The influence of delivery times on repairable k-out-of-N systems with spares

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A B S T R A C T

The k-out-of-N structure is a popular type of redundancy in fault-tolerant systems with wide applications in computer and communication systems, and power transmission and distribution systems, among others, during the past several decades. In this paper, our interest is in such a reliability system with identical, repairable components having exponential life times, in which at least k out of N components are needed for the system to perform its functions. There is a single repairman who attends to failed components on a first-come-first-served basis. The repair times are assumed to be of phase type. The system has K spares which can be tapped to extend the lifetime of the system using a probabilistic rule. We assume that the delivery time of a spare is exponentially distributed and there could be multiple requests for spares at any given time. Our main goal is to study the influence of delivery times on the performance measures of the k-out-of-N reliability system. To that end, the system is analyzed using a finite quasi-birth-and-death process and some interesting results are obtained.

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1. Introduction

During the last few decades evaluating the network reliability has become an important topic in planning, designing and control of real-world systems such as computer and communication systems [1,2], power transmission and distribution systems, transportation systems and oil/gas production systems [3], among others.

Many systems cannot achieve their intended reliability in commercial systems, control programs or operating systems without the use of subsystem redundancy; see e.g. [4]. Classical examples of redundancy requirement occur in aircrafts, nuclear power plant control, satellites, electric generators, design of VLSI (very large scale integrated) circuits, space shuttles and computer systems.

A commonly used form of redundancy is a k-out-of-N system in which at least k of N components must be active for the system to perform its functions. These systems are frequently classified as follows: (i) active redundant systems, in which all N components are active even though only k of them are required for the system to be active; (ii) cold standby systems, where N – k components will not be active and upon failure of one of the k active components, one cold standby component will instantly replace the failed component; (iii) warm standby systems, in which N – k components will have a different but less failure rate compared to the active ones; and (iv) hot standby systems, where N – k components will have a higher failure rate as compared to the k active ones. Besides, a k-out-of-N system can have a repair facility consisting of one or more servers attending to failed components. A variety of related models have been developed and many results on k-out-of-N systems...
have been derived, thus resulting in a large body of literature; for a review of the main bibliography and a comprehensive review of results, the reader is directed to [5]. Recent work can be found, for instance, in [6–9].

This paper deals with an active \( k \)-out-of-\( N \) system with \( K \) spares which are useful to extend the life time of the system. When a component fails, it must be removed from the system and repaired to a good as new state to become part of the set of active components. There is a single reliable repairman attending to failed components on a first-come-first-served basis. We assume a decision rule for replacing a failed component by a spare one based on a probabilistic rule and we allow the time to deliver a spare component to be significant by assuming it to be random.

The work presented here is part of an ongoing study on the behavior of \( k \)-out-of-\( N \) reliability systems with spares and repairs [10] and the use of the matrix-analytic formalism to evaluate their performance numerically. In particular, Chakravarthy [10] explores the above repairable system with spares under the assumption that the times to deliver spares are negligible. Unlike the generalized use of spare parts in inventories [11], up until recently [10], relatively little work has been reported on \( k \)-out-of-\( N \) reliability systems with spares. Fawzi and Hawkes [12] analyze the case of hot standby redundancy for \( k \)-out-of-\( N \) systems with a single repairman and \( K \) spares, where each operational component has a constant failure rate, while installation and repair times have arbitrary distributions. Installation has priority over repair and, when a repair is interrupted, it has to be started once again from the beginning. For this system, Fawzi and Hawkes [12] derive the equilibrium availability and downtime distributions. Frostig and Levikson [13] give a method to calculate the availability for \( k \)-out-of-\( N \) systems with both cold and warm standby redundancy, where spares are sent to the repair facility immediately upon failure. Specifically, they consider a single-server repair facility for the \( k \)-out-of-\( N \) case and a multi-server repair facility for the special case of \( 1 \)-out-of-\( N \) systems only. Recently, de Smidt-Destombes et al. [14] study a \( k \)-out-of-\( N \) system with identical, repairable components, in which maintenance is initiated when the number of failed components exceeds a certain threshold and all failed components are replaced by spares after a possible set-up time. As new aspects compared to [12,13], de Smidt-Destombes et al. [14] address bulk arrival of failed components at the repair facility, a multi-server repair facility and maintenance lead times. They give an exact algorithm to determine the operational availability of \( k \)-out-of-\( N \) systems, depending on the maintenance policy and the resources needed, and study some model variants, such as the inclusion of replacement times and cold standby redundancy. Various results for the optimal allocation of redundant spares in coherent systems are given by Boland et al. [15], with particular reference to \( k \)-out-of-\( N \) systems. Related results can be found in [16].

The outline of the paper is as follows. In Section 2, we describe the mathematical model along with the underlying Markov process. The steady-state analysis of the reliability system is presented in Section 3. In Section 4, several system performance measures are given and interesting representative numerical examples to bring out the qualitative nature of the model are discussed in Section 5.

Before we list the basic assumptions of the model, we setup some notations. We will denote by \( e_m \) a column vector of order \( m \) of 1’s; by \( e_m(i) \) we will denote a column vector of order \( m \) with 1 in the \( i \)th position and 0 elsewhere; by \( O_m \) and \( O_{m,n} \) we will denote a column vector of order \( m \) of 0’s and the null matrix of dimension \( m \times n \), respectively; and by \( I_m \) an identity matrix of order \( m \). The notation \( ^t \) will stand for the transpose of a matrix, and the symbols \( \otimes \) and \( \oplus \) will denote the Kronecker product and sum of matrices, respectively; for details and properties on Kronecker products, we refer the reader to [17]. Let \( \delta_d \) be the Kronecker’s delta. Dimensions of vectors and matrices will be suppressed if they can easily be understood from the context.

2. Model description and underlying Markov process

2.1. Model description

The assumptions of our model are given as follows:

(i) The system consists of \( N \) components and \( K \) spares. At the start of a system uptime, all \( N \) components are as good as new. The failure process of each component is characterized by an exponential distribution with rate \( \lambda > 0 \). We assume that the component failure processes are mutually independent.

(ii) The system functions properly as long as at least \( k \) of the \( N \) components function. When the system is functioning with \( i \) components, for \( k \leq i \leq N \), a spare component will be requested with probability \( p_i \) upon failure of a component. Thus, with probability \( q_i = 1 - p_i \), a spare component will not be requested when a failure of a component occurs. We assume that \( p_i = 1 \), so that the system will always request a spare component (assuming that at least one spare component is available at that instant).

(iii) The time to deliver a spare component is assumed to be exponentially distributed with parameter \( \theta > 0 \). It is possible to have more than one outstanding request for the delivery of spares at any given time. This may happen when a failure occurs before a spare is delivered. We also assume that a repair completion will result in a cancellation of a delivery if one is pending. For example, if at a repair completion \( r > 1 \) requests for delivery are pending, then the number of requests will be reduced to \( r - 1 \).

