into radiative and convective parts. Of course, expressions (48) and (49) can be expressed with quantities relative to the current configurations and the integration might be over the current volume. However, this is omitted for brevity (the implementation is done using the spatial representation, which is fully equivalent, only its representation is slightly more difficult in view of applying time-derivatives).

The discretization draws on shape functions for the displacements, the virtual displacements, the temperature and the virtual temperature, as well as their time-derivatives,

$$\mathbf{u}_h(X, t) = \mathbf{N}_u^e(\varphi^e(X)) \mathbf{u}^e(t) = \mathbf{N}_u^e(\varphi^e(X)) \mathbf{Z}_u^e \mathbf{U}_a = \mathbf{N}_u^e(\varphi^e(X)) \left\{ \mathbf{Z}_u^e \mathbf{u} + \mathbf{Z}_u^e \mathbf{u}(t) \right\}$$ (50)

$$\dot{\mathbf{u}}_h(X, t) = \mathbf{N}_u^e(\varphi^e(X)) \dot{\mathbf{u}}^e(t) = \mathbf{N}_u^e(\varphi^e(X)) \mathbf{Z}_u^e \dot{\mathbf{U}}_a = \mathbf{N}_u^e(\varphi^e(X)) \left\{ \mathbf{Z}_u^e \dot{\mathbf{u}} + \mathbf{Z}_u^e \dot{\mathbf{u}}(t) \right\}$$ (51)

$$\delta \mathbf{u}_h(X) = \mathbf{N}_u^e(\varphi^e(X)) \delta \mathbf{u}^e = \mathbf{N}_u^e(\varphi^e(X)) \mathbf{Z}_u^e \delta \mathbf{U}_a = \mathbf{N}_u^e(\varphi^e(X)) \mathbf{Z}_u^e \delta \mathbf{u}$$ (52)

$$\Theta_h(X, t) = \mathbf{N}_\Theta^e(\varphi^e(X)) \Theta^e(t) = \mathbf{N}_\Theta^e(\varphi^e(X)) \mathbf{Z}_\Theta^e \Theta_a = \mathbf{N}_\Theta^e(\varphi^e(X)) \left\{ \mathbf{Z}_\Theta^e \Theta + \mathbf{Z}_\Theta^e \Theta(t) \right\}$$ (53)

$$\dot{\Theta}_h(X, t) = \mathbf{N}_\Theta^e(\varphi^e(X)) \dot{\Theta}^e(t) = \mathbf{N}_\Theta^e(\varphi^e(X)) \mathbf{Z}_\Theta^e \dot{\Theta}_a = \mathbf{N}_\Theta^e(\varphi^e(X)) \left\{ \mathbf{Z}_\Theta^e \dot{\Theta} + \mathbf{Z}_\Theta^e \dot{\Theta}(t) \right\}$$ (54)

$$\delta \Theta_h(X) = \mathbf{N}_\Theta^e(\varphi^e(X)) \delta \Theta^e = \mathbf{N}_\Theta^e(\varphi^e(X)) \mathbf{Z}_\Theta^e \delta \Theta_a = \mathbf{N}_\Theta^e(\varphi^e(X)) \mathbf{Z}_\Theta^e \delta \Theta$$ (55)
For the three-dimensional case we have the matrix and vector of shape functions \( \mathbf{N}_u^{\tau} \in \mathbb{R}^{3 \times n_u^{\tau}} \), and \( \mathbf{N}_{\Theta} \in \mathbb{R}^{n_{\Theta}} \), where \( n_u^{\tau} \) and \( n_{\Theta} \) are the displacement and temperature element degrees of freedom (DOF). \( \mathbf{u}^e \in \mathbb{R}^{n_u^{\tau}} \) and \( \Theta^e \in \mathbb{R}^{n_{\Theta}} \) describe the element DOF. In the case of h-elements these are displacements and nodal temperatures. In the case of hierarchical shape functions based on Legendre-polynomials they represent generalized unknowns as well, see (Szabo and Babuska, 1991; Szabó et al., 2004). A detailed explanation of shape functions in the sense of p-FEM can be found in (Dütcher, 2001; Netz, 2013). Here, we restrict ourselves to the case of the trunk-space so that the computations are more efficient. \( \xi = \varphi_e(\mathbf{X}) \) defines the (inverse) coordinate transformation of the global material coordinates \( \mathbf{X} \) to the local coordinates \( \xi \) in the reference element. \( \mathbf{u}_a \in \mathbb{R}^{n_u} \) symbolizes all displacement DOF of the entire structure containing the unknown quantities \( \mathbf{u} \in \mathbb{R}^{n_u} \) and the known, i.e. prescribed displacement DOF \( \mathbf{u}_p \in \mathbb{R}^{n_{up}} \). Obviously, \( n_{ua} = n_{uu} + n_{up} \) for \( \mathbf{u}_a^T = \{ \mathbf{u}^T \mathbf{u}_p^T \} \) holds. The same decomposition can be carried out for the temperature DOF, \( \Theta_a \in \mathbb{R}^{n_{\Theta}} \), \( \Theta_p \in \mathbb{R}^{n_{\Theta}} \), \( \Theta \in \mathbb{R}^{n_{\Theta}} \), i.e. \( \Theta_a^T = \{ \Theta^T \Theta_p^T \} \) and \( n_{\Theta_a} = n_{\Theta_u} + n_{\Theta_p} \). For \( n_{ij} \), the first index denotes the mechanical and the second index the thermal quantities, \( i = u, \Theta \). The second subscript, \( j = a, u, p \), stands for all, unknown and prescribed. \( \mathbf{u}(t) \) and \( \Theta(t) \) are given functions of the time \( t \) representing displacement-control and prescribed temperature boundary conditions. The incidence matrices \( \mathbf{Z}_u^e \in \mathbb{R}^{n_u^{\tau} \times n_u} \), \( \mathbf{Z}_{\Theta}^e \in \mathbb{R}^{n_{\Theta} \times n_{\Theta}} \), and \( \mathbf{Z}_{\Theta}^e \in \mathbb{R}^{n_{\Theta} \times n_{\Theta}} \) are related to the assemblage procedure, which is frequently symbolized by \( \bigcup_{\tau=1}^{n_{el}} \) or \( \mathbf{A}^{\tau \Theta}_{e=1} \) in (Wriggers, 2001; Hughes, 1987). Accordingly, the incidence matrices \( \mathbf{Z} \) are not programmed explicitly, but symbolize the assemblage procedure.

