Applications of the Lie Algebraic Formulas of Baker, Campbell, Hausdorff, and Zassenhaus to the Calculation of Explicit Solutions of Partial Differential Equations

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We apply Lie algebraic methods of the type developed by Baker, Campbell, Hausdorff, and Zassenhaus to the initial value and eigenvalue problems for certain special classes of partial differential operators which have many important applications in the physical sciences. We obtain detailed information about these operators including explicit formulas for the solutions of the problems of interest. We have also produced a computer program to do most of the intermediate algebraic computations.

1. INTRODUCTION

In this paper we study how to use Lie algebraic methods to obtain explicit solutions to partial differential equations. In particular, throughout this paper we consider an example generated by

\[ Q = \left\{ A = a \frac{\partial^2}{\partial x^2} + bx \frac{\partial}{\partial x} + cx^2 + \alpha \frac{\partial}{\partial x} + \beta x + \gamma ; \ a, b, c, \alpha, \beta, \gamma \in \mathbb{C} \right\}. \]

Here \( \mathbb{C} \) is the set of complex numbers.

One of our goals is to solve all initial value problems for operators in \( Q \). Thus if we are given \( A \in Q \) and a sufficiently nice function \( g(x) \) we wish to find a function \( f(x, t) \) satisfying

\[ \frac{\partial f(x, t)}{\partial t} = Af(x, t), \quad f(x, 0) = g(x). \] (1.2)

We write the solution of (1.2) as \( f(x, t) = \exp(At)g(x) \) and then use the properties of Lie algebras to study the operator

\[ \exp(At) = e^{At}. \]
At this point one should note that Q is a Lie algebra under the bracket operation \([A, B] = A \circ B - B \circ A\), \(A, B \in Q\), where \(\circ\) is the operator composition (see Sect. 4). Thus we should be able to interpret \(\exp(At)\) as an element in the Lie group associated with Q.

We also study the eigenvalue and eigenfunction problems for operators in Q. Thus if we are given A in Q we shall find all \(\lambda \in \mathbb{C}\) such that there exists a sufficiently nice function \(\varphi_\lambda(x)\) satisfying

\[
(A - \lambda) \varphi_\lambda(x) = 0. \tag{1.3}
\]

We would like to indicate why we do not use the usual methods to study (1.2) and (1.3). First, it is our opinion that the results in this paper give substantially more detailed insight into problems (1.2) and (1.3) than the usual methods. Second, our methods clearly generalize to the study problems (1.2) and (1.3) for operators A which are complex linear combinations of operators from the set

\[
\left\{ \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial y^2}, x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y}, y \frac{\partial}{\partial x}, y \frac{\partial}{\partial y}, x^2, xy, y^2, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x, y, 1 \right\}
\]

or any higher-dimensional analog, while the usual methods offer no hope of handling such a general class of problems.

We also note that problems of the above type occur frequently in the science literature both when (1.2) is of classical type, that is, parabolic, and when (1.2) is not of classical type. We present a short list of references [2, 4, 6, 10–14, 17, 19] which contain examples of this type and reference many other papers that again contain examples of this type. Also, it is possible to find some version of many of the formulas in this paper in the cited literature. Thus what is new in this paper is not any given formula or fact, but the completeness and generality of our approach to these problems.

Our methods proceed as follows. First we choose a basis for the Lie Algebra, i.e.,

\[
A_1 = I, \quad A_2 = x, \quad A_3 = \frac{\partial}{\partial x}, \quad A_4 = x^2, \quad A_5 = x \frac{\partial}{\partial x}, \quad A_6 = \frac{\partial^2}{\partial x^2}
\]

and then compute \(\exp(tA_i)\) for \(1 \leq i \leq 6\). This is a relatively easy task. Next, if \(A = \sum a_iA_i, \quad a = (a_1, ..., a_6)\), then we attempt to find six functions \(f_j(t, a)\) such that

\[
e^{tA} = e^{tA_1}e^{tA_2} ... e^{tA_6}. \tag{1.4}
\]

One natural method of calculating the \(\{f_i\}\) is to differentiate (1.4) with respect to \(t\) and then solve the result for \(df_i/dt\), which yields a system of nonlinear
ordinary differential equations. In this process we discover that we need formulas for expressions of the form

$$e^{tA_i}A_je^{-tA_i}, \quad 1 \leq i, \ j \leq 6,$$  \hspace{1cm} (1.5)

which, again, are easily calculated.

In these and other calculations it becomes evident that a general formula for rearranging products of exponentials would be extremely useful. If $a = (a_1, \ldots, a_n), \ b = (b_1, \ldots, b_n)$ with $a_i$ and $b_i \in \mathbb{C}$ for $1 \leq i \leq 6$, then we will find $g_i(a, b), \ 1 \leq i \leq 6$ such that

$$\prod_{i=1}^{6} \exp(a_iA_i) \prod_{i=1}^{6} \exp(b_iA_i) = \prod_{i=1}^{6} \exp(g_iA_i).$$ \hspace{1cm} (1.6)

Finally, if we are given $A$ and $B \in \mathcal{Q}$ we would like to find $D \in \mathcal{Q}$ and $A_i \in \mathcal{Q}, \ 2 \leq i < \infty,$ such that

$$e^Ae^B = e^D$$ \hspace{1cm} (1.7)

$$e^{A+B} = e^Ae^Be^{C_2}e^{C_3} \ldots.$$ \hspace{1cm} (1.8)

Such formulas are helpful in simplifying expressions that occur in our calculations.

We refer to formulas (1.4) through (1.8) as Baker, Campbell, Hausdorff, Zassenhaus (BCHZ) type formulas although it is usual to refer only to the last two formulas by these names. We note that we have found the last two formulas relatively less useful than the first three formulas.

To complete our methods for the initial value problem we introduce a technique that is called the symbolic calculus and which allows us to explicitly calculate the time dependent Green's function for operators in $\mathcal{Q}$.

To study the eigenvalue problem for an operator $A \in \mathcal{Q}$ we first calculate a sequence of Lie similarity transforms,

$$U_i(A) = \exp(\alpha A_i)A \exp(-\alpha A_i), \quad \alpha \in \mathbb{C},$$

such that if $U = U_1 \circ \cdots \circ U_n$, then $U(A)$ is some simple operator. We then study the eigenproblem for $U(A)$ and transform back these results to a solution of the problem for $A$. This similarity transform technique can also be used to study the initial value problem. Again we find that formula (1.5) plays a critical role. Finally we apply these methods to operators from $\mathcal{Q}$ and then end by giving an application of the entire theory to an operator from $\mathcal{Q}$ (the harmonic oscillator) and a list of open questions.

We prefer to take the point of view that formulas (1.4) through (1.6) are formal power series identities (free Lie algebraic identities) which may or may not converge depending on which representation of the underlying Lie
algebra we are considering. For a discussion of this point, see [9, p. 170; 12]. References [3, 16] provide the theory necessary for discussing the validity of these formulas in representations generated by differential operators, a question we do not discuss in this paper.

There is an alternate method for deriving BCHZ formulas given in [4]. The method consists of finding a low-dimensional faithful representation of the given Lie algebra and then exponentiating the matrices to obtain the desired formula. This method has the advantage of being entirely algebraic and easily carried out in low dimensions. However, knowing that the algebras are isomorphic is not enough; one must know that the associated Lie groups are isomorphic, a difficult question often ignored when this method is applied in the literature. We have used the methods in Section 2 in preference to the matrix method because of the difficulties in finding the faithful matrix representations for the group and because the method presented is a complete algorithm for finding the desired formulas.

Again we would like to emphasize that Sections 2 and 3 are meant as an exposition of a general approach to problems of the type we wish to study. The material in Section 2 is well known. As far as we know the material in Sections 3 and Appendix A is new but not particularly profound. Sections 4, 5, and 6 are meant as an illustrative example for the previous sections and this material is well known. Theorems 7.3 and 7.5 are new although the general ideas used in Section 7 are well known, as are the ideas in Section 8. In Section 9 the idea of using the Symbolic Calculus familiar from the theory of pseudodifferential operators in this Lie algebraic setting is new, although the ideas are based on the paper of Steinberg and Treves [17].

Although some of the methods in this paper have been used from time to time to study partial differential equations we believe this is the first attempt at such a general and thorough attack on these problems.

2. Formulas

In this section we present certain formulas usually referred to by the names of Baker, Campbell, Hausdorff, and Zassenhaus (BCHZ). If we have a finite-dimensional Lie algebra \( \mathfrak{g} = \text{span}\{A_i, 1 \leq i \leq n\} \), where \( A_i \) is a linear basis, then the basic formulas of our theory are

\[
e^{A}Be^{-A} = C = \sum_{i=1}^{n} c_i A_i, \quad \text{(2.1)}
\]

\[
e^{A^T} = \prod_{i=1}^{n} \exp(f_i A_i), \quad \text{(2.2)}
\]
\[
\left( \prod_{i=1}^{n} \exp(a_i A_i) \right) \left( \prod_{i=1}^{n} \exp(b_i A_i) \right) = \prod_{i=1}^{n} \exp(g_i A_i),
\]
(2.3)

\[e^A e^B = e^{D},\]
(2.4)

\[e^{A+B} = e^A e^{C_2 e^{C_3} \ldots},\]
(2.5)

where \( A = \sum a_i A_i \) and \( B = \sum b_i A_i \). Here we consider \( A, B \in \mathcal{O}, a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{C}^n, \) and \( A_i \in \mathcal{O}, 1 \leq i \leq n, \) to be given and \( c_i, f_i, g_i, d_i, \) and \( c_i^{(j)}, 1 \leq i \leq n, j \geq 2, \) to be computed.