(iv) A single repairman will attend to failed components on a first-come-first-served basis. The repair times are assumed to follow a continuous-time phase type distribution (PH-distribution) with representation \( (\beta, S) \) of order \( m \). The mean
service rate $\mu$ is then given by $\mu = (\beta (-S^{-1})e)^{-1}$. For later use, we denote by $S^0$ the column vector $-Se$. Repaired items are considered as new and are sent to the inventory of spares only if the system has $N$ working components. Otherwise the repaired items are sent directly to the system.

(v) The failure times, the repair times and the delivery times are assumed to be mutually independent.

It should be noted that, although our model assumes negligible delivery times when repaired components are sent to the inventory of spares, it can easily be modified to include positive delivery times of the spares as well as to use these spares directly to replace the failed ones. The details are omitted.

It is worth mentioning that a continuous-time PH-distribution can be thought of as the distribution of the time until absorption in a finite-state continuous-time Markov chain (CTMC) with one absorbing state. PH random variables play an important role in stochastic modelling. Indeed, Erlang, generalized Erlang, exponential and hyperexponential distributions are all special cases of PH-distributions. For details on PH-distributions, see [18, Chapter 2]. The reader is also referred to [19] for the use of PH-distributions in optimal redundancy allocation.

2.2. Markov process

The reliability system as described above can be studied as a block-structured CTMC. To see this, we define $D_1(t)$ to be the number of components working, $D_2(t)$ to be the number of requests for spares pending, $D_3(t)$ to be the number of components under repair and $D_4(t)$ to be the phase of the repair process at time $t$. Note that $D_4(t)$ is not defined when the repairman is idle, which will be simply denoted by $a^*$. In the sequel we let $r_i = \min(N-i,K)$.

The 4-tuple process $\mathcal{X} = \{(D_1(t),D_2(t),D_3(t),D_4(t)) : t \geq 0 \}$ is a CTMC with state space given by

$$\Omega = \{(N,0,0,0) \} \cup \{(i,j,j,j),(1 \leq j \leq K, 1 \leq j \leq m) \} \cup \{(i,j,j,j),(k-1 \leq i \leq N-1, 0 \leq j \leq r_i, N-i \leq j_3 \leq n-i + K - j_2, 1 \leq j_4 \leq m)\}$$

Denote by level $N$ the set of states given by $\{(N,0,0,0)\}$ and by $i$, for $k-1 \leq i \leq N-1$, the set of states given by $\{(i,j,j,j),(0 \leq j_2 \leq r_i, N-i \leq j_3 \leq N-i + K - j_2, 1 \leq j_4 \leq m)\}$. Note that while the level $N$ is of dimension $J_N = 1 + K m$, the level $i$, for $k-1 \leq i \leq N-1$, is of dimension $J_i = (r_i+1)(2(K+1)-r_i)m/2$.

The infinitesimal generator of $\mathcal{X}$ is given by

$$Q = \begin{pmatrix} A_N & B_N \\ C_{N-1} & A_{N-1} & B_{N-1} \\ C_{N-2} & A_{N-2} & B_{N-2} \\ \vdots & \vdots & \vdots \\ C_k & A_k & B_k \\ C_{k-1} & A_{k-1} & B_{k-1} \end{pmatrix},$$

from which it follows that we deal with a finite quasi-birth-and-death (QBD) process; see e.g. [18, Chapter 3]. The block matrices $A_i$ are square of order $J_i$, and $B_i$ and $C_i$ are of dimensions $J_i \times J_{i-1}$ and $J_i \times J_{i+1}$, respectively.

The entries of $Q$ depend on whether $K \geq N-k+1$ or $K < N-k+1$. In the former case (which we will refer to as Case 1), these are defined as follows:

$$A_N = \begin{pmatrix} -Ni \\ S^0 \\ S^0 \beta \\ \vdots \\ S^0 \beta \\ S^0 \beta \\ S^0 \beta \\ S^0 \beta \end{pmatrix},$$

$$B_N = Ni (H_1,H_2),$$

$$C_{N-1} = \begin{pmatrix} S^0 \\ S^0 \beta \\ \vdots \\ S^0 \beta \\ S^0 \beta \\ \vdots \\ S^0 \beta \end{pmatrix},$$

where $H_1$ and $H_2$ are defined as follows:

$$H_1 = \begin{pmatrix} \ldots \\ 0 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} \ldots \\ 0 \end{pmatrix}.$$
specifically, they have the block form

\[
A_{N-i} = \begin{pmatrix}
I_{K+1} \otimes S \\
\vdots \\
(-\theta I_k) \oplus S \\
(-2\theta I_{K-1}) \oplus S \\
\vdots \\
(-i\theta I_{K+i-1}) \oplus S
\end{pmatrix},
\]

\[
B_{N-i} = \begin{pmatrix}
I_{K+1} \otimes (S^0\beta) \\
E_{N-i}(K+1) & F_{N-i}(K+1) \\
E_{N-i}(K) & F_{N-i}(K) \\
\vdots & \vdots \\
E_{N-i}(K+1-i) & F_{N-i}(K+1-i)
\end{pmatrix},
\]

\[
C_{N-i} = \begin{pmatrix}
(I_{K+1} \otimes (S^0\beta)) G_1 \\
G_2 \\
\vdots \\
G_i \\
G_i \\
\end{pmatrix},
\]

where the block matrices \(H_1\) and \(H_2\) are defined as

\[
H_1 = \begin{pmatrix}
qu_i \beta \\
q_i N (j-1)m \\
I_m
\end{pmatrix},
\]

\[
H_2 = \begin{pmatrix}
p_i \beta \\
p_i N (j-1)m \\
0_{m \times (j-1)m}
\end{pmatrix},
\]

and \(E_{N-i}(j), F_{N-i}(j)\) and \(G_i\) are matrices of respective dimensions \(jm \times jm, jm \times (j-1)m\) and \((K+1-i)m \times (K+2-i)m\). Specifically, they have the block form

\[
E_{N-i}(j) = (N-i)\alpha \begin{pmatrix}
qu_i (j-1)m \\
I_m
\end{pmatrix},
\]

\[
F_{N-i}(j) = (N-i)\alpha \begin{pmatrix}
p_i (j-1)m \\
0_{m \times (j-1)m}
\end{pmatrix},
\]

\[
G_i = \begin{pmatrix}
S^0\beta \\
i_0 l \\
\vdots \\
S^0\beta \\
i_0 l
\end{pmatrix}.
\]