If we insert the shape functions into the principle of virtual displacements and evaluate the property \( \delta \mathbf{u} \) to be arbitrary, we arrive at the discretized weak formulation (often called equilibrium conditions)

\[
g_u(t, \mathbf{u}, \Theta, \mathbf{q}) = \sum_{e=1}^{n_{el}} \mathbf{Z}_u^e \left( \sum_{j=1}^{n_{GP}} \mathbf{B}_u(\xi^{(j)}, \mathbf{u}^e(\xi^{(j)}, t)) \mathbf{h}(\mathbf{E}^{(j)}(\xi^{(j)}, t), \Theta^h(\xi^{(j)}, t), \mathbf{q}^{(j)}(\xi^{(j)})) \det \mathbf{J}_R^e(\xi^{(j)}) \right) - \mathbf{p}(t) = \mathbf{0}
\]

representing the difference of the internal and external forces, which are equivalent if equilibrium is present (in the quasi-static sense). \( n_{el} \) is the number of elements, \( n_{GP}^e \) the number of Gauss-points in element \( e \), and \( \mathbf{J}_R^e \) is the Jacobian of the coordinate transformation to the reference element (in the reference configuration). \( \mathbf{B}_u \) defines the deformation-dependent strain-displacement matrix resulting from \( \delta \mathbf{E}^h = \mathbf{B}_u \delta \mathbf{u}^e \). \( \mathbf{p}(t) \in \mathbb{R}^{n_u} \) symbolizes the equivalent force vector containing only known quantities. Now, the 2nd Piola-Kirchhoff tensor in Eq.(24) at the Gauss-point \( \xi^{(j)} \) depends on the Green strain tensor in Voigt notation \( \mathbf{E} \) (or, equivalently, on the the right Cauchy-Green tensor in vector form \( \mathbf{C} \))

\[
\mathbf{T}^{(j)} = \mathbf{h}(\mathbf{C}^{(j)}, \Theta^{(j)}, \mathbf{q}^{(j)}) \in \mathbb{R}^6,
\]

where the internal variables \( \mathbf{q}^{(j)} \) are given by the constitutive equations in Eq.(29), \( \mathbf{q}^{(j)} = \mathbf{C}^{(j)} \in \mathbb{R}^6 \). In this paper, we draw on hierarchical shape functions (integrated Legendre-polynomials),
which are contained in the matrices of shape functions $N_e$ and the strain-displacement matrix $B_u$.

The internal variables, which are evaluated at the Gauss-points, can formally be assembled to a large system of ordinary differential equations as discussed in (Ellsiepen and Hartmann, 2001),

$$\dot{q}(t) = r_Q(t, u, \Theta, q).$$ \hspace{1cm} (58)

This is only a formal step, which can be exploited in a particular solution scheme of systems of non-linear equations after the time discretization. The assembling step reads

$$q = \sum_{e=1}^{n_{el}} \sum_{j=1}^{n_{GP}} Z_e^{(j)} q_e^{(j)}. $$ \hspace{1cm} (59)

This can also be applied to the evolution equations in the problem under consideration, see Eq.(29), leading to Eq.(58).

The discretized heat equation reads

$$C_\Theta(t, \Theta) \dot{\Theta}(t) = -C_\kappa(t, u, \Theta) \Theta + R_\Theta(t, u, \dot{u}, \Theta, q) + R_{\Theta}^e(t, u, \Theta)$$ \hspace{1cm} (60)

with the heat capacity matrix

$$C_\Theta(t, \Theta) = \sum_{e=1}^{n_{el}} Z^e T \left[ \sum_{j=1}^{n_{GP}} \rho_R C_\Theta(\Theta_e^{(j)}) N_e^{(j)} N_e^{(j)T} (\det J^{(j)}_R) \right] Z_e^e,$$ \hspace{1cm} (61)

the heat conductivity matrix

$$C_\kappa(t, u, \Theta) = \sum_{e=1}^{n_{el}} Z^e T \left[ \sum_{j=1}^{n_{GP}} B_e^{(j)T} \kappa_\Theta(t, u) B_\Theta^{(j)} (\det J^{(j)}_R) \right] Z_e^e,$$ \hspace{1cm} (62)

the dissipation vector

$$R_\Theta(t, u, \dot{u}, \Theta, q) = \sum_{e=1}^{n_{el}} \left\{ \sum_{j=1}^{n_{GP}} \rho_R (p(t, u, \dot{u}, \Theta) + d(t, u, \dot{u}, q)) N_e^{(j)} (\det J^{(j)}_R) \right\} ,$$ \hspace{1cm} (63)

and the surface contribution $R_{\Theta}^e(t, u, \Theta)$ resulting from convection, radiation as well as the Dirichlet contribution $\overline{\Theta}(t)$. In conclusion, we have to solve the coupled system of differential-algebraic equations, see Eqns.(56), (58) and (60),

$$0 = g_u(t, u, \Theta, q)$$
$$C_\Theta(t, \Theta) \dot{\Theta}(t) = r_\Theta(t, u, \dot{u}, \Theta, q)$$
$$\dot{q}(t) = r_Q(t, u, \Theta, q)$$ \hspace{1cm} (64)
with the initial conditions $\Theta(0) = \Theta_0$, $u(0) = u_0$ and $q(0) = q_0$. In the theory of DAE-systems the initial conditions must be consistent, i.e. the algebraic part must be satisfied at $t = 0$, $g_u(0, u_0, \Theta_0, q_0) = \mathbf{0}$. In some situations – e.g. a jump in the temperature – this requires one additional computation of a system of non-linear equations.