The theory presented in [8, 12] implies that \( c_i, g_i, d_i, c_i^{(j)}, 1 \leq i \leq n, j \geq 2, \) are formal power series in \( a \) and \( b \) and in addition \( f_i \) is a formal power series in \( a \) and \( b \) and \( t \). Moreover, it is known that

- \( c_i \) is an entire function of \( a \) and linear in \( b \),
- \( f_i \) is an analytic function of \( t \) near \( t = 0 \) and analytic in \( a \) and \( b \),
- \( g_i \) is an analytic function of \( a, b \) near \( a = b = 0 \),
- \( d_i \) is an analytic function of \( a, b \) near \( a = b = 0 \),
- \( c_i^{(j)} \) is a polynomial in \( a \) and \( b \).

We also note that the infinite product in (2.5) may be divergent. The following formulas for computing the unknowns in formulas (2.1) through (2.5) are well known:

\[C = e^{ad_A(B)} = \sum_{n=0}^{\infty} [A, B]^n/n!\]
(2.6)

where \( ad_A(B) = [A, B], [A, B]^0 = I, [A, B]^1 = [A, B], ad_A^n(B) = [A, B]^n = [A, [A, B]^{n-1}], n \geq 1, \)

\[D = A + B + \frac{1}{2}[A, B] + \frac{1}{3}[A, [B, [B, A]]] + \text{H.O.T.},\]
(2.7)

\[C_2 = -\frac{1}{3}[A, B], \quad C_3 = -\frac{1}{3}[A, B]B - \frac{1}{3}[[A, B], A], \ldots .\]
(2.8)

There are recursive expressions known for \( D \) and \( C_j \) and thus we can list [6] as many terms in (2.7) and (2.8) as we wish. However, it is not known if there are closed form expressions for \( D \) or \( C_j \) nor even if there are estimates for the coefficients in the expressions for \( C_j \).

Also, if \( f(X_1, X_2, \ldots) \) is any formal power series in \( X_1, X_2, \ldots \in \mathcal{O}, \) then

\[e^A f(X_1, X_2, \ldots) e^{-A} = f(e^A X_1 e^{-A}, e^A X_2 e^{-A}, \ldots).\]
(2.9)

This formula helps simplify many computations involving (2.1).
Finally, we note that the $f_i$ satisfy a system of nonlinear ordinary differential equations and the $g_i$ satisfy a system of nonlinear partial differential equations. We now describe how to derive these differential equations and note that many of our calculations rely on solving these equations. The derivation of the differential equations uses the knowledge of

$$e^{\alpha A_i} A_j e^{-\alpha A_i}$$

(2.10)

for $\alpha \in \mathbb{C}$, $1 \leq i \leq n$, $1 \leq j \leq n$, which are calculated from formula (2.6). Thus when we study a Lie algebra we always provide a table of the expressions given by (2.10).

The differential equations for the $f_i$ are derived by differentiating (2.2) with respect to $t$ and then multiplying the result on either the left or the right by $\exp(-tA)$. The results of this calculation are

$$A = \sum_{j=1}^{n} \left( \prod_{i=1}^{j-1} \exp(f_i A_i) \right) f'_j A_j \left( \prod_{i=j+1}^{n} \exp(-f_i A_i) \right).$$

(2.11)

If one applies the results of (2.10) and uses the fact that $\{A_i\}$ is a basis, then one obtains an implicit system of ordinary differential equations for the $f'_i = \frac{\partial f_i}{\partial t}$. Also note that $f_i(0) = 0$.

Next, if we differentiate (2.3) with respect to $a_i$ or $b_i$ and then multiply the result either on the left or right by (2.3) we obtain

$$\left( \prod_{i=1}^{k-1} \exp(a_i A_i) \right) A_k \left( \prod_{i=k+1}^{n} \exp(-a_i A_i) \right)$$

$$= \sum_{j=1}^{n} \left( \prod_{i=1}^{j-1} \exp(g_i A_i) \right) \frac{\partial g_j}{\partial a_k} A_j \left( \prod_{i=j+1}^{n} \exp(-g_i A_i) \right),$$

(2.12)

$$\left( \prod_{i=1}^{k+1} \exp(-b_i A_i) \right) A_k \left( \prod_{i=k+1}^{n} \exp(b_i A_i) \right)$$

$$= \sum_{j=1}^{n} \left( \prod_{i=n}^{j+1} \exp(-g_i A_i) \right) \frac{\partial g_j}{\partial b_k} A_j \left( \prod_{i=j+1}^{n} \exp(g_i A_i) \right).$$

(2.13)

Again, if we use (2.10) and the fact that $\{A_i\}$ is a basis, then we obtain an implicit system of equations for the gradient of each $g_i$. Also note that $g_i = 0$ when $a = b = 0$.

The above provides a comprehensive list of the techniques we use in our study. Also, note that the formulas depend not only on a choice of basis but also on the choice of an ordering for that basis.
3. **Linear Algebra**

In this section we show that the problems in the previous section can be reduced to problems in linear algebra, ordinary differential equations, and the Frobenius theorem. One important observation is that if \( \mathcal{O} \) is an \( n \)-dimensional algebra and \( X \in \mathcal{O} \), then \( \text{ad}_X \) defined by \( \text{ad}_X(Y) = [X, Y] \) is a linear operator on \( \mathcal{O} \) and consequently can be represented by a matrix \( M_X \sim \text{ad}_X \).

If we choose a basis \( \{A_i\} \) for \( \mathcal{O} \), then the special matrices

\[
M^{(i)} = (M^{(i)}_{jk})
\]

defined by

\[
\text{ad}_{A_i}(A_j) = [A_i, A_j] = \sum_j M^{(i)}_{jk} A_j
\]

completely describe the algebra \( \mathcal{O} \), and in addition, satisfy a Jacobi identity and are skew symmetric in \( (i, j, k) \), \( M^{(i)}_{jk} = -M^{(j)}_{ik} \).

We begin with formula (2.1). We note that \( B \mapsto \exp(A)B \exp(-A) \) is another linear operator on \( \mathcal{O} \) and thus has a matrix representation. If \( \alpha \in \mathbb{C} \) and \( C^{(\alpha)}(\alpha) = (C^{(\alpha)}_{jk}(\alpha)) \) then

\[
e^{\alpha A_i} A_k e^{-\alpha A_i} = \sum_j C^{(\alpha)}_{jk}(\alpha) A_j.
\]  

Formula (2.6) shows that

\[
C^{(\alpha)}(\alpha) = \exp(\alpha M^{(\alpha)}).
\]  

Thus, if we reduce \( M^{(\alpha)} \) to Jordan form we see that the elements of \( C^{(\alpha)} \) are polynomials in \( \alpha \) multiplied by exponentials in \( \alpha \).

Before we consider the remaining formulas we need to discuss the following intermediate calculation. Suppose that \( x = (x_1, \ldots, x_n) \in \mathbb{C}^n \), \( 0 \leq j \leq n+1 \) and \( 1 \leq k \leq n \). We define the following elements of \( \mathcal{O} \):

\[
X_{0,k} = A_k, \quad X_{n+1,k} = A_k,
\]

\[
X_{i,k} = \left( \prod_{i=1}^j \exp(x_i A_i) \right) A_k \left( \prod_{i=j}^n \exp(-x_i A_i) \right),
\]

\[
X_{i,k} = \left( \prod_{i=-n}^j \exp(-x_i A_i) \right) A_k \left( \prod_{i=-j}^n \exp(-x_i A_i) \right).
\]  

If we use (3.1) we obtain

\[
X_{j,k} = \sum_l C_{i,k}^{(\alpha)}(x_i) X_{j-1,l},
\]

\[
X_{j,k} = \sum_l C_{i,k}^{(\alpha)}(-x_i) X_{j+1,l},
\]
and if we assume that

\[ X_{i,k} = \sum_{l} B_{i,k}^{(l)}(x) A_{l}, \quad B'(x) = (B_{i,k}^{(l)}(x)) \]

\[ \bar{X}_{j,k} = \sum_{l} B_{j,k}^{(l)}(x) A_{l}, \quad \bar{B}'(x) = (B_{j,k}^{(l)}(x)), \]

then we obtain

\[ B^{(0)} = \bar{B}^{(n+1)} = I, \]

\[ B^{(l)}(x) = C^{(1)}(x_{1}) C^{(2)}(x_{2}) \cdots C^{(l)}(x_{l}), \] \( \bar{B}^{(l)}(x) = C^{n}(-x_{n}) C^{n-1}(-x_{n-1}) \cdots C^{(l)}(-x_{l}). \] \( (3.5) \)

We now consider formula (2.2). If we use (3.4), then we can rewrite (2.10) as

\[ A - \sum_{j=1}^{n} f'_{j} X_{j-1,j}(f) = \sum_{j=1}^{n} f'_{j} \sum_{l=1}^{n} B_{i,j}^{l-1}(f) A_{l} \]

or for \( 1 \leq l \leq n \) we obtain

\[ a_{l} = \sum_{j=1}^{n} B_{i,j}^{l-1}(f) f'_{j}. \] \( (3.6) \)

This is an implicit set of ordinary differential equations describing the set of functions \( \{ f_{i} \} \).