When \(K < N - k + 1\) (which is denoted by Case 2), the entries that differ from Case 1 are as follows:

\[
A_{N-i} = \begin{pmatrix}
I_{K+1} \otimes S \\
\vdots \\
(-\theta I_k) \oplus S \\
(-2\theta I_{K-1}) \oplus S \\
\vdots \\
(-K_0 I_1) \oplus S
\end{pmatrix},
\]

\[
B_{N-i} = \begin{pmatrix}
I_{K+1} \otimes (S^0\beta) \\
E_{N-i}(K+1) & F_{N-i}(K+1) \\
E_{N-i}(K) & F_{N-i}(K) \\
\vdots & \vdots \\
E_{N-i}(K+1-i) & F_{N-i}(K+1-i)
\end{pmatrix},
\]

\[
C_{N-i} = \begin{pmatrix}
(I_{K+1} \otimes (S^0\beta)) G_1 \\
G_2 \\
\vdots \\
G_i \\
G_i \\
\end{pmatrix},
\]
3. Steady-state analysis

In this section we perform the steady-state analysis of the $k$-out-of-$N$ reliability system with $K$ spares, including the derivation of the probability density functions of the time until failure of the system, the downtime of the system, the time between failures, the idle time of the repairman and the length of a busy period of the repairman, as well as the analysis of a series $(k$-out-of-$k)$ system.

3.1. Steady-state probability vector

Let $x$ be the steady-state probability vector of $Q$. That is, $x$ satisfies

$$xQ = 0' \quad \text{and} \quad xe = 1. \quad (1)$$

Partition $x$ by levels into subvectors $x(i)$, for $k - 1 \leq i \leq N$, where $x(i)$ has $J_i$ entries. For $k - 1 \leq i \leq N - 1$, we further partition $x(i)$ as

$$x(i) = (y_{01}(i), \ldots, y_{0K-1}(i), y_{11}(i), \ldots, y_{1K}(i), \ldots, y_{J_i1}(i), \ldots, y_{J_iK+1-i}(i)),$$

and we let $x(N)$ be

$$x(N) = (y^*, y_{01}(N), \ldots, y_{0K}(N)),$$

where $y^*$ is a scalar and the row vectors $y_{J_i1}(i)$ are of dimension $m$. To be concrete, $y^*$ records the steady-state probability that the reliability system has $N$ working components, no request pending for spares and no component under repair, and the $J_i$th component of $y_{J_i1}(i)$ gives the probability that, at an arbitrary time, the system has $i$ working components, $J_2$ requests pending for spares and $N - i - (1 - \delta_N) + J_3$ components under repair, with the current repair in phase $J_4$, for $k - 1 \leq i \leq N$, $0 \leq J_2 \leq J_3 \leq K + 1 - \delta_N - J_2$ and $1 \leq J_4 \leq m$.

The system of equations given in Eq. (1) can be solved by a variety of general-purpose iterative techniques, such as (block) Gaussian elimination, Gauss-Seidel and aggregate/disaggregate methods, among others; see e.g. [20] for details. Because of the tridiagonal block structure of $Q$, it might be advisable to solve (1) in an efficient manner by using the technique of Gaver et al. [21, Lemma 2, Theorem 1]. Precisely, in Algorithm 1, we give the process for recursively computing the subvectors $x(i)$:

Algorithm 1. Computation of the subvectors $x(i)$.

Step 1. $L_{k-1} := A_{k-1}$;
from $i = k$ to $N$, do
$$L_i := A_i - B_iL_{i-1}^{-1}C_{i-1};$$
enddo.

Step 2. Solve $x(N)L_N = 0'$ and $x(N)e = 1$;
$$\eta := 1.$$

Step 3. From $i = N - 1$ to $k - 1$, do
$$x(i) := -x(i + 1)B_{i+1}L_i^{-1};$$
$$\eta := \eta + x(i)e;$$
enddo.

Step 4. From $i = k - 1$ to $N$, do
$$x(i) := \eta^{-1}x(i);$$
enddo.

A point worth mentioning is that, in order to compute the inverse matrices $L_{i}^{-1}$ in Steps 1 and 3, Algorithm 1 can reduce to evaluating inverse matrices of order $m$. We give in the next lemma the fundamental structural form of all these inverse matrices $L_{i}^{-1}$. The proof of Lemma 1 is a short algebraic proof with a mathematical derivation based on [22, Theorem 4.2.4].

Lemma 1. By writing the matrix $M_i$, for $k - 1 \leq i \leq N - 1$, as

$$
\begin{pmatrix}
M_{11} & M_{21} & M_{22} \\
M_{31} & M_{32} & M_{33} \\
\vdots & \ddots & \ddots \\
M_{p1} & M_{p2} & M_{p3} & \cdots & M_{pp}
\end{pmatrix},
$$

we have
where $M_d$ are appropriate block matrices of order $m$ (depending on $i$), it results that the inverse matrix $L^{-1}$ has the block form

$$
\begin{pmatrix}
N_{11} & N_{12} & \cdots & N_{1p} \\
N_{21} & N_{22} & \cdots & N_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
N_{p1} & N_{p2} & \cdots & N_{pp}
\end{pmatrix},
$$

where $N_{qq} = M_{qq}^{-1}$, for $1 \leq q \leq p$, and $N_{dq} = -M_{qq}^{-1} \sum_{n=1}^{l} M_{q,n} N_{n,d}$, for $1 \leq i \leq q \leq p$.

### 3.2. The time until failure of the system

Suppose that $X_{TTF}$ denotes the time until the first failure of the system starting with $N$ working components and an inventory of $K$ spares. The following theorem shows that $X_{TTF}$ is of phase type.

**Theorem 1.** $X_{TTF}$ follows a PH-distribution with representation $(e_{m'}(1), T)$ of dimension $n_T$, where $T$ is given by

$$
\begin{pmatrix}
A_N & B_N \\
C_{N-1} & A_{N-1} & B_{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{k-1} & A_{k-1} & B_{k-1} \\
C_k & A_k
\end{pmatrix}
$$

When $K \geq N - k + 1$, $n_T$ is given by

$$
n_T = 1 + m \left( (K(N - k + 1) \left( 1 + \frac{N - k}{2} \right) + (N - k) \left( \frac{1 + Nk}{2} - \frac{N + k - 1}{4} \right) + \frac{(k - 1)k(2k - 1) - (N - 1)N(2N - 1)}{12} \right),
$$

and, when $K < N - k + 1$,

$$
n_T = 1 + m \left( K + \frac{(K + 1)(K + 2)(N - K - k)}{2} + \frac{K(K + 1)(2K + 7)}{6} \right).
$$

**Proof.** The failure of the system corresponds to the finite QBD process defined by $Q$ visiting the level $k - 1$ for the first time. Thus, since we are interested only in ever reaching $k - 1$, we may lump all states of the level $k - 1$ together to make one absorbing state, say $(k - 1)$. The infinitesimal generator of the resulting process has the form

$$
\overline{Q} = \begin{pmatrix}
T & T^0 \\
0 & 0
\end{pmatrix},
$$

with $T^0 = -Te$. The absorbing process $\overline{Q}$ is then defined on the state space $\overline{S} = \cup_{i=1}^{l} i \cup \{k - 1\}$. In view of (2) we need only start the system with all $N$ components working and $K$ spares remaining to be used, which readily yields the initial probability vector $(e_{m'}(1), 0)$, where $n_T = \sum_{i=1}^{l} J_i$. This completes the proof. □

Although expressions for the probability distribution function and the density function of $X_{TTF}$ are routinely derived as $F_{TTF}(x) = 1 - e_{m'}(1) \exp\{Tx\}e$ and $f_{TTF}(x) = e_{m'}(1) \exp\{Tx\}T^0e$, respectively, these cannot be notably simplified, with the exception of particular cases. To face this problem, a satisfactory method based on the uniformization technique, which indeed allows us to obtain global error control, can be found in [23, Section 2.8].

