Long-time limit of world subway networks

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We study the temporal evolution of the structure of the world’s largest subway networks. We show that, remarkably, all these networks converge to a shape which shares similar generic features despite their geographic and economic differences. This limiting shape is made of a core with branches radiating from it. For most of these networks, the average degree of the core has a value of order 2.5, slowly increases with time and displays small fluctuations. The current proportion of branches represents about 40% of the total number of stations and the average diameter of branches is about twice the average radial extension of the core. Spatial measures such as the number of stations at a given distance to the barycenter display a first regime growing as $r^2$ followed by another regime with different exponents. These results – which are difficult to interpret in the framework of fractal geometry – confirm and find a natural explanation in the core and branches picture: the first regime corresponds to a uniform core, while the second regime is controlled by the interstation spacing on branches. The existence of a unique network shape in the temporal limit suggests the existence of dominant, universal mechanisms governing the evolution of these structures.

INTRODUCTION

Transportation systems, especially mass transit, determine the extent to which cities are able to expand. In a world where more than 50% of the population lives in urban areas [1], and where individual transportation increases in cost as cities grow larger, mass transit and in particular, subway networks, are central to the evolution of cities, their spatial organization [2, 3] and dynamical processes occuring in them [4, 5]. Both casual and informed analysis of city size suggests that the larger the city, the more likely it is to have some form of mass transit system and once a population threshold between 1 and 5 million persons has been passed, the likelihood of a city having a subway system is extremely high.

For some cities, subway systems have existed for more than a century. Fascination with the apparent diversity of their structure has led to many studies and to particular abstractions of their representation in the design of idealized transit maps [6]. However, subway networks are an intrinsic response for transportation needs in large urban areas, and despite their apparent diversity, they are in principle all driven by the same mechanisms. We would thus expect general, universal features, whose detection and characterization require us to understand the evolution of these spatial structures. Indeed, subway networks are spatial [7] in the sense that they form a graph where stations are the nodes and links represent rail connections. We now understand quite well how to characterize a spatial network but we still lack tools for studying their temporal evolution. The present article tackles this problem, proposing various measures for these time dependent, spatial networks.

Here we focus on the largest networks in major world cities and thus ignore currently developing, smaller networks in many mid-sized cities. We thus consider most of the largest metro networks (with at least one hundred stations) which exist in major world cities. These are: Barcelona, Berlin, London, Madrid, Mexico, Moscow, New York City (NYC), Osaka, Paris, Seoul, Shanghai, and Tokyo (a sample of these networks is shown in Fig. 1). Additionally, we focus on urban subway systems and do not consider longer-distance heavy and light-rail commuting systems, such as RER in Paris or overground NetworkRail in London.

Static properties of transportation networks had been studied for many years [8] and in particular simple connectivity properties were studied in [9] and fractal aspects were considered in [10]. With the recent availability of new data, studies on transportation systems have accelerated [7] and this is particularly so for subway systems [11,19]. These studies have revealed some significant similarities between different networks, despite differences in their historical development and in the cultures and economies in which they have been developed. In particular, the average shortest path seems to scale
with the number of stations and the average clustering coefficient is large, consistent with general results associated with two-dimensional spatial networks. In [12], a strong correlation between the number of stations and population size has been observed for 22 Polish cities, but such correlation is not observed at the world level [17].

Our empirical analysis of the evolution of these transportation networks is in line with approaches developed in the 1970’s (see [20] and references therein) but we take advantage here of recent progress made in the understanding of spatial networks in general and new historical data sources which provide us with detailed chronologies of how these networks have developed.

Data

The network topologies at various points in time were built using two main data sources. First, current network maps as of 2009 were used to define lines for each network, and then to define line-based topologies, i.e. which station(s) follow(s) which other station(s) on each line. This information was then combined with opening dates for lines and stations. This second type of data has been gathered into the subway graph corresponding to year \( t \). Note that we used 2009 topologies as it was relatively difficult to find and also process network maps for all these networks for each year of existence. As a result, topologies for any given year before 2009 may overlook topology features pertaining to station or line closures: for instance, a station which existed between 1950 and 1970 and which remained closed until now will not appear in any of our network datasets. We suggest however that the effect of this bias is limited: on one hand, generally few stations undergo closure in the course of the network evolution; on the other hand, these stations are rarely hubs, most often intermediary stations (of degree two), thus their non-inclusion bears little topological impact.