The second step of the method of vertical lines requires the time-discretization, here, by applying a solution scheme to the DAE-system (64). The application of stiffly accurate diagonally-implicit Runge-Kutta methods (SDIRK) offers the possibility to extend classical Backward-Euler implementations to high-order time integration, for details see (Ellsiepen and Hartmann, 2001; Hartmann, 2002) and the literature cited therein. We assume that the displacements, temperatures and internal variables are known at time $t_{n}$, meaning that $y_{T_{n}} = \{u_{T_{n}}, \Theta_{T_{n}}, q_{T_{n}}\}$ is given. In the case of high-order SDIRK-methods, we obtain the new quantities at time $t_{n+1}$, i.e. $y_{T_{n+1}} = \{u_{T_{n+1}}, \Theta_{T_{n+1}}, q_{T_{n+1}}\}$ by

$$
y_{n+1} = y_n + \Delta t_n \sum_{i=1}^{s} b_i \dot{Y}_{ni}, \quad (65)
$$

$\Delta t_n = t_{n+1} - t_n$ is the step-size, $b_i$, $i = 1, \ldots, s$, defines the weighting factors of the underlying method, $s$ the number of stages and $\dot{Y}_{ni}$ the stage derivatives. The stage derivatives

$$
\dot{Y}_{ni} = \frac{Y_{ni} - S_{ni}}{\Delta t_n a_{ii}}, \quad (66)
$$

are stored after each stage computation, where $a_{ij}$ are further weighting factors ($a_{ij} = 0$ for $j > i$, $i = 1, \ldots, s$ and $j = 1, \ldots, s$),

$$
S_{ni} = y_n + \Delta t_n \sum_{j=1}^{i-1} a_{ij} \dot{Y}_{nj}, \quad (67)
$$

symbolizes the starting vector, which is a known quantity at each stage computation, and $Y_{ni}$ represent the stage quantities resulting from the computation of the coupled system of non-linear equations

$$
\mathbf{0} = g_u(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}),
$$

$$
\mathbf{C}_{\Theta}(T_{ni}, \Theta_{ni}) \frac{\Theta_{ni} - \Theta_{ni}^S}{\Delta t_n a_{ii}} = r_{\Theta}(T_{ni}, u_{ni}, \frac{u_{ni} - u_{ni}^S}{\Delta t_n a_{ii}}, \Theta_{ni}, q_{ni}), \quad (68)
$$

$$
\frac{q_{ni} - q_{ni}^S}{\Delta t_n a_{ii}} = r_{Q}(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}).
$$

The starting vector is decomposed into the different physical variables, $S_{ni} = \{u_{ni}^S, \Theta_{ni}^S, q_{ni}^S\}$. $T_{ni} = t_n + c_i \Delta t_n$ is the stage time and the $c_i$, $i = 1, \ldots, s$, are further parameters of the method. The parameters $b_i$, $a_{ij}$ and $c_i$ are commonly compiled in the so-called Butcher array. Eqns.(68) define a non-linear system

$$
\begin{align*}
g_u(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}) & = \mathbf{0} \\
g_{\Theta}(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}) & = \mathbf{0} \\
l_{Q}(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}) & = \mathbf{0}
\end{align*} \quad (69)
$$
with
\[
\begin{align*}
g\Theta(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}) &= C\Theta(T_{ni}, \Theta_{ni}) \left\{ \Theta_{ni} - \Theta_{ni}^S \right\} \\
&\quad - \Delta t_n a_{ii} r\Theta(T_{ni}, u_{ni}, u_{ni} - u_{ni}^S, \Theta_{ni}, q_{ni}), \\
l_Q(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}) &= q_{ni} - q_{ni}^S - \Delta t_n a_{ii} r_Q(T_{ni}, u_{ni}, \Theta_{ni}, q_{ni}).
\end{align*}
\]

In other words, we have to solve at each point in time (stage) the coupled system
\[
\begin{align*}
G_u(u, \Theta, q) &= 0 \\
G\Theta(u, \Theta, q) &= 0 \\
L_Q(u, \Theta, q) &= 0
\end{align*}
\]
where we have omitted the time \(T_{ni}\) and the index \(ni\) for brevity. In a natural manner, a Newton-Raphson method would be the first choice. However, the Multilevel-Newton algorithm (MLNA) has turned out to be the method frequently applied in finite elements, see for general aspects (Rabbat et al., 1979; Hoyer and Schmidt, 1984) or in the context of finite elements (Hartmann, 2005; Kulkarni et al., 2007). According to (Quint, 2012) we decompose the displacements and temperature degrees of freedom into one part, \(v = \{u, \Theta\}\) and the internal variables into a remaining part. This leads us to the system
\[
\begin{align*}
G(v, q) &= \left\{ \begin{array}{c} G_u(u, \Theta, q) \\ G\Theta(u, \Theta, q) \end{array} \right\} = 0, \\
L(v, q) &= L_Q(u, \Theta, q) = 0.
\end{align*}
\]
The basic ingredient of the Multilevel-Newton algorithm is the application of the implicit function theorem. On Gauss-point level, as in classical finite elements, the internal variables are solved using Eq.(73) for given displacements and temperatures. On global level, Eq.(72) is computed as in a “classical” Newton-Raphson iteration step, see, (Quint, 2012; Netz, 2013) for details in thermo-mechanics.