Next, we consider formula (2.3). If we substitute (3.3) into (2.12) and (2.13) we obtain

\[ X_{i-1,k}(a) = \sum_{j=1}^{n} X_{j-1,j}(a)(\partial g_{j}/\partial a_{k}), \]

\[ \bar{X}_{i+1,k}(b) = \sum_{j=1}^{n} X_{j+1,j}(b)(\partial g_{j}/\partial b_{k}). \]

Using (3.14) we obtain

\[ B_{i,k}^{(k-1)}(a) = \sum_{j=1}^{n} B_{i,j}^{j-1}(g)(\partial g_{j}/\partial a_{k}), \]

\[ \bar{B}_{i,k}^{(k+1)}(b) = \sum_{j=1}^{n} \bar{B}_{i,j}^{j+1}(g)(\partial g_{j}/\partial b_{k}). \]

This is a system of equations for the gradient of \( g \), that is, a system where one would apply the Frobenius theorem.
Theorem 3.1. If we set

\[ f' = \text{column}(f'_1, ..., f'_n), \]
\[ g(a) = (\frac{\partial g}{\partial a_i}) = (\frac{\partial g}{\partial a_j}), \]
\[ B(x) = (B_{ij}(x)) = (B_{ij}^{(j-1)}(x)), \]
\[ a = \text{column}(a_1, ..., a_n), \]

then our equations for \( f \) and \( g \) can be written as \( B(f)f' = a \) and \( B(g)(\frac{\partial g}{\partial a_i}) = B(a), B(g)(\frac{\partial g}{\partial b}) = \bar{B}(b) \). Moreover \( B(0) = \bar{B}(0) = I, f(0) = 0, \) and \( g(0,0) = 0. \) As we have yet to find a use for the matrix version of (2.4) and (2.5), we omit these formulas.

4. The Algebra

In this section we wish to describe the algebra \( Q = \text{span}\{I, x, \partial, x^2, xa, \partial^2\} \), where \( \partial = \frac{d}{dx} \). We note that the algebra \( Q \) is six dimensional and the operators listed are a linear basis. We list in Table I the commutators of all the basis elements. For this purpose set

\[ A_1 = I, \quad A_2 = x, \quad A_3 = \partial, \quad A_4 = x^2, \]
\[ A_5 = xa, \quad A_6 = \partial^2, \quad A_{ij} = [A_i, A_j]. \]

Thus \( A_{ij} \) is the entry in Table I, where \( i \) is the row index and \( j \) is the column index.

The above choice of basis is natural from a mathematical point of view; however, from a Lie algebraic point of view a choice of basis that simplifies the commutator table would be more reasonable. The following choice and

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>( I )</th>
<th>( x )</th>
<th>( \partial )</th>
<th>( x^2 )</th>
<th>( xa )</th>
<th>( \partial^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x )</td>
<td>0</td>
<td>0</td>
<td>(-I)</td>
<td>0</td>
<td>(-x)</td>
<td>(-2\partial)</td>
</tr>
<tr>
<td>( \partial )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2x</td>
<td>( \partial )</td>
<td>0</td>
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<tr>
<td>( x^2 )</td>
<td>0</td>
<td>0</td>
<td>(-2x)</td>
<td>0</td>
<td>(-2x^2)</td>
<td>(-(2 + 4x\partial))</td>
</tr>
<tr>
<td>( xa )</td>
<td>0</td>
<td>0</td>
<td>(-\partial)</td>
<td>2x^2</td>
<td>0</td>
<td>(-2\partial^2)</td>
</tr>
<tr>
<td>( \partial^2 )</td>
<td>0</td>
<td>2\partial</td>
<td>0</td>
<td>2 + 4x\partial</td>
<td>2\partial^2</td>
<td>0</td>
</tr>
</tbody>
</table>
Table II illustrate this point. The choice is related to the creation and annihilation operators of quantum mechanics and is useful for some later computations. Again we set

\[ A_1 = I, \quad A_2 = \alpha = (x + \partial)/2^{1/2}, \]
\[ A_3 = \alpha^+ = (x - \partial)/2^{1/2}, \quad A_4 = \beta = (x\partial + \partial x)/2 = (x\partial + I/2), \]
\[ A_5 = \gamma = (x^2 + \partial^2)/2, \quad A_6 = \gamma^+ = (x^2 - \partial^2)/2. \]

**Table II**

<table>
<thead>
<tr>
<th>( A_i )</th>
<th>( I )</th>
<th>( \alpha )</th>
<th>( \alpha^+ )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \gamma^+ )</th>
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<tbody>
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<td>0</td>
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<tr>
<td>( \alpha )</td>
<td>0</td>
<td>0</td>
<td>( I )</td>
<td>( -\alpha )</td>
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<tr>
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<td>( \alpha^+ )</td>
<td>( -\alpha )</td>
<td>0</td>
<td>2( \gamma^+ )</td>
<td>2( \gamma )</td>
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<tr>
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<td>( \alpha^+ )</td>
<td>( \alpha )</td>
<td>( -2\gamma^+ )</td>
<td>0</td>
<td>2( \beta )</td>
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<tr>
<td>( \gamma^+ )</td>
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<td>( -\alpha )</td>
<td>( \alpha^+ )</td>
<td>( -2\gamma )</td>
<td>( -2\beta )</td>
<td>0</td>
</tr>
</tbody>
</table>

We now note that the following are subalgebras of \( Q \):  
\[ \text{span}\{I, x, \partial\}, \quad \text{span}\{\partial^2, x\partial\}, \]
\[ \text{span}\{x^2, \partial, x\partial + I/2\}, \quad \text{span}\{x^2, x\partial\}, \]
\[ \text{span}\{x^2, x, \partial, I\}, \quad \text{span}\{x\partial, x\}, \]
\[ \text{span}\{\partial^2, x, \partial, I\}, \quad \text{span}\{x\partial, \partial\}. \]

The knowledge of these subalgebras helps in simplifying some later computations.

Because many of our later computations rely on formula (2.10) we provide a table of these results for \( Q \) (Table III) Because of formula (2.9) we need only record these results for

\[ x' = e^{xA}xe^{-xA}, \quad \partial' = e^{xA}\partial e^{-xA}, \]

where \( \alpha \in \mathbb{C} \) and \( A \) is a basis element. The derivation of the formulas in Table III is straightforward using formula (2.6) and Table I. For example,

\[ e^{x\partial x}e^{-x\partial} = x + \alpha[x\partial, x] + \alpha^2[x\partial, [x\partial, x]]/2 + \cdots \]
\[ = x + \alpha x + \alpha^2 x/2 + \alpha^3 x/3! + \cdots \]
\[ = e^\alpha x. \]
The calculation of the analog of Table III for any Lie algebra is a straightforward use of the Jordan form theorem and formula (3.2).

5. Two Subalgebras of $Q$

We are interested in calculating the parameters in formulas (2.1) through (2.5), that is, $e_i, f_i, g_i, d_i, c_i^{(i)}$ for two subalgebras of $Q$, before we turn to the full algebra $Q$.

The subalgebra $\mathcal{B} = \text{span}\{I, x, \partial\}$ is particularly simple to analyze because it is nilpotent. In fact, if $A, B, C \in \mathcal{B}$, then $[[A, B], C] = 0$. Consequently all the infinite expressions in Section 2 become finite. If $A = \alpha I + \beta x + \gamma \partial, B = \alpha I + \beta x + \gamma \partial$, then formulas (2.1) through (2.5) become

$$e^{A} B e^{-A} = (\alpha + \beta c - \gamma b)I + \beta x + \gamma \partial,$$
$$e^{A t} = \exp((at + bct^2/2)I) \exp(btx) \exp(ct\partial),$$
$$e^{A} e^{B} = e^{(a + \alpha + \beta + \beta b + \beta c + \beta c^2)I/2},$$
$$e^{A} e^{B} = e^{(a + \alpha + \beta + \beta b + \beta c + \beta c^2)I/2}.\quad (5.1)$$

The formulas for the algebra $\mathcal{E} = \text{span}\{x^2, Y, \partial^2\}$ where $Y = (x\partial + \partial x)/2 = x\partial + I/2$ are straightforward using the results of Section 3. The calculations are somewhat lengthy even for an algebra as small as $\mathcal{E}$. We include some of the intermediate results of the calculations for formula (2.3) at the end of this section. A major portion of these calculations was done using the MACSYMA program in the Appendix. We also extend some of these results in Section 6.

In the case of formula (2.1) it is sufficient to know Table IV.

In the case of formula (2.2) if $A = \alpha x^2 + bY + c\partial^2$ and

$$e^{tA} = e^{tx^2} e^{bY} e^{c\partial^2},$$
then \( f, g, h \) satisfy the system of differential equations

\[
\begin{align*}
f' &= 4cf^2 + 2bf + a, \\
g' &= b + 4cf, \\
h' &= ce^{2g}.
\end{align*}
\]

If \( v = (4ac - b^2)^{1/2} \), then

\[
\begin{align*}
f &= a \tan(\nu t)/(\delta + b \tan(\nu t)) & \text{if } \nu \neq 0, \ c \neq 0, \\
f &= b^2 t/(bt - 1) 4c & \text{if } \nu = 0, \ c \neq 0, \\
f &= a(e^{bt} - 1)/2b & \text{if } c = 0, \ b \neq 0, \\
f &= at & \text{if } c = 0, \ a = 0.
\end{align*}
\]

Finally \( g \) and \( h \) can be obtained by simple integration.