**FIG. 1:** A sample of large subway networks in large urban areas, all displaying a core and branches structure. From left to right and top to bottom: Shanghai, Madrid, Moscow, Tokyo, Seoul, Barcelona (Figures from Wikimedia Commons [22]).

### SOME STATIC PROPERTIES

The main characteristics of the networks we have chosen are shown in Table I where it is clear that Shanghai and Seoul are the most recent subway networks experiencing a most rapid expansion that has elevated them to amongst the largest networks in the world. We also observe that the number of different lines appears to increase incrementally with the number of stations and that on average for these world networks, there are approximately 18 stations per line. Also, the average interstation distance is on average \( \overline{\ell_1} \approx 1\) km with Moscow showing the longest one (1.67kms) and Paris displaying the shortest one (570 meters). These fluctuations can probably be explained in terms of cost, planning and the different historical paths of these networks. Other quantities such as the average number \( P \) of individual served by one station (as \( P = \rho \ell_2^2 \) where \( \rho \) is the population density) can be computed and in this case follows the density: for decreasing densities the number of people dependent on a station also decreases. With one major exception, however, the urban area of Paris has an average density comparable to Shanghai but has a very dense network which gives a number of individuals per station about 5 times less than the one for Shanghai. More generally speaking, many parameters such as the population density, land use activity distribution, and traffic are important drivers in the evolution of those networks, but we will focus in this first study on the characterization of these networks in terms of space and topology, independently of other socio-economical considerations.

In order to get some initial insight into the topology of these networks, one can first compare the total length \( \ell_P \) of these networks to the corresponding quantity computed for a regular graph \( \ell_P^{GR} \) with same average degree, area, and number of stations. For a regular graph of degree \( \langle k \rangle \), and constituted of \( N \) nodes dispersed uniformly,
indicators, which we consider to be the most informative. We thus have to compute more globally structured indicators of various quantities for the results are too noisy. Networks have a relatively small number of stations (allways smaller than 500) which implies that we cannot use useful information from them. In addition, the largest networks have a relatively small number of stations (always smaller than 500) which implies that we cannot expect to extract useful information from the distributions of various quantities for the results are too noisy. We thus have to compute more globally structured indicators which are, however, sensitive to the usually small temporal variations associated with these networks. In the following, we will focus on a certain number of these indicators, which we consider to be the most informative at this point.

Finally, we will focus in this study on purely spatial and topological properties: we will consider the evolution in space of these subway networks and we will not consider any other parameters which might be used to characterize urban growth. We believe that our study is a first step towards the integration of the most important factors into this research and despite its simplicity in that we focus almost entirely on geometrical attributes, we think that the evolution of the topology encodes many different factors and that its study can point to some important general mechanisms governing the evolution of these networks.

### NETWORK DYNAMICS

In order to get an initial impression of the dynamics of these networks, we first estimate the simplest indicator $v = dN/dt$ which represents the number of new stations built per year. From the instantaneous velocity, we can compute the average velocity over all years. This average can however be misleading as there are many years where no stations are built and thus we describe this by the fraction of ‘inactivity’ time $f$. We provide results for the networks considered in the Table II from which some interesting facts are revealed.