**Remark 1.** The mean $\mu_{TTF}$ of the time until failure of the system is given by

$$
\mu_{TTF} = e_{m'}(1)(-T^{-1})e.
$$

However, due to a large dimension of $T$, an efficient way to calculate this mean value is done as follows:

**Algorithm 2.** Computation of $\mu_{TTF}$.

1. **Step 1.** $L_k := A_k$;
   from $i = k + 1$ to $N$, do
   $L_i := A_i - BL_{i-1}C_{i-1}$;
   enddo
2. **Step 2.** $\delta(N) := -e_{m'}(1)L_N^{-1}$;
   $\mu_{TTF} := \delta(N)e$.
Step 3. From $i = N - 1$ to $k$, do

\[
\delta(i) := -\delta(i+1)B_{i+1}\tilde{L}^{-1};
\]

\[
\muTTF := \muTTF + \delta(i)e;
\]

dendo.

To verify the validity of Algorithm 2, we first notice that $\muTTF$ can be evaluated as

\[
\muTTF = \frac{1}{\delta T},
\]

where $\delta$ denotes the steady-state probability vector of the irreducible infinitesimal generator $T + T^0e_n(1)$ of the phase process. If we partition $\delta$ by levels into subvectors $\delta(i)$ of dimension $j_i$, for $k \leq i \leq N$, then the matrix equation $\delta(T + T^0e_n(1)) = 0$ can be rewritten as

\[
\delta(k+1)B_{k+1} = -\delta(k)A_k,
\]

\[
\delta(i+1)B_{i+1} = -\delta(i)A_i - \delta(i-1)C_{i-1}, k+1 \leq i \leq N - 1,
\]

\[
\delta(N)A_N = -\delta(N-1)C_{N-1} - \delta(k)B_ke_n(1).
\]

Straightforward algebra then yields

\[
\delta(i) = -\delta(i+1)B_{i+1}\tilde{L}^{-1}, \quad k \leq i \leq N - 1.
\]

Since $\delta T^0 = \delta(k)B_k$, we have that

\[
\delta(N) = -\muTTF e_n(1)\tilde{L}^{-1}.
\]

Thus, instead of $\delta(N)$, we may compute $\delta(N) = \muTTF^{-1}\delta(N)$ and recursively $\delta(i) = -\delta(i+1)B_{i+1}\tilde{L}^{-1}$, for $k \leq i \leq N - 1$. The value of $\muTTF$ at Step 3 is then derived by normalization as $\muTTF = \sum_{i=1}^{N}\delta(i)e$.

Since $X_{TTF}$ is the first-passage time of the finite QBD process $Q$ to reach the level $k - 1$, starting from the state $(N, 0, 0, +)$, its first moment $\muTTF$ can be also computed by applying Algorithm B of Gaver et al. [21]. As the reader may verify, Algorithm 2 has less complexity because it is built by exploiting the special block structure of $Q$.

### 3.3. The downtime of the system

Let $X_{DT}$ be the duration during which the system is down. Then we have

**Theorem 2.** The random variable $X_{DT}$ follows a PH-distribution with representation $(\zeta, U)$ of dimension $n_U = \min\{N - k + 2, K + 1\}$. When $K \geq N - k + 1$, the vector $\zeta$, partitioned as $(\zeta_0^{(1)}, \ldots, \zeta_{N-k+1}^{(1)})$, is given by

\[
\begin{align*}
\zeta_0^{(1)} & = c \left( q_k \sum_{j=1}^{K} y_{j0}(k) + y_{0,K+1}(k) \right), \\
\zeta_{j}^{(1)} & = c \left( q_k \sum_{j=1}^{K-j} y_{jj}(k) + y_{j,K+1-j}(k) + p_{k} \sum_{j=1}^{K-k-j} y_{j-1, j}(k) \right), \quad 1 \leq j \leq N - k, \\
\zeta_{N-k+2}^{(1)} & = c p_{k} \sum_{j=1}^{K} y_{N-k-j}(k),
\end{align*}
\]

where $c$ is the normalizing constant. When $K < N - k + 1$, the vector $\zeta$, partitioned as $(\zeta_0^{(2)}, \ldots, \zeta_{K}^{(2)})$, can be expressed as

\[
\begin{align*}
\zeta_0^{(2)} & = c \left( q_k \sum_{j=1}^{K} y_{j0}(k) + y_{0,K+1}(k) \right), \\
\zeta_{j}^{(2)} & = c \left( q_k \sum_{j=1}^{K-j} y_{jj}(k) + y_{j,K+1-j}(k) + p_{k} \sum_{j=1}^{K-k-j} y_{j-1, j}(k) \right), \quad 1 \leq j \leq K,
\end{align*}
\]

where $c$ is the normalizing constant. Furthermore, the matrix $U$ is a (block) diagonal matrix whose $j_2$th (block) element is given by $S - j_2 O$, for $0 \leq j_2 \leq \min\{N - k + 1, K\}$.

**Proof.** The result follows by observing that $X_{DT}$ is the duration that the finite QBD process $Q$ spends in states of the level $k - 1$ before hitting the level $k$, and the $j_2$th entry of the $(j_2, j_2)$th (block) component vector of $\alpha(k)B_k$ gives the probability that, at the time of the system failure, the repair process is in phase $j_2$, with $N - k + j_2$ components under repair and $j_2$ requests for spares pending. □

The preceding result has the following immediate consequence.
Corollary 1. When $K \geq N - k + 1$, the distribution function of $X_{\text{DT}}$ is given by

$$
F_{\text{DT}}(x) = 1 - \sum_{j_2=0}^{N-k+1} \zeta_{j_2}^{(1)} \exp\{(S - j_2 \theta l)x\} e,
$$

for $x \geq 0$, its density function has the form

$$
f_{\text{DT}}(x) = \sum_{j_2=0}^{N-k+1} \zeta_{j_2}^{(1)} \exp\{(S - j_2 \theta l)x\}(S^0 + j_2 \theta e),
$$

for $x > 0$, and its moments are given by

$$
E[X_{\text{DT}}^l] = l! \sum_{j_2=0}^{N-k+1} \zeta_{j_2}^{(1)} (j_2 \theta l - S)^{-1} e,
$$

for $l \geq 1$. Similarly, when $K < N - k + 1$, we find that

$$
F_{\text{DT}}(x) = 1 - \sum_{j_2=0}^{K} \zeta_{j_2}^{(2)} \exp\{(S - j_2 \theta l)x\} e,
$$

$$
f_{\text{DT}}(x) = \sum_{j_2=0}^{K} \zeta_{j_2}^{(2)} \exp\{(S - j_2 \theta l)x\}(S^0 + j_2 \theta e),
$$

$$
E[X_{\text{DT}}^l] = l! \sum_{j_2=0}^{K} \zeta_{j_2}^{(2)} (j_2 \theta l - S)^{-1} e.
$$

The following lemma establishes a simple formula for computing the mean downtime of the system in terms of the steady-state probability vector $\mathbf{x}$.