Thus, the entire scheme is provided. Here, one more aspect concerning the choice of the time step-sizes \(\Delta t_n\) should be mentioned: \(\Delta t_n\) can be determined by an error estimation of the local integration error, which is based on the difference of the solution of a method of order \(q\) and a method of \(\hat{q}, \hat{q} < q\). The method of lower order draws on the same Butcher-array coefficients – only the final step is different – and results in the product of a scalar with a (difference) vector. This is a very fast possibility to determine the local error, see (Diebels et al., 1999; Ellsiepen and Hartmann, 2001).

4. Examples

This section discusses several examples. First, a fully three-dimensional rod is investigated with respect to the error resulting from different spatial discretization schemes. Additionally, several high-order time integration schemes are studied. In the second section, a bearing used in the automotive industry is considered.
The unit system employed for all thermo-mechanically coupled computations is given in Tab. A.6, see Appendix A. The chosen material parameters of the rubber material are $K = 1000\text{MPa}$, $c_{10} = 0.1788\text{MPa}$, $c_{01} = 0.1958\text{MPa}$, and $\alpha = 3.67 \times 10^{-3}\text{MPa}$ for the equilibrium stress state, see Hartmann and Neff (2003). The parameters of the overstress-part are provided in Tab. 1. The material parameters of the thermal part of the model are compiled in Tab. 2.

Table 1: Mechanical material parameters of constitutive model (overstress-part)

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\bar{\eta}$</th>
<th>$\bar{m}_0$</th>
<th>$m_\infty$</th>
<th>$\beta$</th>
<th>$\omega_m$</th>
<th>$r_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MPa</td>
<td>MPa s</td>
<td>MPa</td>
<td>MPa</td>
<td>K</td>
<td>K $^{-1}$</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>180</td>
<td>$1 \times 10^{-3}$</td>
<td>$1.5 \times 10^{-3}$</td>
<td>$8 \times 10^3$</td>
<td>$8 \times 10^{-2}$</td>
<td>4</td>
</tr>
</tbody>
</table>

For our investigations, we have to consider the following two aspects: first, in view of the concept of verification and validation – see, for example, (Babuska and Oden, 2004; Oberkampf and Trucano, 2002; ASME, 2006) – we have to investigate the magnitude of the discretization error, which is related to spatial and temporal influences. In this respect, a method with highly accurate results is of particular interest. Second, in view of the uncertainties resulting from the boundary conditions and the material properties, we need methods that are accurate and fast, because most stochastic programs call (deterministic) programs – for a basic concept, see for example, (Todor and Schwab, 2007) and the references cited therein. Thus, the following investigations are performed.

4.1. Rod under different mechanical and thermal boundary conditions

The basic purpose of this section is to investigate the spatial and the temporal discretization applied to the thermo-mechanically coupled problem. For this purpose a simple three-dimensional rod is chosen, see Fig. 1. To begin with, we assume thermo-elasticity ($\mu = 0$, see Eq.(28)) in the first example. On the one hand, two versions of the h-version (Q1: 8-noded hexahedral element, Q2: 20-noded hexahedral element) and, on the other hand, the p-version of the finite element method are compared with respect to accuracy and computational efficiency.

Afterwards, different temporal discretization schemes of DIRK-type are investigated. For this purpose, thermo-viscoelastic material behavior is assumed.

Table 2: Thermal material parameters of constitutive model for elastomer

<table>
<thead>
<tr>
<th>$c_{d0}$</th>
<th>$c_{dk}$</th>
<th>$\kappa$</th>
<th>$\alpha_\Theta$</th>
<th>$\Theta_0$</th>
<th>$\Theta_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{mm}^2/(s^2\text{K})$</td>
<td>$K^{-1}$</td>
<td>$\text{tmm}/(s^3\text{K})$</td>
<td>$K^{-1}$</td>
<td>K</td>
<td>t/$\text{mm}^3$</td>
</tr>
<tr>
<td>$1.539 \times 10^9$</td>
<td>$3.75 \times 10^{-3}$</td>
<td>0.2595</td>
<td>$2.06 \times 10^{-4}$</td>
<td>20</td>
<td>$1.12 \times 10^{-9}$</td>
</tr>
</tbody>
</table>
4.1.1. Finite strain thermo-elasticity

In order to study the influence of different space discretization schemes the structure given in Fig. 1 is considered. Here, due to symmetry conditions, only a quarter of a rod is discretized.

![Figure 1: Geometry and dimensioning of rod discretized with 30 elements with p-extension](image)

The displacement as well as temperature degrees of freedom on the plane given by \( z = 200 \) are held constant. On its opposite plane, \( z = 0 \), the Cauchy-stress in \( z \)-direction as well as the temperature, both increasing linearly in time, are prescribed according to

\[
\bar{p}_z(t) := \sigma_{xx}(X,Y,0,t) = \sigma_0 \frac{t}{t_e} \text{MPa}, \\
\bar{\Theta}_1(t) := \Theta(X,Y,200,t) = 293.15 \text{K}, \\
\bar{\Theta}_2(t) := \Theta(X,Y,0,t) = 293.15 + 80 \frac{t}{t_e} \text{K},
\]

with \( \sigma_0 = 1 \text{MPa} \) and \( t_e = 1000 \text{s} \), see Fig. 2 for the boundary conditions (in view of displacement-dependent surface traction for p-FEM, see (Yosibash et al., 2007; Heisserer, 2008)). Except the