In the case of formula (2.3), if we set \( A = ax^2 + bY + c\partial^2, \ B = \alpha x^2 + \beta Y + \gamma \partial^2 \) and

\[
e^A e^B = \exp(e^{\alpha x^2} e^\beta Y e^{\gamma \partial^2}),
\]

then we obtain

\[
\begin{align*}
f &= a + \alpha e^{\beta t}/(1 - 4\alpha c), \\
g &= b + \beta + \ln(1 - 4\alpha c), \\
h &= \gamma + ce^{\beta t}/(1 - 4\alpha c).
\end{align*}
\]

The calculation of (5.2) is actually a special case of the calculation of the more complicated calculation of (5.4). We now give some of the intermediate results of the computation of formulas (5.4) using the methods outlined in Section 3. If we set

\[
\begin{align*}
a_1 &= a, & a_2 &= b, & a_3 &= c, \\
b_1 &= \alpha, & b_2 &= \beta, & b_3 &= \gamma, \\
g_1 &= f, & g_2 &= g, & g_3 &= h, \\
A_1 &= x^2, & A_2 &= Y, & A_3 &= \partial^2.
\end{align*}
\]
then we obtain

\[
M^{(1)} = \begin{bmatrix}
0 & -2 & 0 \\
0 & 0 & -4 \\
0 & 0 & 0
\end{bmatrix}, \quad M^{(2)} = \begin{bmatrix}
2 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{bmatrix}, \quad M^{(3)} = \begin{bmatrix}
0 & 0 & 0 \\
4 & 0 & 0 \\
0 & 2 & 0
\end{bmatrix},
\]

\[
C^{(1)}(\alpha) = \begin{bmatrix}
1 & -2\alpha & 4\alpha^2 \\
0 & 1 & -4\alpha \\
0 & 0 & 1
\end{bmatrix}, \quad C^{(2)}(\alpha) = \begin{bmatrix}
\alpha^2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{-2\alpha}
\end{bmatrix}, \quad C^{(3)}(\alpha) = \begin{bmatrix}
1 & 0 & 0 \\
4\alpha & 1 & 0 \\
4\alpha^2 & 2\alpha & 1
\end{bmatrix},
\]

\[
B^{(1)}(\alpha) = \begin{bmatrix}
1 & -2x_1 & 4x_1^2 \\
0 & 1 & -4x_1^2 \\
0 & 0 & 1
\end{bmatrix}, \quad B^{(2)}(\alpha) = \begin{bmatrix}
\exp(2x_3) & -2x_4 & 4x_1^2 \exp(-2x_2) \\
0 & 1 & -4x_1 \exp(-2x_2) \\
0 & 0 & \exp(-2x_2)
\end{bmatrix},
\]

\[
B^{(3)} = \begin{bmatrix}
1 & 0 & 0 \\
-4x_3 & 1 & 0 \\
4x_3^2 & -2x_3 & 1
\end{bmatrix}, \quad \mathcal{B}^{(3)} = \begin{bmatrix}
\exp(-2x_2) & 0 & 0 \\
-4x_3 \exp(-2x_2) & 1 & 0 \\
4x_3^2 \exp(-2x_2) & -2x_3 \exp(2x_3)
\end{bmatrix},
\]

\[
B(x) = \begin{bmatrix}
1 & -2x_1 & 4x_1^2 \exp(-2x_2) \\
0 & 1 & -4x_1 \exp(-2x_2) \\
0 & 0 & \exp(-2x_2)
\end{bmatrix}, \quad \mathcal{B}(x) = \begin{bmatrix}
\exp(-2x_2) & 0 & 0 \\
-4x_3 \exp(-2x_2) & 1 & 0 \\
4x_3^2 \exp(-2x_2) & -2x_3 \exp(2x_3)
\end{bmatrix},
\]

Finally we obtain the system of partial differential equations

\[
\begin{bmatrix}
f_a & f_b & f_c \\
g_a & g_b & g_c \\
h_a & h_b & h_c
\end{bmatrix} = \begin{bmatrix}
1 & 2(f - a) & 4(f - a)^2 e^{-2b} \\
0 & 1 & 4(f - a) e^{-2b} \\
0 & 0 & e^{2b}
\end{bmatrix},
\]

\[
\begin{bmatrix}
f_a & f_\beta & f_\gamma \\
g_a & g_\beta & g_\gamma \\
h_a & h_\beta & h_\gamma
\end{bmatrix} = \begin{bmatrix}
\alpha & 0 & 0 \\
4(h - \gamma) e^{-2b} & 1 & 0 \\
4(h - \gamma)^2 e^{-2b} & 2(h - \gamma) & 1
\end{bmatrix}.
\]

These equations can be integrated to obtain

\[
f = a + \frac{e^{2b}}{K - 4c}, \quad g = b - \ln(K - 4c) + L, \quad h = \frac{\frac{1}{3}e^{2\alpha} L}{K - 4c} + M,
\]

\[
f = \frac{\frac{1}{3}e^{2\alpha L}}{K_1 - 4\alpha} + M_1, \quad g = \beta - \ln(K_1 - 4\alpha) + L_1, \quad h = \gamma + \frac{e^{2\beta}}{K_1 - 4\alpha},
\]

where $K, L, M$ do not depend on $a, b, c$, and $M_1, L_1, K_1$ do not depend on \(\alpha, \beta, \gamma\). For appropriate choices of the constants we obtain (5.4).

We do not attempt formulas (2.4) and (2.5). We also note that Gilmore [4, p. 149] extensively studies the algebra \(\mathfrak{g} \cong \mathfrak{su}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})\) from a point of view different from ours and has obtained information about (2.4). If we combine (5.3) and (5.4) we obtain similar information about (2.4).
6. Formulas for $Q$

In this section we consider extending the results in Section 5 to the full algebra $Q$. Again the calculations are based in Section 3 and are done using the MACSYMA program listed in the Appendix.

We begin with formula (2.1). Thus, we need to calculate $\exp(A)B\exp(-A)$ for every $A, B \in Q$. However, formula (2.9) shows that we need only consider $B = x$ or $B = \partial$. If we set $A = ax^2 + bx\partial + c\partial^2 + \alpha x + \beta \partial + \gamma I$, then in matrix notation

$$
\text{ad}_A \begin{pmatrix} x \\ \partial \\ I \end{pmatrix} = \text{ad}_A \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix} \begin{pmatrix} b & 2c & \beta \\ -2a & -b & \alpha \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \partial \\ I \end{pmatrix},
$$

as is easily calculated from Table I. We now wish to calculate $\exp(\text{ad}_A)$ which is done by diagonalizing the matrix in (6.1) with a matrix similarity transform and then back transforming the exponential of the diagonalized matrix. The matrix has eigenvalues, $\delta, -\delta, 0$ where $\delta = (b^2 - 4ac)^{1/2}$. The results of this calculation give

$$
x' = \left( \cosh(\delta) + \frac{b}{\delta} \sinh(\delta) \right) x + \left( \frac{2c}{\delta} \sinh(\delta) \right) \partial + R_1 I, \\
\partial' = \left( \frac{-2a}{\delta} \sinh(\delta) \right) x + \left( \cosh(\delta) - \frac{b}{\delta} \sinh(\delta) \right) \partial + R_2 I,
$$

(6.2)

where

$$
R_1 = \frac{\beta}{\delta} \cosh(\delta) + \frac{2ac - b\beta}{\delta^2} \left( 1 - \sinh(\delta) \right), \\
R_2 = \frac{\alpha}{\delta} \cosh(\delta) + \frac{2a\beta - b\alpha}{\delta^2} \left( 1 + \sinh(\delta) \right).
$$

Next, we consider formula (2.2). If we use MACSYMA to perform the linear algebra calculations given in Section 3 for the basis

$$
\{A_1 = I, A_2 = x, A_3 = x^2, A_4 = x\partial + I/2, A_5 = \partial, A_6 = \partial^2\},
$$

(6.3)

then the matrix $B(x)$ is given by

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & -x_2 e^{-x_3} & x_2^2 e^{-2x_3} \\
0 & 1 & 0 & -x_3 & -2x_3 e^{-x_3} & (4x_3^2 - 2x_2) e^{-2x_3} \\
0 & 0 & 1 & -2x_3 & 0 & 4x_3^2 e^{-2x_3} \\
0 & 0 & 0 & 1 & 0 & 4x_3 e^{-2x_3} \\
0 & 0 & 0 & 0 & e^{-x_3} & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-2x_3}
\end{pmatrix}
$$
Consequently the functions $f_i$ satisfy the following nonlinear system of differential equations.

$$
\begin{align*}
f'_1 &= a_1 + a_5f_2 - a_6f_2^2, & f'_2 &= a_2 + a_4f_2 + 2a_5f_3 + 2a_6f_2, \\
f'_3 &= a_3 + 2a_4f_3 + 4a_6f_3^2, & f'_4 &= a_4 + 4a_6f_3, \\
f'_5 &= a_5 \exp(f_4), & f'_6 &= a_6 \exp(2f_4).
\end{align*}
$$

We note that the equation for $f_5$ is uncoupled from the remaining equations and that we solved this equation in Section 5. The remaining equations can be solved in terms of simple integrals.

We record the above as a theorem because we use the information in some later problems.

**Theorem 6.1.** There exist functions $f_i(a,t)$, $1 \leq i \leq 6$, $a = (a_1, \ldots, a_6)$, such that formula (2.2) holds for the algebra $\mathcal{Q}$ with the basis (6.2). Moreover, the functions $f_i$ are (not necessarily single valued) analytic functions for all $t$ and $a$ with the exception of certain isolated points.

**Proof.** This is clear from the above discussion.

At this point we would like to emphasize that if we are given numerical values for all of the parameters $a$ and $b$ in (2.1), (2.2), or (2.3), we can easily calculate the unknown functions. However, if we leave $a$ and $b$ as parameters then listing the various algebraic forms for the remaining parameters seems not to be worthwhile because it is possible to find the required system of differential equations with the parameters evaluated, and then it is an elementary problem to solve for the unknown functions. Thus we do not record the results for the remaining formulas.