Lemma 2. The mean downtime $\mu_{\text{DT}} = E[X_{\text{DT}}]$ of the system can be calculated as

$$
\mu_{\text{DT}} = \frac{\mathbf{x}(k-1) e}{k \mathbf{x}(k) e}.
$$

Proof. We outline the proof for the case $K \geq N - k + 1$ as it is similar for the other case. We first note that the steady-state equation $\mathbf{x}(k) B_k + \mathbf{x}(k-1) A_{k-1} = 0'$ can be rewritten as

$$
k \mathbf{q}_j \mathbf{y}_{0j_2} (k) = -\mathbf{y}_{0j_2} (k-1) S, \quad 1 \leq j_2 \leq K,
$$

$$
k \mathbf{q}_j \mathbf{y}_{0, k-1} (k) = -\mathbf{y}_{0, k-1} (k-1) S,
$$

$$
k \mathbf{p}_j \mathbf{y}_{j_2, 1-j_2} (k) = -k \mathbf{q}_j \mathbf{y}_{j_2, 1-j_2} (k) - \mathbf{y}_{j_2, 1-j_2} (k-1) (S - j_2 \theta l), \quad 1 \leq j_2 \leq N - k, \quad 1 \leq j_2 \leq K - j_2,
$$

$$
k \mathbf{p}_j \mathbf{y}_{j_2, 1-k-1} (k) = -\mathbf{y}_{j_2, 1-k-1} (k-1) (S - (N - k + 1) \theta l), \quad 1 \leq j_2 \leq N - k,
$$

$$
k \mathbf{p}_j \mathbf{y}_{N-k-j_2} (k) = -\mathbf{y}_{N-k-j_2} (k-1) (S - (N - k + 1) \theta l), \quad 1 \leq j_2 \leq K - N + k.
$$

Now, post-multiplying the first two equations by $(-S^{-1}) e$, the next two equations by $(j_2 \theta l - S)^{-1} e$, and the last one by $((N - k + 1) \theta l - S)^{-1} e$, and adding the resulting equations yields the stated result. \(\square\)

3.4. Successive times between working of the system

Suppose that $Z_i$ denotes the $i$th duration during which our reliability system functions starting in level $k$. Note that the system gets back to working condition, from being in failed state, either through a completion of a repair or through an arrival of a spare, if any request is pending. By defining $X_i = Z_i + X_{\text{DT}}$, we can model the sequence $\{X_i : i \geq 1\}$ of successive times between system functioning by being in level $k$ as a Markovian arrival process (MAP). The MAP, a special class of tractable Markov renewal process, is a rich class of point processes that includes many well-known processes such as the scalar Poisson process, PH-renewal processes and Markov-modulated Poisson process, among others. The MAP process is described by two parameter matrices $(D_0, D_1)$, such that $D_0$ governs the transitions corresponding to no arrival and $D_1$ governs those corresponding to an arrival. For further details on the MAP and their usefulness in stochastic modelling, we refer to [24–26,18]; for a review and recent work on the MAP, we refer the reader to [27].
In our context here, the parameter matrices \((D_0, D_1)\) governing the successive cycle times \(\{X_i : i \geq 1\}\) of the reliability system are given by

\[
D_0 = \begin{pmatrix}
A_N & B_N \\
C_{N-1} & A_{N-1} & B_{N-1} \\
& C_{N-2} & A_{N-2} & B_{N-2} \\
& & \ddots & \ddots & \ddots \\
& & & C_k & A_k & B_k \\
& & & & \ddots & \ddots & \ddots \\
& & & & & & C_{k-1} & 0
\end{pmatrix},
\]

\[
D_1 = \begin{pmatrix}
0 & 0 \\
0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & & 0 & 0 & 0 \\
& & & & & & & & C_{k-1} & 0
\end{pmatrix}.
\]

Then, on noting that \(x\) is the steady-state probability vector of \(D_0 + D_1\), the mean cycle time is given by

\[
\mu_{CT} = \frac{1}{x D_1 e},
\]

where \(x D_1 e = x(k - 1) C_{k-1} e\). By (1), we can readily derive the equality \(x(k - 1) C_{k-1} e = -x \beta_0 A_{k-1}^{-1} C_{k-1} e\), so that

\[
\mu_{CT} = \frac{1}{k \beta \alpha},
\]

since \(-A_{k-1}^{-1} C_{k-1} e = e\) and \(B_k e = k \beta e\).

### 3.5. The busy period of the repairman

The busy period of the repairman is defined as the interval starting with the repairman getting busy, and ending when for the first time the repairman becomes idle. Let \(X_{BPR}\) denote the length of a busy period of the repairman. Then we have the following result, which follows on noting that the busy period of the repairman starts when the reliability system gets either into the state of states \(\{(N - 1, 1, 1, j_a), 1 \leq j_a \leq m\}\), with probability \(p_n\), or the state of states \(\{(N - 1, 0, 1, j_a), 1 \leq j_a \leq m\}\), with probability \(q_N\).