![Figure 2: Reference solution with 112 elements with p-extension and boundary conditions](image)

planes defined by \( z = 0 \) and \( z = 200 \text{mm} \), where a temperature is prescribed, the surface of the structure is considered to be adiabatic \((q_n = 0)\). The intention is to compare p-version as well as h-version finite elements. p-discretizations employing a polynomial degree from 1 to 9 are constructed on the basis of the mesh given in Fig. 1. The corresponding number of unknowns is
Table 3: p-level of rod with 30 elements and the total number of unknowns $n_{uu} + n_{Θu}$

<table>
<thead>
<tr>
<th>p-level</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_{uu} + n_{Θu}$</td>
<td>213</td>
<td>744</td>
<td>1275</td>
<td>2243</td>
<td>3648</td>
<td>5610</td>
<td>8249</td>
<td>11658</td>
<td>16038</td>
</tr>
</tbody>
</table>

Figure 3: h-version meshes compiled in Tab. 3. The h-version meshes are depicted in Fig. 3 and the corresponding number of unknowns are assembled in Tab. 4. Apart from the boundary conditions, Fig. 2 illustrates

Table 4: Number of unknowns $n_{uu} + n_{Θu}$ of the chosen meshes of low order elements

<table>
<thead>
<tr>
<th></th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>M4</th>
<th>M5</th>
<th>M6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>971</td>
<td>1301</td>
<td>2643</td>
<td>3313</td>
<td>3983</td>
<td>10109</td>
</tr>
<tr>
<td>Q2</td>
<td>3611</td>
<td>4831</td>
<td>10060</td>
<td>12600</td>
<td>15104</td>
<td>38891</td>
</tr>
</tbody>
</table>

the reference mesh too, which is used to compare the results of the h- and p-version discretizations. The reference mesh contains 112 hexahedrals with a uniform polynomial degree of 10, yielding 74865 unknowns. For the temporal discretization, the second order accurate method of Ellsiepen (1999), with an embedded step-size control is applied for each discretization. For the displacements, we use the absolute error $ε^u_a = 10^{-4}$ and the relative error $ε^r_u = 10^{-6}$, see (Ellsiepen and Hartmann, 2001; Hartmann, 2002) for further details. The prescribed error tolerances for the temperatures and the internal variables, respectively, are chosen in such a way that the local time integration error is controlled by the step-size selection: $ε^Θ_a = 10^{-3}$, $ε^r_Θ = 10^{-5}$, $ε^q_a = 10^{-5}$ and $ε^r_q = 10^{-5}$. In order to assess the quality of the discretization schemes, we draw
on the error measures

\[
\varepsilon(u) = \frac{1}{n_{\text{evp}}} \sqrt{n_{\text{evp}} \sum_{k=1}^{n_{\text{evp}}} (\mathbf{u}_k - \mathbf{u}_{\text{ex}})^T (\mathbf{u}_k - \mathbf{u}_{\text{ex}})},
\]

(77)

\[
\varepsilon(T) = \frac{1}{n_{\text{evp}}} \sum_{k=1}^{n_{\text{evp}}} \max |T_k - T_{\text{ex}}|,
\]

(78)

\[
\varepsilon(\Theta) = \frac{1}{n_{\text{evp}}} \sum_{k=1}^{n_{\text{evp}}} \max |\Theta_k - \Theta_{\text{ex}}|.
\]

(79)

In other words, we compare the computed values of the displacements, stresses and temperatures with reference solution at the evaluation points of a background grid, which is shown in Fig. 4. \(n_{\text{evp}}\) is the number of points of the evaluation (background) grid. Since this investigation

![Figure 4: Evaluation grid with 33431 points](image)

is based on finite strain thermo-elasticity implicating \(\tilde{T}_{\text{ov}} = 0\), the stress state at a given node of the evaluation grid is only dependent on the temperature and the displacement at the considered point. Having evaluated these quantities, the corresponding stress state can be derived without any interpolation of the internal variables.

First of all, the results are discussed. The resulting displacement field in \(z\)-direction is displayed in Fig. 5(a), whereas the stress state and the temperature distribution can be found in Figs. 5(c) and 5(e).

In the following, we take a look at the convergence of the different discretizations. The errors in the displacement, stress and temperature field with respect to the number of unknowns and the CPU-time can be found in Fig. 6. The errors in the temperature and stress field can be found in Figs. 6(a) and 6(b). No matter which error measure is taken into consideration, the linear elements exhibit the poorest behavior. This holds both for accuracy and efficiency considerations, meaning that the error for a given number of unknowns, respectively a given CPU-time, is by far larger if linear instead of quadratic elements are used. However, the best choice of the considered discretizations is the p-version above a p-level of 5, which exhibits both a steeper slope, especially for the error in the displacement and the stress field, as well as a smaller error – meaning that it yields the smallest absolute value of all considered error measures for a given number of unknowns or for a given CPU-time. In this example, a superior accuracy and efficiency of the p-version can thus be observed, which is mainly attributed to the property of the smooth problem under consideration.
Figure 5: Numerical results of the rod under thermal and mechanical agencies ($p = 6$)
(a) Error in displacement field

(b) Error in displacement field over CPU-time

(c) Error in temperature field

(d) Error in temperature field over CPU-time

(e) Error in stress field

(f) Error in stress field over CPU-time
4.1.2. Finite strain thermo-viscoelasticity

In this example the application of different temporal discretization schemes of DIRK-type to the thermo-viscoelastic problem is investigated. The material parameters are given in Tabs. 1 and 2 and the boundary conditions are illustrated in Fig. 7. The prescribed Dirichlet boundary conditions of the displacements at \( z = 0 \) and the temperature at \( z = 0 \) are linearly increased within \( 0 \leq t \leq 40s \),

\[
\overline{u}_z = u(X, Y, 0, t) = -0.5t \text{ mm}, \quad \overline{\Theta} = \Theta(X, Y, 200, t) = 293.15 + t \text{ K}.
\]