### 7. Similarity Transforms

We recall from the introduction that if $H \in \mathcal{O}$, a finite-dimensional Lie algebra, then we are interested in solving problems of the type

$$
\frac{df(t)}{dt} = H f(t), \quad f(0) = f_0,
$$

and

$$
(H - \lambda) \phi_\lambda = 0
$$

for $f(t)$, $\lambda$, and $\phi_\lambda$. Here we assume that all $H \in \mathcal{O}$ are defined on some subset of a given space of function $\mathcal{X}$ and $f(t), f_0, \phi_\lambda \in \mathcal{X}$. We use similarity transforms to study these problems.

**Definition 7.1.** If $\mathcal{O}$ is a Lie algebra and $X_i \in \mathcal{O}$, $1 \leq i \leq k$, and

$$
U = \prod_{i=1}^{k} \exp(X_i),
$$

we have

$$
\prod_{i=1}^{k} \exp(X_i) = U.
$$
then the mapping $A \rightarrow A' = UAU^{-1}$ is a linear mapping of $\mathcal{O}$ that we call a Lie similarity transform generated by $U$. If $k = 1$, then the Lie similarity transform is called simple.

We note that if $Y_1, Y_2, \ldots \in \mathcal{O}$ and $f(Y_1, Y_2, \ldots)$ is a formal power series in $Y_1, Y_2, \ldots \in \mathcal{O}$, then according to (2.9) the similarity transform can be extended to $f$ by setting $f' = UfU^{-1} = f(UY_1U^{-1}, UY_2U^{-1}, \ldots)$.

Our approach to problems (7.1) and (7.2) is to choose $H_0 \in \mathcal{O}$ for which the problems (7.1) and (7.2) are easily solved; that is, we can easily calculate $\exp(H_0t)$ and the $\mu$ and $\psi = \psi_\mu$ such that $(H_0 - \mu)\psi_\mu = 0$. We then calculate $X_i \in \mathcal{O}, 1 \leq i \leq k$, such that if $U$ is given by (7.3), then

$$H = UH_0U^{-1}, \quad e^{Ht} = Ue^{H_0t}U^{-1},$$

$$\lambda = \mu, \quad \varphi_\lambda = U\psi_\mu. \quad (7.4)$$

This section is organized in the following way. In Theorems 7.1 and 7.2 we discuss certain general properties of Lie similarity transforms and their relation to contact transforms. We next attempt to find similarity transforms that would reduce every operator in $\mathcal{Q}$ to the operator $x\partial$. Theorem 7.1 implies that the operator $x\partial$ is a bit too simple for our purposes. Theorem 7.3 gives the best possible reductions to operators like $x\partial$ and Theorem 7.4 analyzes the simple operators to which we can reduce all problems for the algebra $\mathcal{Q}$.

Finally we note that the operator (the Harmonic Oscillator Hamiltonian) $\partial^2 - x^2$ belongs to $\mathcal{Q}$ and has, as is well known, the Hermite functions as eigenfunctions. These functions are the exponential of a quadratic function times a polynomial. Theorem 7.5 shows that many of the operators in $\mathcal{Q}$ share this property.

**Theorem 7.1.** Let $U$ generate a Lie similarity transform of $\mathcal{Q}$. If

$$X = ax + b\partial + \alpha I, \quad Y = cx + d\partial + \beta I$$

for $a, b, c, d, \alpha, \beta \in \mathbb{C}$ and

$$X' = UXU^{-1} = a'x + b'\partial + \alpha'I, \quad Y' = UYU^{-1} = c'x + d'\partial + \beta'I,$$

then $a'd' - b'c' = ad - bc$. Moreover, if

$$X = ax^2 + bx\partial + c\partial^2 + ax + \beta\partial + \gamma I, \quad \delta = b^2 - 4ac,$$

and

$$X' = UXU^{-1} = a'x^2 + b'x\partial + c'\partial^2 + a'x + b'\partial + \gamma'I, \quad \delta' = (b')^2 - 4a'c',$$

then $\delta' = \delta$. 

Proof. Let $A \in Q$ and consider the simple Lie similarity transform generated by $U = \exp(A)$ where $A = ax^2 + bx\partial + cx^3 + \alpha x + \beta \partial + \gamma YI$. In formula (6.2) we give explicit expressions for $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ in the following:

$$U \begin{bmatrix} x \\ \partial \end{bmatrix} U^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ \partial \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$ 

Now

$$\begin{bmatrix} x \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ \partial \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta \end{bmatrix} I$$

and

$$U \begin{bmatrix} X \\ Y \end{bmatrix} U^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ \partial \end{bmatrix} + R$$

$$- \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \begin{bmatrix} x \\ \partial \end{bmatrix} + R,$$

where $R$ involves only scalars and $I$. The product property of determinants and the formulas for $a_{ij}$ in (6.2) give the first part of the theorem. The second part is a standard result about the changes in quadratic forms under transformations with determinant one and can also be verified using a straightforward calculation.

We note that it is usual to call a transformation of variables of the type $x' = ax + b\partial, \; \partial' = cx + d\partial$ with $ad - bc = 1$, and $a, b, c, d \in \mathbb{C}$, a complex contact transform. Our next theorem shows that these transformations can all be generated by Lie similarity transforms.

**Theorem 7.2.** If $a, b, c, d, \alpha, \beta$ are complex numbers satisfying $ad - bc = 1$, then any transform of the type

$$x' = ax + b\partial + \alpha, \; \partial' = cx + d\partial + \beta$$

can be given as a Lie similarity transform of $Q$, i.e.,

$$x' = UxU^{-1}, \; \partial' = U\partial U^{-1}.$$

**Proof.** We first note that the condition $ad - bc = 1$ is essential, as was shown in Theorem 7.1. If we set

$$U = \exp(x^2/2) \exp(\partial^2/2) \exp(x^3/2)$$

then Table III gives

$$UsU^{-1} = \partial, \; \quad U\partial U^{-1} = x$$

which is "naturally" called the Fourier transform. If

$$U = \exp(\alpha Y) \exp(\beta D^2/2) \exp(-\gamma x^2/2)$$

then
\[ U_x U^{-1} = e^x + \beta e^{-x} \partial, \]
\[ U_x U^{-1} = \gamma e^x + (1 + \gamma \beta) e^{-x} \partial. \]

If \( a \neq 0 \) and we choose
\[ e^x = a, \quad \beta = \beta e^x, \quad \gamma = c e^{-x}, \]
then
\[ d = (1 + \beta \gamma) \gamma e^{-x}, \]
and we have the desired result. If \( a = 0 \), then \( c \neq 0 \), and if we apply the Fourier transform defined above, then we reduce this problem to the previous case. Finally, if we set
\[ U_1 = \exp((\alpha - \beta \gamma)x) \exp((\beta \gamma - \beta \gamma \gamma) \gamma) \]
then \( U_1 U \) gives the desired result as can be easily checked using Table III.

**Theorem 7.3.** If
\[ A = a x^2 + b x \partial + c \partial^2 + \alpha x + \beta \partial + \gamma \partial, \]
\[ \delta = (b^2 - 4ac)^{1/2} \]
for \( a, b, c, \alpha, \beta, \gamma \in \mathbb{C} \) and
\[ |a| + |b| + |c| > 0, \]
than there exist \( r, s, t \in \mathbb{C} \) and a Lie similarity transformation \( U \) such that
\[
(i) \quad U A U^{-1} = \delta x \partial + s I \quad \text{if} \quad \delta \neq 0, \\
(ii) \quad U A U^{-1} = x^2 + r \partial + t I \quad \text{if} \quad \delta = 0.
\]
Moreover, the transform involves only exponentials of multiples of the basis elements \( \{x^2, x \partial + I/2, \partial^2, x, \partial, I\} \). The set of elements of \( Q \) similar to a given element are called complex orbits.

**Proof.** We first consider a simpler problem. As in Section 4 we set \( Y = (x \partial + \partial x)/2 = x \partial + \frac{1}{2} \) and \( A = ax^2 + b Y + c \partial^2 \) and then attempt to find a contact transform \( U \) of the algebra \( \mathcal{B} = \text{span}\{x^2, Y, \partial^2\} \) such that \( A \) is reduced to a simple form. For these calculations we continually use the formulas in Table IV. We also note that the importance of \( \delta \) was shown in Theorem 7.1.

We consider several cases
\[
(i) \quad b^2 - 4ac \neq 0, \quad ac \neq 0, \\
(ii) \quad b^2 - 4ac \neq 0, \quad a = 0, \\
(iii) \quad b^2 - 4ac \neq 0, \quad c = 0, \\
(iv) \quad b^2 - 4ac = 0.
\]
If we are in case (i), then we let \( a \) be one of the values of \( 4a = \ln(c/a) \) and then set \( a_1 = ae^{2x} = ce^{-2x}, U_1 = \exp(\alpha Y) \). Thus

\[
A_1 = U_1 A U_1^{-1} = a_1(x^2 + \delta^2) + bY.
\]

Next set \( \nu = b/a_1 \) and note that \( \nu^2 = b^2/ac \) so that \( \nu^2 = 4 \) if and only if \( b^2 - 4ac = 0 \). Choose \( \alpha \) a root of \( \alpha^2 + \nu \alpha + 1 = 0 \) and \( \beta = 1/(\nu + 2\alpha) \). Note that \( \nu^2 \neq 4 \) implies \( \nu + 2\alpha \neq 0 \). A straightforward calculation using Table I shows that if \( U_2 = \exp(\beta x^2/2) \exp(\alpha \delta^2/2) \), then \( U_2 A_1 U_2^{-1} = \delta Y \) for \( \delta \in \mathbb{C} \).