<table>
<thead>
<tr>
<th>City</th>
<th>$t_0$</th>
<th>$\bar{v}$</th>
<th>$\sigma_v$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokyo</td>
<td>1927</td>
<td>1.8</td>
<td>3.4</td>
<td>58%</td>
</tr>
<tr>
<td>Seoul</td>
<td>1974</td>
<td>11.2</td>
<td>14.9</td>
<td>20%</td>
</tr>
<tr>
<td>Mexico</td>
<td>1969</td>
<td>3.7</td>
<td>5.9</td>
<td>55%</td>
</tr>
<tr>
<td>New York city</td>
<td>1878</td>
<td>3.3</td>
<td>8.3</td>
<td>68%</td>
</tr>
<tr>
<td>Shanghai</td>
<td>1995</td>
<td>14.9</td>
<td>20.2</td>
<td>31%</td>
</tr>
<tr>
<td>Moscow</td>
<td>1936</td>
<td>1.7</td>
<td>1.9</td>
<td>43%</td>
</tr>
<tr>
<td>London</td>
<td>1863</td>
<td>2.3</td>
<td>3.8</td>
<td>48%</td>
</tr>
<tr>
<td>Paris</td>
<td>1900</td>
<td>2.6</td>
<td>5.1</td>
<td>60%</td>
</tr>
<tr>
<td>Madrid</td>
<td>1919</td>
<td>2.3</td>
<td>4.6</td>
<td>59%</td>
</tr>
<tr>
<td>Berlin</td>
<td>1901</td>
<td>1.6</td>
<td>3.3</td>
<td>65%</td>
</tr>
<tr>
<td>Barcelona</td>
<td>1914</td>
<td>1.4</td>
<td>4.8</td>
<td>78%</td>
</tr>
<tr>
<td>Osaka</td>
<td>1934</td>
<td>1.4</td>
<td>4.1</td>
<td>79%</td>
</tr>
</tbody>
</table>

Table II: $t_0$ is the initial year considered here for the different subways networks. $\bar{v}$ is the average velocity (number of stations built per year), $\sigma_v$ is the standard deviation of $v$, and $f$ is the fraction of years of inactivity (no stations built).

For most of these networks the average velocity is in a small range (typically $\bar{v} \in [1.7, 3.7]$) except for Seoul and Shanghai which are more recently developed networks. This is however an average velocity and we observe that (i) for all networks, larger velocities occur at earlier stages of the network and (ii) large fluctuations occur from one year to another. Interestingly, the fraction of inactivity time (i.e. the time when no stations are built) is similar for all these networks with an average of about 58%. We also show in Fig. 2 the time evolution of the percentage of new stations for each decade.

### TABLE I

<table>
<thead>
<tr>
<th>City</th>
<th>$N_L$</th>
<th>$N$</th>
<th>$\ell_1$</th>
<th>$\ell_T$</th>
<th>$\ell_T/\ell^{reg}_T$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokyo</td>
<td>13</td>
<td>229</td>
<td>1.07</td>
<td>311</td>
<td>0.18</td>
<td>34</td>
</tr>
<tr>
<td>Seoul</td>
<td>9</td>
<td>392</td>
<td>1.39</td>
<td>609</td>
<td>0.39</td>
<td>38</td>
</tr>
<tr>
<td>Mexico</td>
<td>11</td>
<td>147</td>
<td>1.04</td>
<td>170</td>
<td>0.15</td>
<td>39</td>
</tr>
<tr>
<td>New York city</td>
<td>24</td>
<td>433</td>
<td>0.78</td>
<td>373</td>
<td>0.12</td>
<td>36</td>
</tr>
<tr>
<td>Shanghai</td>
<td>11</td>
<td>148</td>
<td>1.47</td>
<td>233</td>
<td>0.21</td>
<td>61</td>
</tr>
<tr>
<td>Moscow</td>
<td>12</td>
<td>134</td>
<td>1.67</td>
<td>260</td>
<td>0.16</td>
<td>71</td>
</tr>
<tr>
<td>London</td>
<td>11</td>
<td>266</td>
<td>1.29</td>
<td>397</td>
<td>0.20</td>
<td>47</td>
</tr>
<tr>
<td>Paris</td>
<td>16</td>
<td>299</td>
<td>0.57</td>
<td>205</td>
<td>0.18</td>
<td>38</td>
</tr>
<tr>
<td>Madrid</td>
<td>13</td>
<td>209</td>
<td>0.90</td>
<td>215</td>
<td>0.42</td>
<td>46</td>
</tr>
<tr>
<td>Berlin</td>
<td>10</td>
<td>170</td>
<td>0.77</td>
<td>141</td>
<td>0.30</td>
<td>60</td>
</tr>
<tr>
<td>Barcelona</td>
<td>11</td>
<td>128</td>
<td>0.72</td>
<td>103</td>
<td>0.32</td>
<td>38</td>
</tr>
<tr>
<td>Osaka</td>
<td>9</td>
<td>108</td>
<td>1.12</td>
<td>137</td>
<td>0.88</td>
<td>43</td>
</tr>
</tbody>
</table>

Table I: List of various indicators (for the year 2009) for the major subway networks considered in this study (and sorted according to their metro population). $N_L$ is the number of lines, $N$ the number of physical stations, $\ell_1$ is the average inter-station distance, $\ell_T$ total route length, $\ell^{reg}_T$ the total route length for a regular graph with same average degree, area, and number of stations, and $\beta$ the final ratio between branch and core stations.