All dissipative parts are included, so that a small additional temperature evolution takes place. In view of the discussion of the spatial discretization schemes performed in the previous section, a polynomial degree of \( p = 6 \) is considered to be sufficiently accurate. In the following, the behavior of the time discretization schemes is studied, considering first, second and third order methods. The first order scheme is given by the classical Backward-Euler method \((s = 1)\), Ellsiepen’s method is of second order \((s = 2)\), see (Elsiepen, 1999), and the third order method goes back to Alexander (1977) \((s = 3)\). In order to assess the quality of the temporal discretization schemes, the error measures

\[
\text{abserr}_u := \max_n \left( \| \overline{u}_n^{\text{ref}} - u_n \| \right), \quad \text{abserr}_q := \max_n \left( \| \overline{q}_{n}^{\text{ref}} - q_n \| \right), \quad \text{abserr}_\Theta := \max_n \left( \| \overline{\Theta}_n^{\text{ref}} - \Theta_n \| \right)
\]

are considered. The highly accurate solution (reference solution \( \overline{u}_n^{\text{ref}}, \overline{q}_{n}^{\text{ref}} \) and \( \overline{\Theta}_n^{\text{ref}} \)) is provided by means of the 3rd order method of Alexander (1977), with a constant step-size of \( \Delta t_n = 0.25 \times 10^{-2} \text{s} \). Here, we consider both the accuracy as well as the efficiency of the time integration schemes. For accuracy considerations the errors in displacements, temperatures and internal variables are plotted over the step-size, see Figs. 8(a), 8(c) and 8(e). The first observation is that all methods reach their expected order.
(a) Error in displacements

(b) Error in displacements

(c) Error in temperatures

(d) Error in temperatures

(e) Error in internal variables

(f) Error in internal variables
With respect to a given step-size $\Delta t_n$, the second order accurate method of Ellsiepen is more than two orders more accurate than the Backward-Euler scheme. Better results are obtained using the third order accurate method of Alexander. In view of the efficiency of the time-integration procedures, the errors in displacements, temperatures and internal variables versus the CPU-time are shown in Figs. 8(b), 8(d) and 8(f). Again, the time integration schemes of higher order prove to be more efficient than the Backward-Euler scheme. The second-order scheme seems to be more efficient than the third-order method.

So far, accuracy and efficiency considerations for thermo-mechanically coupled problems have been discussed. It turns out that the advantageous characteristics of the considered approach carry over from the isothermal case presented in (Hartmann, 2002) and (Netz et al., 2013b) to the thermo-mechanically coupled case. Similarly to the spatial error behavior of the p-version in comparison to h-elements, high-order time integration methods prove to be superior to the commonly order one methods in the time domain. In the case of non-smooth problems, this statement cannot be guaranteed anymore. Accordingly, the combination of the p-version for the spatial discretization and high-order time integration schemes of DIRK-type turns out to be a very competitive approach if highly accurate solutions of thermo-mechanically coupled problems are sought.

4.2. Elastomeric bearing

This section treats a more practical example, in which different time-scales are inherent in the loading process. The focus of the example lies in the description of the numerical behavior of the time-integration methods when different weather conditions (cold, hot, ...) are applied on a bearing (wishbone suspension). The bearing is composed of three regions. Two of them are made of steel, one consists of rubber. The inner core and the outer ring are the steel regions, see Fig. 9. The geometry is summarized in Figs. 10(a) and 10(b). The elastomeric material is modeled by the constitutive model of finite strain thermo-viscoelasticity proposed in Section 2. The material parameters are those of the previous section.

The steel is modeled by a temperature-dependent elastic material, according to data proposed in (Quint et al., 2011a). The material parameters are compiled in Tab. 5 and are taken
from (Quint, 2012). There is no plastification, due to the fact that the yield stress is not reached

Table 5: Material parameters of constitutive model for steel taken from (Quint, 2012)

<table>
<thead>
<tr>
<th>$K_0$ (GPa)</th>
<th>$c_k$ (MPa/K)</th>
<th>$G_0$ (GPa)</th>
<th>$c_g$ (MPa/K)</th>
<th>$\lambda_0$ (mm/(s$^3$K))</th>
<th>$c_{d0}$ (mm$^2$/(s$^2$K))</th>
<th>$\alpha_{\Theta}$ (1/K)</th>
<th>$\varrho_R$ (t/mm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>167</td>
<td>−91</td>
<td>76.9</td>
<td>−42</td>
<td>35</td>
<td>6e+08</td>
<td>1.2e−05</td>
<td>7.815e−09</td>
</tr>
</tbody>
</table>

within the steel layers. According to (Quint, 2012), the free energy (Neo-Hookean type)

$$ \varrho_R \psi = U(J_M, \Theta) + \bar{v}(I_C, \Theta) $$

with

$$ U(J_M, \Theta) = \frac{K(\Theta)}{50\varrho_R} (J_M^5 + J_M^{-5} - 2) \quad \text{and} \quad \bar{v}(I_C, \Theta) = \frac{G(\Theta)}{2\varrho_R} (I_C - 3) $$

is chosen. The bulk and shear moduli are assumed to be temperature-dependent

$$ K(\Theta) = K_0 + c_k(\Theta - \Theta_0) \quad \text{and} \quad G(\Theta) = G_0 + c_g(\Theta - \Theta_0). $$

4.2.1. Bearing subjected to different thermal boundary conditions

In order to simulate the cooling or heating of the bearing a heat flux is applied to its surface. The heat flux in Eq.(49) is modeled as a superposition of convection and radiation, $q_n = q = \eta_{\text{con}} + \eta_{\text{rad}}$, with $\eta_{\text{con}} = h_c(\Theta(X, t) - \Theta_r(t))$ and $\eta_{\text{rad}} = e\sigma(\Theta^4(X, t) - \Theta_\infty^4(t))$. On the symmetry planes $q_n = 0$ is assumed. These thermal boundary conditions are depicted in Fig. 11(a). We investigate two different loading cases, a cooling process, where $\Theta_\infty(t) = \Theta_r(t) = 253.15$K, $0 \leq t \leq 1000$s, is assumed, and a heating process, $\Theta_\infty(t) = \Theta_r(t) = 323.15$K, $0 \leq t \leq 1000$s.