For case (ii) we have \( b \neq 0 \) and \( A = bY + c\theta \). If we set \( U = \exp(-c\delta^2/2b) \), then \( UAU^{-1} = bY \). Similarly, in case (iii) \( b \neq 0 \) and \( A = ax^2 + bY, U = \exp(ax^2/2b) \) and \( UAU^{-1} = bY \).

If we are in case (iv), and \( a \neq 0 \) then

\[
A = a(x + (b/2a)\theta)^2,
\]

and if \( U = \exp(-\ln(a)Y/2) \exp(-b\delta^2/4a) \), then (see Table III), \( UAU^{-1} = x^2 \). Similarly, if \( c \neq 0 \), then

\[
A = c(\theta + (b/2c)x)^a
\]

and if \( U = \exp(-\ln(c)Y/2) \exp(bx^2/4c) \), then \( UAU^{-1} = \delta^2 \).

Next we note that

\[
\exp(-x^2/2)\delta^2 \exp(x^2/2) = \delta^2 + 2Y + x^2,
\]

\[
\exp(\delta^2/2)x^2 \exp(-\delta^2/2) = \delta^2 + 2Y + x^2,
\]

so that

\[
x^2 = \exp(-\delta^2/2) \exp(-x^2/2)\delta^2 \exp(x^2/2) \exp(\delta^2/2).
\]

This concludes case (iv).

Next we consider

\[
A = a_1 x^2 + a_2 x \theta + a_3 \delta^2 + a_4 x + a_5 \theta + a_6 I.
\]

Because the algebra \( \text{span}(I, x, \theta) \) is sent into itself by any of the above transformations we are reduced to considering the two cases:

(i) \( A = \delta x \theta + ax + b \delta + c, \quad \delta \neq 0, \)

(ii) \( A = x^2 + ax + b \delta + c. \)

In case (i) if \( U = \exp(-a\delta/\delta) \exp(bx/\delta) \), then \( UAU^{-1} = \delta x \theta + \beta \) for some \( \beta \in \mathbb{C} \).

In case (ii) if we choose \( U = \exp(-a\delta/2) \) then \( UAU^{-1} = x^2 + a \theta + \gamma \) for some \( a, \gamma \in \mathbb{C} \).

We now describe the solution of (7.1) and (7.2) for our reduced problem.
THEOREM 7.4. For every \( \lambda \in \mathbb{C} \), \( x^3 \) satisfies
\[
(\alpha \partial + \beta)x^3 = (\alpha \lambda + \beta)x^3,
\]
and
\[
e^{t(\alpha \partial + \beta)}f(x) = e^{\delta t}f(e^{\delta t}x).
\]
Moreover, if \( \beta = 0 \) and if \( H(x) = 0 \) for \( x \leq 0 \) and \( H(x) = 1 \) if \( x > 0 \), then \( \alpha \partial H(x) = 0 \). If \( \gamma \neq 0 \), then for every \( \lambda \in \mathbb{C} \),
\[
\psi = \exp\left(-x^3/3\gamma + (\lambda - \delta)x/\gamma + c\right)
\]
satisfies
\[
(x^2 + \gamma \delta + \delta)\psi = \lambda \psi
\]
and
\[
e^{(x^2 + \gamma \delta + \delta)t}f(x) = e^{(tx^2 + t\delta + 4t^3\gamma^2/3)}f(x + \gamma t).
\]
Finally if \( \gamma = 0 \), then \( \psi = \psi(x \pm (\lambda - \delta)^{1/2}) \) is a Dirac delta distribution at \( \pm(\lambda - \delta)^{1/2} \) and \( \exp(x^2 + \delta)t \) is just a multiplier.

Proof. The first equation is trivial. Next
\[
e^{t(\alpha \partial + \beta)}f(x) = e^{(\alpha \lambda + \beta)}g(x, t), \quad \text{where} \quad g(x, t) = e^{tx^2}f(x).
\]
Thus
\[
(\partial^2 g/\partial t) = \alpha x(\partial g/\partial x).
\]
This equation can be solved by the method of characteristics giving the second formula.

The eigenfunctions for \( x^2 + \gamma \delta + \delta, \gamma \neq 0 \) are solutions of a trivial ordinary differential equation. We give the following interesting proof of last formula in the theorem. If in (2.5) and (2.8) we set \( A = tx^2, B = t\gamma \delta \) then
\[
C_2 = t\gamma x, \quad C_3 = t^3\gamma^2 I/3, \quad C_4 = 0, \ldots,
\]
and consequently
\[
\exp(x^2 + \gamma \delta + \delta)t = \exp(\delta t) \exp(tx^2) \exp(t\gamma \delta) \exp(t^2\gamma x) \exp(t^3\gamma^2/3).
\]
Clearly
\[
e^{t\gamma \delta}f(x) = f(x + \gamma t)
\]
so that
\[
(\exp(x^2 + \gamma \delta + \delta)t)f(x) = \exp(tx^2 + t\delta + 4t^3\gamma^2/3)f(x + \gamma t).
\]
The remaining parts of the theorem are clear.
THEOREM 7.5. If \( a, b, c, \alpha, \beta, \gamma \in \mathbb{C} \),
\[
A = ax^2 + bx + c + \alpha x + \beta + \gamma I,
\]
and
\[
\delta = (b^2 - 4ac)^{1/2} \neq 0
\]
and \( a, b, c, \alpha, \beta, \gamma \) is not one of the exceptional points described below, then there exist a sequence of complex numbers
\[
z_n = n\delta + \beta, \quad \beta \in \mathbb{C},
\]
and a sequence of complex polynomials \( p_n(x) \) of degree \( n \) and \( r, s, t \in \mathbb{C} \) such that
\[
\varphi_n(x) = p_n(x) \exp(rx^2 + sx + t)
\]
satisfies
\[
A\varphi_n = z_n\varphi_n.
\]

Proof. Theorem 7.3 implies that we can find a Lie similarity transform \( U \) such that
\[
UAU^{-1} = \delta x + \beta.
\]
We set \( z_n - n\delta + \beta, \varphi_n = Ux^n \) and then clearly
\[
A\varphi_n = z_n\varphi_n.
\]

We now need to show that \( \varphi_n \) has the correct form. We first observe that if \( q(x, \partial) \) is a quadratic expression in \( x \) and \( \partial \), i.e., \( q(x, \partial) \in Q \), \( p_n(x) \) is a polynomial of degree \( n \) in \( x \), and \( U - \exp(A) \) is a simple similarity transform, then
\[
U \exp(q(x, \partial)) p_n(x) = \exp(q'(x, \partial)) p'_n(x),
\]
where \( q' \) is again a quadratic expression in \( x \) and \( \partial \) and \( p'_n \) is a polynomial of degree \( n \). This follows by checking each entry in Table III. For those entries containing \( \partial \) one must rewrite the above expression as
\[
\exp(q(UxU^{-1}, U\partial U^{-1})) Up_n(x).
\]
Thus \( \varphi_n \) has the form
\[
\varphi_n(x) = \exp(q(x, \partial)) p_n(x).
\]

Next, use Theorem 7.1 with \( t = 1 \) to rewrite
\[
\exp(q(x, \partial)) = \exp(rx^2 + sx + t) \exp(fx\partial) \exp(g\partial + h\partial^2)
\]
for some \( r, s, t, f, g, h \in \mathbb{C} \). We note that there are certain exceptional values of the coefficients of \( q \) given by Theorem 6.1 which depend on the coefficients of \( A \) such that the last formula does not hold. As before, \( \exp(fx\partial) \) and \( \exp(g\partial + h\partial^2) \) operating on a polynomial of degree \( n \) gives a polynomial of degree \( n \).
We return to the problem of describing $\exp(At)$ in Section 9, where we develop the techniques to say something more about this problem.

8. THE OPERATORS ASSOCIATED WITH A HAMILTONIAN

In this section we explore the following situation. We assume that we are given a Lie algebra $\mathcal{G}$ and a particular element $H \in \mathcal{G}$ and then attempt to find other elements in $\mathcal{G}$ that are useful in understanding the properties of $H$. We consider two cases, the recursion elements and the symmetry algebra. These elements are characterized by their commutation relation with $H$ and correspond to the two possibilities

\[
[H, A] = \mu A, \quad [H, A] = \mu H,
\]

for some $\mu \in \mathbb{C}$.

**Definition 8.1.** If $H \in \mathcal{G}$, then we call $A \in \mathcal{G}$ a recursion element with respect to $H$ with value $\mu \in \mathbb{C}$ if $A \neq 0$ and $[H, A] = \mu A$. If $\mu > 0$, then $A$ is called a raising element with value $\mu$ and if $\mu < 0$, then $A$ is called a lowering element with value $\mu$.

The reason for calling such elements recursion elements is that they generate recursion among the eigenfunctions of $H$ in the following sense. Suppose that

\[
H\varphi = \lambda \varphi, \quad [H, A] = \mu A.
\]

Then

\[
H(A\varphi) = (\lambda + \mu) A\varphi,
\]

and consequently if $A\varphi \neq 0$, then $A\varphi$ is an eigenfunction of $H$ with eigenvalue $\lambda + \mu$. Thus, if $\mu \neq 0$ and we have one eigenfunction $\varphi$ for $H$ and a recursion operator $A$ for $H$, then we can generate infinitely many eigenfunctions.