A simple calculation gives $\ell^{reg}_T \sim \frac{\langle k \rangle}{2} \sqrt{AN}$ (1) where $A$ is the area of the city. As shown in the Table I the ratio $\ell_T/\ell^{reg}_T$ varies from 0.12 to 0.88, has an average of order 0.29 and displays essentially three outliers. First, Osaka has a very large value indicating a highly reticulated structure. In contrast, NYC and Mexico City have a much smaller value ($\approx 0.1$) signaling a more heterogeneous structure. This observation is consistent with the fact that large population densities imply a more reticulated network as the value of $P$ for Osaka is very large and very small for NYC.

The total length and the comparison with a regular structure gives a first hint about the structure of these networks but we need other indicators in order to get a more accurate view of these networks. There exist many different indicators and variables that describe these networks and their evolution. An important difficulty thus lies in the choice of these indicators and how to extract useful information from them. In addition, the largest networks have a relatively small number of stations (always smaller than 500) which implies that we cannot expect to extract useful information from the distributions of various quantities for the results are too noisy. We thus have to compute more globally structured indicators which are, however, sensitive to the usually small temporal variations associated with these networks. In the following, we will focus on a certain number of these indicators, which we consider to be the most informative at this point.
FIG. 2: Evolution of the number of new stations (as a percentage of existing ones) for various large world subway networks. The relative percentage goes to zero for all these networks signalling the appearance of a stationary limit.

This figure shows convincingly that as these networks become large, then for a few decades, new stations represent an almost negligible percentage of existing ones. This first result anticipates the fact that these large networks reach some kind of limiting shape that we will characterize in the next section.

Core and branches structure

The large subway networks considered here thus converge to a long time limit where there is always an increasingly smaller percentage of new stations added through time. The remarkable point that we will show below is that all these networks, despite their geographical and economical differences, converge to a shape which exhibits several typical topological and spatial features. Indeed, by inspection, we observe that in most large urban areas, the network consists of a set of stations delimited by a ‘ring’ which constitute the ‘core’. From this core, quasi-one dimensional branches grow and reach out to areas of the city further and further from the core. In Fig. 1, we show these networks as they currently exist. We note here that the ring, which is defined topologically as the set of core stations which are either at the junction of branches or on the shortest geodesic path connecting these junction stations, exists or not as a subway line. For instance, for Tokyo, there is a such a circular line (called the Yamanote line), while for Paris the topological ring does not correspond to a single line.

More formally, branches are defined as that set of stations which are iteratively built from a “tail” station, or a station of degree 1. New neighbors are added to a given branch as long as their degree is 2 – continuing the line, or 3 – defining a fork. In this latter case, the aggregative process continues if and only if at least one of the two possible new paths stemming from the fork is made up of stations of degree 2 or less.

The general structure can schematically be represented as in Fig. 3.

FIG. 3: Schematic structure of subway networks. A large ‘ring’ encircles a core of stations. Branches radiate from the core and reach further areas of the urban system. The branches are essentially characterized by their size (parameter $\beta(t)$, Eq. 2), and their spatial extension (parameter $\eta(t)$ in Eq. 3). The core is characterized by its average degree $(\langle k_{\text{core}} \rangle(t)$ defined in Eq. 4), its number of station $N_C(t)$, its size $r_C(t)$, and the parameter $\alpha(t)$ (Eq. 5).

We first characterize this branch and core structure with the parameter $\beta(t)$ defined as

$$\beta(t) = \frac{N_B(t)}{N_B(t) + N_C(t)}$$

where $N_B(t)$ and $N_C(t)$ respectively represent the number of stations on branches and the number of stations in the core at time $t$.