We draw on an emissivity of $e = e_r = 0.95$ for the rubber, see (Banic et al., 2012), and for
the steel $\varepsilon = \varepsilon_s = 0.2$, see (Quint, 2012). The heat transfer coefficient for the rubber and steel surface amounts to $h_c = 2.5 \times 10^{-2} t/(s^3K)$.

Following this process, the mechanical boundary conditions are applied, as illustrated in Fig. 11(b). For $t \geq 1000s$ we apply

$$
\begin{align*}
\bar{\pi}_x &= 0 \text{ mm for } 0 \leq t \leq 1000s, \\
\bar{\pi}_x &= -1.25(t - 1000) \text{ mm for } 1000 s < t \leq 1002s, \\
\bar{\pi}_x &= -2.5 \text{ mm} - 0.5 \sin(\text{f}_{\text{bearing}} (t - 1002)) \text{ mm for } 1002 < t \leq 1100s \quad (84)
\end{align*}
$$

see Fig. 12 as well. Apart from the ongoing cooling/heating process of the bearing, 30 sinusoidal cycles are carried out for a time period of 98s, which is defined by a frequency of $f_{\text{bearing}} = 0.306s^{-1}$. The spatial discretization is carried out using a p-level of $p = 6$ and the time discretization is performed using Ellsiepen’s method with an embedded step-size control, employing the tolerances $\epsilon^u_a = 10^{-3}$, $\epsilon^u_r = 10^{-5}$, $\epsilon^\Theta_a = 10^{-4}$, $\epsilon^\Theta_r = 10^{-6}$, $\epsilon^q_a = 10^{-5}$ and $\epsilon^q_r = 10^{-5}$. The temperature fields, which occur after applying the heat flux and the linearly increasing displacements on the inner radius of the bearing, $t = 1002s$, are given in Figs. 13(a) and 13(c) for the heating and the cooling process. The displacements at $t = 1002s$ are shown in Fig. 14, which only illustrates the cooling process. The resulting force at the inner ring – for applying the displacement at the inner radius – is shown in Fig. 15(a). At the beginning of the process, the reaction force of the bearing – which is subjected to a heat flux that results in a temperature increase – is approximately 50N larger than the reaction force resulting from the cooling process. A computation assuming room temperature ($\Theta_\infty(t) = \Theta_f(t) = 293.15K$) shows that the reaction force lies in between both other computations.

The step-sizes chosen by the adaptive time-step control are given in Fig. 15(b). Within the first period of the process, $0 \leq t \leq 1000s$, the step-sizes of the process at room temperature are by far the largest, because there is neither an evolution in the temperature nor in the displacement field. Furthermore, it can be observed that the heating process requires a slightly smaller step-size than the cooling process. Due to the rapidly changing variables in the period of the cyclic loading, small step-sizes are required.
Obviously, an automatic step-size selection is necessary in such processes to ensure that the local time-integration error stays below prescribed error tolerances, as well as for efficiency considerations, because constant step-sizes or even step-size estimations based on the number of global Newton-iterations in the MLNA do not lead to efficient computations.
Figure 13: Temperature distribution (shown in °C) in bearing at \( t = 1002 \)s displayed on deformed configuration (\( p = 6 \), Ellsiepen’s method, time-adaptive computation)

Figure 14: Displacement field (in mm) in bearing at \( t = 1002 \)s displayed on deformed configuration
Figure 15: Reaction force of bearing for different temperature levels and corresponding step-size behavior

(a) Reaction forces for different temperature levels

(b) Corresponding step-size behavior
4.2.2. Investigation of thermo-mechanical coupling

In order to study the coupling of the mechanical and the thermal field, the temperature increase of the bearing (resulting from a change in the mechanical field) is investigated for different values of the material parameter $\mu$ (overstress part in Eq.(28)), which mainly influences the temperature evolution due to inelastic dissipation.

The mechanical boundary conditions are given by

\[ \bar{u}_x = -1.25t \text{ mm} \quad \text{for} \quad 0 < t \leq 2 \text{s}, \quad (85) \]
\[ \bar{u}_x = -2.5 \text{ mm} - 0.5 \sin (f_{\text{bearing}} (t - 2)) \text{ mm} \quad \text{for} \quad 2 < t \leq 20 \text{s}, \quad (86) \]

see again Fig. 11(b). Accordingly, 40 sinusoidal cycles are conducted within 18s implying a frequency of $f_{\text{bearing}} = 2.2s^{-1}$. For the thermal boundary conditions we refer to Fig. 11(a) with $q_n = 0$, $q_{\text{rad}} = 0$ (heat transfer coefficient $h_c = 0.025t/(s^3\text{K})$ and $\Theta_f = 293.15\text{K}$). Again, we rely on Ellsiepen’s method, with an embedded step-size control for the time-discretization and a p-level of 6. We draw on the tolerances $\varepsilon_0^u = 10^{-3}$, $\varepsilon_0^\Theta = 10^{-4}$, $\varepsilon_0^\phi = 10^{-6}$, $\varepsilon_q^u = 10^{-5}$ and $\varepsilon_q^\Theta = 10^{-5}$.