We note that if $A$ and $B$ are recursion elements with respect to $H$ with values $\mu_1$ and $\mu_2$ and the product $AB$ of $A$ and $B$ is defined, then $[H, AB] = [H, A]B + A[H, B] = (\mu_1 + \mu_2)AB$ and consequently $AB$ is a recursion operator with respect to $H$ with value $\mu_1 + \mu_2$. Also, if $A$ and $B$ are distinct recursion elements with respect to $H$ with different values then $A + B$ is not a recursion operator with respect to $H$. In addition recursion elements have the following Lie properties.

**Proposition 8.1.** If $H, A, B \in \mathcal{G}, \alpha \in \mathbb{C}$, and $[H, A] = \mu_1 A, [H, B] = \mu_2 B$, then

\[
[H, [A, B]] = (\mu_1 + \mu_2)[A, B], \quad [H, \alpha A] = \mu_1(\alpha A).
\]
Moreover, if \( U \) is a Lie similarity transform of \( \mathcal{O} \) and \( H' = UHU\text{-}1, A' = UAU\text{-}1 \), then \([H', A'] = \mu A'\). Thus recursion elements are preserved under products, commutations, scalar multiplication, and Lie similarity transforms.

**Proof.** This is straightforward algebra using the Jacobi identity. Note that if \([A, B] = 0\), then \([A, B]\) is not considered a recursion operator.

**Proposition 8.2.** If \( \mathcal{O} = \mathbb{Q}, r \in \mathbb{C}, \) and \( H = xa, \) then the following is a complete list of recursion elements with respect to \( H \) and their values:

<table>
<thead>
<tr>
<th>elements</th>
<th>values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^0 )</td>
<td>2</td>
</tr>
<tr>
<td>( x )</td>
<td>1</td>
</tr>
<tr>
<td>( \partial^0 )</td>
<td>(-2)</td>
</tr>
<tr>
<td>( \partial )</td>
<td>(-1)</td>
</tr>
<tr>
<td>( xa + Y )</td>
<td>0</td>
</tr>
</tbody>
</table>

**Proof.** This is a straightforward calculation.

We note that the above propositions along with the material in Section 7 imply that "most" operators in \( \mathcal{Q} \) have nontrivial raising and lowering operators. Also, if we choose \( \varphi_n = x^n \) as eigenfunctions of \( x\partial \), then \( x\varphi_n = \varphi_{n+1}, \partial \varphi_n = n\varphi_{n-1} \).

We turn to symmetry elements.

**Definition 8.2.** If \( H \in \mathcal{O} \), then we call \( A \in \mathcal{O} \) a symmetry element with respect to \( H \) if there exists \( \mu \in \mathbb{C} \) such that \([H, A] = \mu H\).

**Proposition 8.3.** If \( H \in \mathcal{O} \), then the symmetry elements with respect to \( H \) form a Lie subalgebra which we call the symmetry algebra of \( H \).

**Proof.** This is an elementary calculation using Jacobi's identity.

**Remark.** Note that if \([H, A] = \mu H\), then \( e^{A}He^{-A} = e^{\mu H} \).

**Proposition 8.4.** The symmetry algebra of \( x\partial \) in \( \mathcal{Q} \) has the basis \( \{x\partial, I\} \).

**Proof.** This is an elementary calculation.

We note that in the case of \( \mathcal{Q} \) the symmetry algebra is trivial. However, in the higher-dimensional analogs of \( \mathcal{Q} \) this is no longer true.

9. **The Symbolic Calculus**

In this section we introduce the symbolic calculus and use the calculus to solve in closed form all initial value problems involving operators from \( \mathcal{Q} \).
DEFINITION 9.1. If $a_{mn} \in \mathbb{C}$ and

$$X := \sum_{m,n=0}^{\infty} a_{mn} x^m \partial^n$$

is a formal series in $x$ and $\partial$, then the symbol of $X$,

$$\sigma X(x, \xi) := \sum_{m,n=0}^{\infty} a_{mn} x^m \xi^n.$$ 

The importance of the symbol arises from the following facts. If

$$\hat{f}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \, dx,$$

then

$$Af(x) = \int_{-\infty}^{\infty} e^{ix\xi} \sigma A(x, i\xi) \hat{f}(\xi) \, d\xi$$

provided that we place appropriate restrictions on $A$ and $f$ so that the above integrals exist. In particular, if we study the initial value problem

$$\frac{\partial f(x, t)}{\partial t} = Af(x, t), \quad f(x, 0) = g(x), \quad (9.1)$$

and

$$G(x, \xi, t) = e^{at}(x, i\xi), \quad (9.2)$$

then

$$f(x, t) = \int_{-\infty}^{\infty} G(x, \xi, t) e^{i\xi\xi} g(\xi) \, d\xi, \quad (9.3)$$

Note that $G(x, \xi, t)$ is usually called the time dependent Green's function for the initial value problem (9.1).

The reason for the term calculus is the two formulas given in Propositions 9.1 and 9.2 that allow us to use our previous theory to calculate the symbol of $\exp(At)$ for $A \in Q$. Although, if $A$ is one of the elements listed in Table V

<table>
<thead>
<tr>
<th>TABLE V</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
</tr>
<tr>
<td>$e^{ax^2}$</td>
</tr>
<tr>
<td>$e^{ax\partial}$</td>
</tr>
<tr>
<td>$e^{a\partial^2}$</td>
</tr>
<tr>
<td>$e^{ax}$</td>
</tr>
<tr>
<td>$e^{a\xi}$</td>
</tr>
<tr>
<td>$T$</td>
</tr>
</tbody>
</table>
then the symbol of \( \exp(At) \) exists for all time, and Theorems 9.1 and 9.2 only guarantee that our formulas hold for \( t \) sufficiently small. Example 9.1 shows, in fact, that some of our series do diverge.

**Remark.** In the terminology of quantum mechanics \( \sigma X \) is the normally ordered form of \( X \); that is, if we are given any expression \( Y \) in \( x \) and \( \partial \), then we use the commutation relation \( \partial x = x \partial + 1 \) to move all \( \partial \) to the right of \( x \) and then replace \( \partial \) by \( \xi \). We always think of \( x \) and \( \xi \) as commuting variables. For example \( \sigma(x\partial) = x\xi \) and \( \sigma(\partial x) = x\xi + 1 \).

**Proposition 9.1.** If \( Y \) is any formal expression in \( x \) and \( \partial \), then

\[
\sigma Y = e^{-\xi x}Y e^{\xi x}(1).
\]

**Proof.** If we normally order \( Y \), then as operators on functions \( X = Y \), i.e.,

\[
e^{-\xi x}X e^{\xi x} = e^{-\xi x}Y e^{\xi x}.
\]

Also,

\[
e^{-\xi x}\partial^n e^{\xi x} = \partial^n(\partial + \xi)^n
\]

and if we let this expression operate on the function which is identically 1, then we obtain

\[
\sigma(x^n\partial^n) = e^{-\xi x}(x^n\partial^n) e^{\xi x}(1) = x^n\xi^n.
\]

Because \( X \) is a sum of such terms we are done.

**Proposition 9.2.** If \( X \) and \( Y \) are two formal series in \( x \) and \( \partial \), and \( X \circ Y \) is defined then

\[
\sigma(X \circ Y)(x, \xi) = \sum_{a=0}^{\infty} \frac{1}{\alpha!} (\partial_1^a \partial^\alpha A(x, \xi))(\partial_2^a \partial^\alpha B(x, \xi)).
\]

**Remark.** We note that \( X \circ Y \) is the operator composition where the variables \( x \) and \( \partial \) do not commute, where as the expression for \( \sigma(X \circ Y) \) has commuting variables and thus must be algebraically more complicated than the product \( XY \).

**Proof.** First we recall Libnitz's rule for derivatives of products

\[
\partial_x^n(f(x) g(x)) = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \partial_x^\beta f e^{\xi x} \partial_x^{\alpha-\beta} g.
\]

If

\[
X = \sum a_{mn} x^n \partial^n,
\]

\[
Y = \sum b_{mn} x^n \partial^n,
\]
then

\[ X \circ Y = \sum a_{mn}b_{ij}x^m \xi^n x^i \xi^j \]

\[ = \sum a_{mn}b_{ij}x^m \sum_{\beta=0}^n \frac{1}{\beta!} \frac{n!}{(n-\beta)!} \frac{i!}{(i-\beta)!} x^{i-\beta} \xi^{n-\beta} x^i \xi^j. \]

Thus

\[ \sigma(X \circ Y) = \sum_{\beta=0}^n \sum_{m,n,i,j=0}^\infty \frac{a_{mn}b_{ij}n!}{\beta!(n-\beta)!} \frac{i!}{(i-\beta)!} x^{n+i-\beta} \xi^{n+i-\beta}, \]

which is just what we obtain if we evaluate the right-hand side of (9.12).

We now wish to calculate the symbols of the exponentials of all elements of \( Q \). First we use Proposition 9.1 to tabulate the symbols in Table V. In fact all but the second formula in Table V are trivial. For example,

\[ \sigma(e^{\alpha \xi^2}) = e^{-\xi^2} e^{\alpha \xi^2} e^{\xi x}(1) \]

\[ = e^{\alpha(\xi+1)^2}(1) \]

\[ = e^{\alpha^2} e^{2 \alpha \xi} e^{\xi^2}(1) \]

\[ = e^{\alpha^2}. \]

To obtain the second formula in Table V we compute

\[ \sigma(e^{ax \xi}) = e^{-\xi^2} e^{ax \xi} e^{\xi x}(1) \]

\[ = e^{ax(\xi+1)}(1) \]

\[ = e^{-a^2} e^{(a-1)\xi x} e^{ax(\xi+1/2)}(1) \]

\[ = \exp((e^a - 1) \xi x). \]

To obtain the next to the last step above we can use the results on formula (2.2) given in Section 6. However, if we note that \( x \xi \) and \( x \xi \) form a closed algebra, then there exist \( f \) and \( g \) such that \( \exp(ax + bx \xi)t = \exp(fx) \exp(gx \xi) \) and we can calculate \( f \) and \( g \) from the theory in Section 3. This indicates that a list of formulas (2.1) to (2.5) for all lower-dimensional algebras would be useful.