**Theorem 3.** The random variable \(X_{BPR}\) follows a PH-distribution with representation \((\varphi, L)\) of dimension \(n_L\), where

\[
\varphi = \left(0_{m \times 1}, q_N e_{k-1}^T(1) \otimes \beta, p_n e_{k-1}^T(1) \otimes \beta, 0^T\right)
\]

and \(L\) is obtained from \(Q\) by deleting its first row and first column. The dimension \(n_L\) is given by

\[
n_L = n_T - 1 + (N - k + 2)(2K - N + k + 1)m/2, \quad \text{when} \ K \geq N - k + 1, \ \text{and} \ n_L = n_T - 1 + (k + 1)(K + 2)m/2, \ \text{when} \ K < N - k + 1.
\]

By noting that

\[
Q = \begin{pmatrix}
-N \lambda & N \lambda \varphi \\
L^0 & L
\end{pmatrix},
\]

where \(L^0 = -L e\), we have that the first moment of \(X_{BPR}\) is given by \(\mu_{BPR} = \varphi(-L^{-1})e\). Now, an appeal to the theory of regenerative processes [\[28], Theorem 2.24] allows us to express the steady-state probability \(1 - y^-\) as

\[
1 - y^- = \frac{\mu_{BPR}}{\mu_{BPR} + (N \lambda)^{-1}},
\]

which leads to a more suitable expression for \(\mu_{BPR}\):

\[
\mu_{BPR} = \frac{1}{N \lambda} \left(\frac{1}{y^-} - 1\right).
\]

### 3.6. A series system

In the special case \(N = k\), a closed-form solution and interesting limiting results can be derived. For convenience, we reorder the state space of the finite QBD process \(Q\) as follows: let \(I_1(t)\), \(I_2(t)\), and \(I_3(t)\) denote, respectively, the number of components under repair, the state of the system, and the phase of the repair process at time \(t\). Here, the state of the system can be in one of three states: \(I_2(t) = 1\) indicates the system is working; and \(I_2(t) = 2\) and \(I_2(t) = 3\) indicate the system is down with no request for a spare component and with one request for spare pending, respectively. Note that if \(I_1(t) = 0\), then \(I_2(t)\) and \(I_3(t)\) will not be defined, which are denoted by \(*\). Similarly, when \(I_1(t) = K + 1\), it is clear that \(I_2(t) = 2\).
The triplet process \( \mathcal{Y} = \{ (l_1(t), l_2(t), l_3(t)) : t \geq 0 \} \) constitutes a finite QBD process defined on the state space \( \Omega_3 = \{(0,*,*) \} \cup \{ (i_1, i_2, i_3) : 1 \leq i_1 \leq K, 1 \leq i_2 \leq 3, 1 \leq i_3 \leq m \} \cup \{ (K+1, 2, i_3) : 1 \leq i_3 \leq m \} \). Its infinitesimal generator \( Q_3 \) has the form

\[
Q_3 = \\
\begin{pmatrix}
-kl & C_0 & 0 \\
C_2 & A_1 & A_0 \\
A_2 & A_1 & A_0 \\
\vdots & \vdots & \vdots \\
A_2 & A_1 & A_0 \\
A_2 & A_1 & B_0 \\
B_2 & S \\
\end{pmatrix},
\]

where

\[
C_0 = \begin{pmatrix}
k \lambda q_i I & k \lambda p_i I \\
0_{m \times m} & 0_{m \times m} \\
0_{2m \times m} & 0_{2m \times m} \\
\end{pmatrix},
\]

\[
A_1 = \begin{pmatrix}
S - k \lambda I \\
0 & S - k \lambda I \\
\end{pmatrix},
\]

\[
A_2 = (e_3 \otimes (S^0 \beta), 0_{3m \times 2m}).
\]

Let \( \mathbf{x} \) be the steady-state probability vector of \( Q_3 \). By partitioning \( \mathbf{x} \) by levels into the scalar \( y^r \) and the subvectors \( z(i) \), for \( 1 \leq i \leq K + 1 \), with \( z(i) = (z_1(i), z_2(i), z_3(i)) \), for \( 1 \leq i \leq K \), we can apply matrix-analytic techniques to \( Q_3 \) in order to get the following result:

**Theorem 4.** For a \( k \)-out-of-\( k \) reliability system with \( K \geq 1 \) spares, we have

\[
z(i) = y^r \beta R_i^{-1} R_1, \quad j \in \{1, 2, 3\}, \quad 1 \leq i \leq K,
\]

\[
z(K + 1) = k \lambda y^r \beta R_k (-S^{-1}).
\]

where \( R_1 = \theta R_1 (k \lambda I - k \lambda \mathbf{e} \beta - S)^{-1}, R_2 = k \lambda q_i (-S^{-1}), R_3 = k \lambda p_i ((0 I - S)^{-1} \mathbf{e}) + k \lambda \beta R_k (-S)^{-1} \mathbf{e} \). And the steady-state probability \( y^r \) is given by

\[
y^r = (1 + \beta (I - R_1)^{-1} (I - R_k^x) (R_1 + R_2 + R_3) \mathbf{e} + k \lambda \beta R_k (-S)^{-1} \mathbf{e})^{-1}.
\]

**Remark 2.** When there are no spares in the system (i.e., \( K = 0 \)), it can be verified that the steady-state distribution of \( \mathcal{Y} \) is given by

\[
y^r = \left( 1 + \frac{k \lambda}{\mu} \right)^{-1},
\]

\[
z(1) = k \lambda y^r \beta (-S^{-1}).
\]

Without going into details, we remark that the study of the distributions of the time until failure, the downtime, successive times between working of the system and the length of a busy period of the repairman can be performed similarly to Sections 3.2–3.5, with minor modifications.

The following result shows that the first moment of the downtime of the system approaches a limit value as the number of spares increases. Let \( \rho \) be the spectral radius of the matrix \( R_1 \).

**Theorem 5.** For the \( k \)-out-of-\( k \) reliability system with \( K \) spares, we have

(i) If \( \rho < 1 \), then

\[
\mu_{ot} = \frac{\mu + (k \lambda - \mu) \beta (I - R_1)^{-1} \mathbf{e}}{k \lambda \mu (I - R_1)^{-1} \mathbf{e}}, \quad \text{as} \quad K \to \infty.
\]

(ii) If \( \rho \geq 1 \), then

\[
\mu_{ot} = \frac{k \lambda - \mu}{k \lambda \mu}, \quad \text{as} \quad K \to \infty.
\]
The stated result follows on noting that the power series \( \sum_{i=0}^{\infty} R_i^e \) converges if and only if the spectral radius \( \rho \) of \( R_1 \) is strictly less than 1. In that case, \( \sum_{i=0}^{\infty} R_i^e \rightarrow \beta(I - R_1)^{-1}e \), as \( K \rightarrow \infty \); see e.g. [22, Theorem 4.5.4].

3.7. The case when \( \theta \rightarrow \infty \)

In the case when \( \theta \rightarrow \infty \), the mean delivery time becomes insignificant and, as a result, our reliability model reduces to that discussed in [10]. In this case the limiting results simply need some minor modifications.

For the sake of brevity, we remark only that \( \theta \rightarrow \infty \) turns the result of Lemma 2 into

\[
\mu_{DT} = \frac{x(k-1)e}{k\lambda_I \left( q_k \sum_{j=0}^{K-1} y_{ij} (k) e + y_{0k} (k) e \right)}.
\]

To prove Eq. (7), we first notice that the underlying PH-representation for the downtime becomes now of order \( m \), whence the vector \( \zeta = \zeta_0^{(1)} = \zeta_0^{(2)} \) is given by

\[
\zeta = c \left( q_k \sum_{j=0}^{k-1} y_{ij} (k) + y_{0k} (k) \right).
\]

where \( c \) is the normalizing constant. Eq. (7) and similar expressions for other descriptors are used in Section 5 as accuracy checks in our numerical computation of the steady-state measures studied in the preceding sections.