The temperature evolution is investigated at four points in the bearing, which are shown in Fig. 16. The evaluation points of temperature field for parameter study

![Figure 16: Evaluation points of temperature field for parameter study](image-url)

its influence. The resulting temperature evolution at the four different points is shown in Fig. 17.

The first observation to be made is that the temperature increase related to a purely thermo-elastic material behavior ($\mu = 0\text{MPa}$) is, of course, the smallest (result of thermo-elastic coupling term). Furthermore, it can be stated that the main temperature increase takes place at point 2, which is not surprising, because this is where the largeste strain-rates occur. Since the steel layers show hardly any deformation, the contribution to the mechanical dissipation in the temperature field is not visible. Since point 3 belongs to the elastomer as well as to the steel layer, there are smaller strains and strain-rates – resulting in a lesser increase in temperature. The temperature increase in points 1 and 4 is caused by convection, which is why only minor oscillations take place here. Only a very small increase in the temperature caused by the inelastic coupling is observed if a physically meaningful value of the material parameter $\mu$ is assumed.
Figure 17: Temperature evolution of bearing at the points given in Fig. 16 ($p = 6$). The corresponding legend is given in Fig. 17(e).

A higher temperature change would only become apparent if considerably higher number of cycles were to be applied.
If $\mu$ is assigned with much larger values, this leads to a considerable increase in the temperature field, see Figs. 17(c) and 17(d) (in this respect see (Amin et al., 2006), who uses high damping rubber to damp seismic agencies).

Depending on the purpose of the computation and the chosen material parameters, it might be adequate to neglect the temperature evolution caused by the inelastic coupling term $d$ if only small strain-rates and a small number of loading-cycles are applied. Furthermore, it can be observed that the higher the value of the parameter $\mu$ is, the smaller the chosen step-size selected by the adaptive scheme, see Fig. 18(a).

![Figure 18: Step-size behavior of displacement induced increase in temperature field in bearing for different values of $\mu$](image)

5. Conclusions

Highly accurate results are required in the verification step of finite element computations, see (Babuska and Oden, 2004) for the basic ideas. High order p-version finite elements are one possibility to obtain accurate results in space. In the time domain, however, it is very common to apply order one methods. This “asymmetry” is solved in this article by coupling high-order time integration and high-order spatial discretization. This concept is applied to the case of thermo-mechanically coupled problems of thermo-viscoelasticity, where the heat equation is an additional differential part of the resulting DAE-system of the entire spatial discretized system. This is done – for the first time, at least to the knowledge of the authors – using a monolithic scheme for p-version finite elements. It turns out that p-version finite elements can also be applied to more realistic geometries and complicated loading paths, where slow processes and cyclic processes are combined. The main ingredient is the time-adaptive solution procedure in
an SDIRK-method, which offers a very efficient procedure to ensure that the step-size is adapted to the difficulty of the problem. In most situations this leads to a higher efficiency.

Appendix A. Unit system

<table>
<thead>
<tr>
<th>Measure</th>
<th>Dim</th>
<th>SI</th>
<th>Factor</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>L</td>
<td>m</td>
<td>$10^3$</td>
<td>mm</td>
</tr>
<tr>
<td>Mass</td>
<td>M</td>
<td>kg</td>
<td>$10^{-3}$</td>
<td>t</td>
</tr>
<tr>
<td>Time</td>
<td>T</td>
<td>s</td>
<td>1</td>
<td>s</td>
</tr>
<tr>
<td>Temperature</td>
<td>θ</td>
<td>K</td>
<td>1</td>
<td>K</td>
</tr>
<tr>
<td>Density</td>
<td>M/L$^3$</td>
<td>kg/m$^3$</td>
<td>$10^{-12}$</td>
<td>t/mm$^3$</td>
</tr>
<tr>
<td>Force</td>
<td>ML/T$^2$</td>
<td>N</td>
<td>1</td>
<td>N</td>
</tr>
<tr>
<td>Stress</td>
<td>M/(L T$^2$)</td>
<td>Pa</td>
<td>$10^{-6}$</td>
<td>MPa</td>
</tr>
<tr>
<td>Energy</td>
<td>ML$^2$/T$^2$</td>
<td>J</td>
<td>$10^3$</td>
<td>tmm$^2$/s$^2$</td>
</tr>
<tr>
<td>Power</td>
<td>ML$^2$/T$^3$</td>
<td>W</td>
<td>$10^3$</td>
<td>tmm$^2$/s$^3$</td>
</tr>
<tr>
<td>Heat capacity</td>
<td>L$^2$/(T$^2$θ)</td>
<td>J/(kgK)</td>
<td>$10^6$</td>
<td>mm$^2$/(s$^2$K)</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>ML/(T$^3$θ)</td>
<td>W/(mK)</td>
<td>1</td>
<td>tmm/(s$^3$K)</td>
</tr>
<tr>
<td>Heat transfer coefficient</td>
<td>M/(T$^3$θ)</td>
<td>W/(m$^2$K)</td>
<td>$10^{-3}$</td>
<td>t/(s$^3$K)</td>
</tr>
<tr>
<td>Stefan Boltzmann constant</td>
<td>M/(T$^3$θ$^4$)</td>
<td>W/(m$^2$K$^4$)</td>
<td>$10^{-3}$</td>
<td>t/(s$^3$K$^4$)</td>
</tr>
</tbody>
</table>

Table A.6: Unit system for thermo-mechanically coupled computations
References


Hartmann, S. (2012). Comparison of the multiplicative decompositions $F = F_\Theta F_m$ and $F = F_m F_\Theta$ in finite strain thermo-elasticity. Technical Report Series Fac3-12-01, Faculty of Mathematics/Computer Sciences and Mechanical Engineering, Clausthal University of Technology (Germany).