**Theorem 9.1.** If \( A = A(x, \xi) \in Q \), then \( \exp(A) \) has a symbol of the form \( \exp(r(x, \xi)) \), where \( r(x, \xi) \) is a quadratic polynomial in \( x \) and \( \xi \), with the possible exceptions given in Theorem 6.1.

**Proof.** We have

\[ \sigma(\exp(A)) = e^{-\xi^2} e^{Ax} e^{\xi x}(1) = e^{A(n, \xi+1)}(1). \]
Then use Theorem 6.1 to write

$$e^{t\omega(x,\dot{\xi})}(1) = e^{t\omega(x,\dot{\xi})}e^{2\omega\dot{\xi}^2}e^{\beta}(1) = e^{t\omega(x,\dot{\xi})},$$

for some $f, g, h \in \mathbb{C}$.

**Theorem 9.2.** If $p(x, \xi)$ and $q(x, \xi)$ are quadratic polynomials in $x$ and $\xi$, and $t$ is sufficiently small, then

$$\sum_{\alpha} \frac{1}{\alpha!} \left( \partial^\alpha_x e^{t p(x,\dot{\xi})} \right) \left( \partial^\alpha_x e^{t q(x,\dot{\xi})} \right)$$

converges for $x$ and $\xi$ sufficiently small.

**Proof.** From Cauchy's integral theorem we obtain the estimates

$$| \partial^\alpha_x e^{t p(x,\dot{\xi})} | \leq K(\alpha)!^1/2 (Mt)^n e^{t(\omega^2 + \dot{\xi}^2)},$$

$$| \partial^\alpha_x e^{t q(x,\dot{\xi})} | \leq K(\alpha)!^1/2 (Mt)^n e^{t(\omega^2 + \dot{\xi}^2)}$$

for some constants $K, M, n$ and thus the convergence is clear.

**Example 9.1.** If $A = \exp(t\dot{\beta})$ and $B = \exp(t\omega^2)$, then

$$\sigma(A \circ B) = \sum_{\alpha} \frac{1}{\alpha!} \left( \partial^\alpha_x e^{t\beta} \right) \left( \partial^\alpha_x e^{t\omega^2} \right).$$

We can use the power series of $\exp(x)$ to evaluate the above expression at $x = \xi = 0$. We obtain

$$\sigma(A \circ B)(0, 0) = \sum \frac{(2n)!}{(n!)^2} t^{2n},$$

which is clearly divergent for $t$ sufficiently large. Thus some restriction on $t$ in Theorem 9.2 is essential.

10. **Example: The Hermite Functions and the Harmonic Oscillator**

We now illustrate our previous theory by choosing

$$H = \beta^2 - x^2 \in Q$$

and, then applying the results of Sections 7, 8, and 9 to $H$. We note that $H$ is the Hamiltonian of the harmonic oscillator.
If we apply Theorem 7.3 to $H$ (use Table IV), we find that if $U = \exp(\partial^2/4) \exp(x^2/2)$, then $UH^{-1} = -2x\partial - 1$. Thus, if we choose

$$\lambda_n = -2n - 1,$$

$$\varphi_n = \exp(-x^2/2) \exp(-\partial^2/4)x^n,$$

then $\varphi_n$ and $\lambda_n$ are eigenpairs for $H$. Let us set

$$p_n(x) = \exp(-\partial^2/4)x^n,$$

which is clearly a polynomial of degree $n$. Proposition 8.2 tells us that

$$R = U^{-1}xU = (x - \partial)/2, \quad L = U^{-1}\partial U = x + \partial,$$

are raising and lowering operators for $H$, i.e.,

$$\varphi_{n+1} = (x - \partial)\varphi_n/2, \quad n \geq 0,$$

$$n\varphi_{n-1} = (x + \partial)\varphi_n, \quad n \geq 1.$$

If we rearrange these last two formulas, we obtain

$$p_{n+1} = x p_n - np_{n-1}/2,$$

$$p_0 = 1, \quad p_1 = x,$$

which yields a recursion for the polynomial $p_n$. This recursion shows that $2^n p_n(x)$ are the usual Hermite polynomials.

Next, if we solve Eqs. (5.4) or (6.4) for the appropriate values of the parameters we obtain

$$\exp(t(\partial^2 - x^2)) = \exp(-\frac{1}{2} \tanh(2t)x^2) \exp(-\ln(cosh(2t))(x\partial + \frac{1}{2})) \exp(\frac{1}{2} \tanh(2t)\partial^2).$$

If we now use the theory in Section 9, we obtain

$$\sigma(\exp(t(\partial^2 - x^2)))$$

$$= e^{-\xi} \exp(t(\partial^2 - x^2)) e^{\xi}(1)$$

$$= \exp(-\frac{1}{2} \tanh(2t)x^2) \exp \ln(cosh(2t))(x(\partial + \xi) + \frac{1}{2})) \exp(\frac{1}{2} \tanh(2t)(\partial - \xi)^2)(1)$$

$$= (\cosh(2t))^{1/2} \exp((\text{sech}(2t) - 1)x\xi + \frac{1}{2} \tanh(2t)(\xi^2 - x^2)).$$

We can now use (9.2) and (9.3) to solve the initial value problem

$$\frac{\partial f}{\partial t}(x, t) = \frac{\partial^2 f(x, t)}{\partial x^2} - x^2 f(x, t), \quad f(x, 0) = g(x).$$

This gives a complete description of the two fundamental problems for the Harmonic oscillator if we replace $t$ by it.
11. Open Problems

1. Calculate all the parameters in formulas 2.1 through 2.5 for all the lower-dimensional Lie algebras for all reasonable choices of basis. Because there are lists of such Lie algebras this seems a fairly practical suggestion. Is it possible to extend the MACSYMA program to do this work?

2. Extend the results on similarity transforms for \( Q \) to a general Lie algebra.

3. How are the formulas for Lie subalgebras related to the formulas for the full Lie algebra? Does the general theory of Lie algebras tell us anything about our formulas?

4. Describe the possible singularities in the functions \( f \) and \( g \) of formulas (2.2) and (2.3). For what Lie algebras does (2.5) converge?

5. Can these results be extended to infinite-dimensional algebras?

6. What are all the finite-dimensional Lie algebras of differential operators with polynomial coefficients?

7. For what types of function spaces can we find representations of the Lie group associated with a given Lie algebra and thus obtain existence and uniqueness theorems for our initial value problems?

8. How do our methods compare to those of Gilmore [3, p. 149] that use matrix representation theory to do the calculations of the parameters in (2.1) through (2.5)? In particular, which methods would be better for higher-dimensional algebras?

APPENDIX: MACSYMA Program

The following is a MACSYMA program that computes the matrices \( B(x), B^{-1}(f)a, B^{-1}(g)B(a) \), and \( \tilde{B}^{-1}(g) \tilde{B}(b) \) given in Theorem 3.1 from the matrices \( C^{(\alpha)}(\alpha) \) given in (3.7). If the dimension of the Lie algebra is \( N \), then one must add to the program the \( N \) statements

\[
C[I][V] := \text{MATRICE}(?)
\]

where ? is the rows of the matrix \( C^{(\alpha)}(\alpha) \) with \( \alpha = V \) and \( 1 \leq I \leq N \). The program is:

\[
\begin{align*}
\text{RHSI(F,A,C,N):=}\text{BLOCK([AA],FOR I THRU N DO AA[I,1]:A[I], AA:GENMATRIX(AA,N,1),B(F,C,N)\uparrow\uparrow(-1)\cdot AA);}
\text{RHSII(G,Y,C,N):=}\text{BLOCK([RT],RT:B(G,C,N)\uparrow\uparrow(-1)\cdot B(Y,C,N),FOR I THRU N DO FOR J THRU N DO RT[I,J]:FACTOR(RT[I,J]),DISP(RT))};
\end{align*}
\]
LIE METHODS AND P.D.E.'S

\[
\text{RHSIII}(G,Z,C,N) := \text{BLOCK}([\text{RT}], \text{RT} : \text{BB}(G,C,N) \uparrow \downarrow (-1) \cdot \text{BB}(Z,C,N), \text{FOR I THRU N DO FOR J THRU N DO RT[I,J] : FACTOR(RT[I,J]), DISP(RT)])}
\]

\[
\text{B}(X,C,N) := \text{BLOCK}([M,BT], \text{FOR I THRU} (-1)+N \text{ DO IF I=1 THEN M[1] : C[1](X[1]) ELSE M[I] : M[(-1)+I] \cdot C[I](X[I]), \text{FOR J THRU N DO FOR I THRU N DO IF J=1 THEN BT[I,J] : \delta(I,J) ELSE BT'[I,J] : M[(-1)+J][I,J], BT' : \text{GENMATRIX}(BT,N,N), \text{RATSIMP}(BT)])}
\]

\[
\]

\[
\text{DELTA}(I,J) := \text{IF I = J THEN 1 ELSE 0;}
\]

\[
\]

where in the last statement \(N\) is replaced by the dimension of the example to be calculated.

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