4. System performance measures

For the qualitative analysis of our \( k \)-out-of-\( N \) reliability system with \( K \) spares and repairs, we consider a number of system performance measures. These and their formulas are given below.

(i) Steady-state probability that the repairman is idle: the probability that the repairman is idle at an arbitrary time is given by \( y^* \).

(ii) Steady-state probability that the system is down: the probability that the system is down at an arbitrary time is given by \( x(k-1)e \).

(iii) Probability mass function of the number of components under repair: let \( a_i^{(l)} \) denote the probability that, at an arbitrary time, the number of components under repair equals \( j \) for Case \( l \), for \( l \in \{1, 2\} \). From Section 3.1, we have

\[
\begin{align*}
a_i^{(0)} &= y^*, \quad l \in \{1, 2\}, \\
a_i^{(l)} &= \sum_{j=0}^{N} \sum_{j_2=0}^{K-1} y_{ij} (j) e, \quad l \in \{1, 2\}, \quad 1 \leq j \leq K, \\
a_i^{(1)} &= \sum_{i=0}^{N-K j} \sum_{j_2=0}^{K-1} y_{ij} (j) e, \quad K + 1 \leq j \leq N + K - 1.
\end{align*}
\]
Example 1. For a fixed pair $(k,N) = (10,15)$, Figs. 1–3 illustrate the variability of $\mu_{\text{TP}}$, the mean number of requests for spares pending and $\mu_{\text{HPR}}$, respectively. In each figure, five curves associated with the values $\theta^{-1} \in \{0.25\mu^{-1}, 0.5\mu^{-1}, \mu^{-1}, 2.0\mu^{-1}, 4.0\mu^{-1}\}$ are displayed.

It is shown in Fig. 1 that, for each fixed value of $\theta^{-1}$, $\mu_{\text{TP}}$ increases with increasing values of $K$. For $K$ fixed, the smaller mean delivery times yield higher magnitudes of $\mu_{\text{TP}}$. From Fig. 2, we see that the mean number of requests for spares pending increases as a function of $K$, irrespective of the value of $\theta^{-1}$; however, the impact of $K$ is more apparent for higher values of $\theta^{-1}$. As $K$ is fixed, the mean increases with increasing values of $\theta^{-1}$.

The behavior of $\mu_{\text{HPR}}$ is shown in Fig. 3. For a fixed value of $\theta^{-1}$, this descriptor increases with increasing values of $K$ and, for a fixed value of $K$, it is a decreasing function of $\theta^{-1}$. It should be noted that the magnitude of the differences becomes more apparent as the number $K$ of spares becomes higher.

![Fig. 1. $\mu_{\text{TP}}$ versus $K$ (from top to bottom) $\theta^{-1} = 0.25\mu^{-1}, 0.5\mu^{-1}, \mu^{-1}, 2.0\mu^{-1}$ and $4.0\mu^{-1}$.

$$a_j^{(2)} = \sum_{i=N-k}^{N-K-j} \sum_{j=0}^{i-j} y_{ij}(i)e, \quad K + 1 \leq j \leq N - k + 1,$$

$$a_j^{(2)} = \sum_{i=k-1}^{N-k-j} \sum_{j=0}^{i-j} y_{ij}(i)e, \quad N - k + 2 \leq j \leq N + K - 1.$$
Fig. 3 allows us to observe that $l_{BPR}$ is not solely influenced by the failure and repair rates, but it does also depend on the number of spares $K$. This may be corroborated by Table 1, where we find that increasing values of $K$ always lead to higher values for the mean number of components under repair and, consequently, to increasing values of $l_{BPR}$. The perceptive reader will have noticed that this behavior is closely related to the dependence of the PH-distribution in Theorem 3 on the number $K$ of spares.

Fig. 3. $l_{BPR}$ versus $K$ for (from top to bottom) $\theta^{-1} = 4.0\mu^{-1}, 2.0\mu^{-1}, \mu^{-1}, 0.5\mu^{-1}$ and $0.25\mu^{-1}$.

Table 1
Mean number of components under repair versus $\theta^{-1}$ and $K$

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\theta^{-1} = 0.25\mu^{-1}$</th>
<th>$\theta^{-1} = 0.5\mu^{-1}$</th>
<th>$\theta^{-1} = \mu^{-1}$</th>
<th>$\theta^{-1} = 2.0\mu^{-1}$</th>
<th>$\theta^{-1} = 4.0\mu^{-1}$</th>
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</tr>
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</table>
**Example 2.** Here we take $N = 20$, $\mu = 10.0$ and $\theta = 4.0$, thus satisfying $\theta^{-1} = 2.5 \mu^{-1}$. Figs. 4–6 show the influence of $K$ on the steady-state probability that the system is down, the mean number of components under repair and $\mu_{DT}$, respectively, for four choices of $k \in \{5, 10, 15, 20\}$.

As intuition tells us, Fig. 4 shows that, for fixed $k$, the probability that the system is down is a decreasing function of $K$, whose higher magnitudes are associated with higher values of $k$. In Fig. 5, the mean number of components under repair is clearly an increasing function of $K$ with the smallest magnitudes in the case of a series system. The mean downtime $\mu_{DT}$ is shown to be a non-increasing function of $K$ (see Fig. 6). This seems to be true for all values of $k$. Although the values of $\mu_{DT}$ seem to be moderate, its variability is more notable for the series system.

![Fig. 4. Steady-state probability that the system is down versus $K$ for (from top to bottom) $k = 20, 15, 10$ and 5.](image)

![Fig. 5. Mean number of components under repair versus $K$ for (from top to bottom) $k = 5, 10, 15$ and 20.](image)

![Fig. 6. $\mu_{DT}$ versus $K$ for (from top to bottom) $k = 5, 10, 15$ and 20.](image)
We stress that Figs. 4–6 show very special trends (seemingly constant or linear increasing) for the descriptors in the cases $k \in \{5, 10, 15\}$. Other experiments not reported in the paper provided identical conclusions for other system parameters and values of $k$, with the exception of the case $k = N$, that is, a series system.

**Example 3** and **Example 4** show how the behavior of the system depends on the mean delivery time $\theta^{-1}$.

**Example 3.** In this example, we fix the pair $(k, N) = (40, 50)$ to study the effect of $\theta^{-1}$ on the mean number of components under repair and $\mu_{\text{TTF}}$, respectively, for five choices of $K \in \{0, 3, 5, 7, 10\}$.

In Fig. 7, it is shown that the mean number of components under repair is an increasing function of $K$, which is essentially insensitive with respect to $\theta^{-1}$. The measure $\mu_{\text{TTF}}$ plotted in Fig. 8 decreases as a function of $\theta^{-1}$, for fixed values of $K \in \{3, 5, 7, 10\}$, and it remains constant when $K = 0$. Furthermore, for fixed $\theta^{-1}$, $\mu_{\text{TTF}}$ is seen to increase with increasing values of $K$.

In Fig. 7, we observe that the mean number of components under repair is always greater than the value $(N - k) + K$, from which it follows that the mean number of components working is less than $K$. This remark may induce us to think that the system is at failure state at every time, which is erroneous. For example, in the case $K = 5$, the steady-state probability that the system is down is not greater than 0.75235. Therefore, the mean number of components under repair does not capture all of the information regarding the steady-state dynamics of the system.

**Example 4.** In this example by fixing $N = 20$ and $\mu = 10.0$, we choose several values of the pair $(k, K) \in \{(5, 1), (10, 2), (15, 3), (20, 4)\}$. In Figs. 9–11, we examine the impact of the mean delivery time on $\mu_{\text{DT}}$, the mean number of spares used and the mean number of requests for spares pending, respectively.

In Fig. 9, the mean downtime $\mu_{\text{DT}}$ behaves as a seemingly constant function of $\theta^{-1}$ in the cases $(k, K) = (5, 1), (10, 2)$ and $(15, 3)$. In contrast, for the series system (i.e., $(k, K) = (20, 4))$, the measure $\mu_{\text{DT}}$ increases with increasing values of $\theta^{-1}$. Fig. 10 plots the mean number of spares used as a decreasing function of $\theta^{-1}$, with a dramatically different behavior for the case $(k, K) = (20, 4)$. Finally, we notice in Fig. 11 that the mean number of requests for spares pending increases with increasing values of the mean delivery time $\theta^{-1}$. Its highest values are associated with the choice $(k, K) = (20, 4)$.

We display in **Example 5** some more numerical results to bring out the effect of the variability in the repair times with significant delivery times.
Example 5. In Tables 2–4, we list values of $\mu_{TTF}$, $\mu_{DT}$ and $\mu_{BPR}$ for various choices of the repair time distributions. We consider models with $\lambda^{-1} = 1.0$, $\mu^{-1} = 0.1$, $p_i = 0.9^{1-k}$, for $i \in \{k,k+1, \ldots, N\}$, $(k,N) = (10,15)$ and Erlang (Erl), exponential (Exp) and $H_2$ repair time distributions. Specifically, we take exponentially distributed repair times of rate 0.1, and Erlang repair times consisting of 5 phases, each taking an independent, identically and exponentially distributed amount of time with rate 50.0. For the case $H_2$, we take the mixture probabilities to be $(p,1-p) = (0.7,0.3)$ with rates, respectively, given by $(v_1,v_2) = (14.5,5.8)$. 

Fig. 9. $\mu_{DT}$ versus $\theta^{-1}$ for (from top to bottom) $(k,K) = (5,1), (10,2), (15,3)$ and (20,4).

Fig. 10. Mean number of spares used versus $\theta^{-1}$.

Fig. 11. Mean number of requests for spares pending versus $\theta^{-1}$ for (from top to bottom) $(k,K) = (20,4), (15,3), (10,2)$ and (5,1).
preceding examples, which were made for the
also smaller than those for

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The entries of Tables 2–4 show that the mean values \(\mu_{\text{STF}}, \mu_{\text{OT}}\) and \(\mu_{\text{HPR}}\) always exhibit the same behavior as functions of \(\theta^{-1}\) and \(K\), irrespective of the repair time distribution. Thus, for the sake of brevity, the reader is referred to our comments in preceding examples, which were made for the \(H_2\) repair times.

We remark that the values of \(\mu_{\text{OT}}\) for Erlang repair times seem to be smaller than those for the exponential case, which are also smaller than those for \(H_2\) repair times, once we fix \(\theta^{-1}\) and \(K\). With the exception of a few examples with \(K = 0\) and 1, the
highest values of $\mu_{\text{err}}$ are associated with the Erlang case, whereas the smallest values correspond to the $H_2$ case. For fixed values of $h/C_0$ and $K$, we observe that the model with Erlang repair times exhibits higher expected values $l_{\text{BPR}}$. Differences among the three repair time distributions under study become more apparent for higher values of $K$.

Finally, in Example 6 we focus on the influence of the probabilities $p_i$ when they depend on the mean delivery time $h/C_0$.

**Example 6.** We fix the pair $(k, N) = (40, 50)$, the mean lifetime $\lambda^{-1} = 1.0$ and for the repair times consider the $H_2$ distribution of Example 5.

Here, we take $p_i = (f(\theta))^{i-k}$, for $i \in \{k, k + 1, \ldots, N\}$, where the function $f(\theta)$ is chosen as $f(\theta) = 0.8\theta^{-1}$ in Scenario I, and $f(\theta) = 1.0 - 0.8\theta^{-1}$ in Scenario II. Hence, $f(\theta)$ tends to 0 and 1 as $\theta^{-1} \to 0$ under Scenarios I and II, respectively.

Similar to Fig. 8, the measure $l_{\text{err}}$ is plotted in Figs. 12 and 13 for Scenarios I and II, respectively. We first observe that, for fixed values of $\theta^{-1}$, $l_{\text{err}}$ is an increasing function of $K$. As a function of $\theta^{-1}$, the descriptor remains constant for the case $K = 0$. These behaviors are in agreement with those remarked in Example 3, where $f(\theta)$ is a constant given by 0.9.

In contrast, the cases $K \in \{3, 5, 7, 10\}$ result in notably different shapes under Scenarios I and II. Specifically, $l_{\text{err}}$ increases with increasing values of $\theta^{-1}$ in Scenario I, and it decreases as a function of $\theta^{-1}$ in Scenario II. Thus, increasing the mean deliv-
ery times yields smaller values of $\mu_{TTF}$, provided the probability $p_i$ of requesting spares increases as the mean delivery time becomes smaller. On the other hand, if $p_i$ decreases (as $\theta^{-1}$ tends to 0), we observe higher values of $\mu_{TTF}$ as $\theta^{-1}$ increases.

The mean number of components under repair, unlike $\mu_{TTF}$, seems to be insensitive with respect to the mean delivery time and the probability of requesting spares. Further numerical experience indicates this measure, for both Scenarios I and II, to be almost identical to the ones plotted in Fig. 7.

6. Concluding remarks

In this paper we consider a $k$-out-of-$N$ system with $K$ spares to extend the life time of the system. Failed components are sent to a repair facility attended by a repairman, and after repair they are sent to the inventory containing the spares. The repair times are assumed to be of phase type. The replacement of failed components with spares from the inventory is done using a probabilistic rule and the delivery times of spares are assumed to be exponentially distributed. Using finite QBD process, the reliability system is studied in steady-state and a number of illustrative examples to bring out the qualitative nature of the system is discussed.

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References