We can also characterize a little further the structure of branches. Their topological properties are trivial and their complexity resides in their spatial structure. We can then determine the average distance from the geographic barycenter of the city to all core and branches stations, respectively: $D_C(t)$ and $D_B(t)$. This last distance provides information about the spatial extension of
the branches when we can form the ratio \( \eta(t) \)
\[
\eta(t) = \frac{D_B(t)}{D_C(t)}
\]
which gives a spatial measure of the amount of extension of the branches.

We also need information on the structure of the core. The core is a planar (which is correct at a good accuracy for most networks), spatial network and can be characterized by many parameters \([7]\). It is important to choose those which are not simply related but ideally which represent different aspects of the network (such as those proposed in the form of various indicators, see for example \([7, 8, 21]\)). At each time step \( t \), we will characterize the core structure by the following two parameters. The first parameter is simply the average degree of the core which characterizes its ‘density’
\[
\langle k_{\text{core}} \rangle(t) = \frac{2E_{\text{C}}(t)}{NC(t)}
\]
where \( NC(t) \) is the number of core nodes and \( E_{\text{C}}(t) \) the number of its edges. The average degree is connected to the standard index \( \gamma(t) = E_{\text{C}}(t)/(3NC(t) - 6) \) where the denominator is the maximum number of links admissible for a planar network \([8]\). Another standard index – the \( \alpha(t) \) index – is defined as the number of elementary loops (the cyclomatic number) divided by the corresponding maximum number for a planar network \([8]\)
\[
\alpha(t) = \frac{E_{\text{C}}(t) - NC(t) + 1}{2NC(t) - 5}
\]
The number of loops is related to the number of different paths going from one station to another. It thus characterizes the redundancy of the network and its resilience to failure \([8, 18]\). Indeed, a tree graph can be disconnected in two parts by the failure of a single link while the existence of loops and thus many paths, allows the network to keep the traffic flowing even if a link (or more) is failing.

Once we know for example the parameter \( \alpha(t) \) which characterizes the core and the parameter \( \eta(t) \) which characterizes the relative spatial extension of branches, we have key information on the intertwinenment of both topological and geographical features in such “core/branch” networks.

**Time evolution of \( \beta, \langle k_{\text{core}} \rangle(t) \text{ and } \eta(t); \text{ convergence}**

The historical development of these networks is very different from one city to another and representing the evolution of a specific quantity versus time would probably not be particularly meaningful. Similarly, city networks often experience significant development in some particular years, while they rather experience little or no evolution for the rest of the time. In order to be able to compare the networks across periods and cities, we propose to study their evolution in terms of the number of stations that are constructed.

We first observe in Fig. 4 that most networks converge to a value of \( \beta \) in a small range \([35\%, 45\%]\), except for Berlin, Moscow, and Shanghai which display large values of \( \beta(t) \) signalling the overrepresentation of the subway network function to connect far remote suburbs to the center.

In Fig. 6 we show the average degree of the core \( \langle k_{\text{core}} \rangle(t) \text{ and } \eta(t) \). In this figure, we see that for the networks at large times (ie. 2009), the average degree of the core \( \langle k_{\text{core}} \rangle \) is of order \( 2.5 \pm 0.2 \) and the value of \( \eta \) is \( 2.3 \pm 0.6 \). We note that this value of 2.5 is relatively small and indicates that most core stations belong to one single line with few that actually allow connections.

At this point, we thus observe a temporal evolution of the parameters \( \beta(t), \langle k_{\text{core}} \rangle(t) \text{ and } \eta(t) \). We show in Fig. 6 the evolution of the relative dispersions (standard deviations around the mean) \( \delta_\beta, \delta_\eta \text{ and } \delta_k \) for \( \beta(t), \eta(t), \text{ and } \langle k_{\text{core}} \rangle(t) \) versus the fraction of stations (compared to the current networks). In this figure, we observe that the relative dispersion of these different parameters are small. The relative dispersion of \( \beta(t) \) is decreasing for the last 80 years, supporting the idea that there might be a ‘universal’ limiting value \( \beta_\infty \approx 40\% \).

The relative dispersion for \( \eta(t) \) is on average constant but we observe in the last half of the temporal evolution, a small decrease which is also consistent with a limiting value (whose average so far is \( \eta_\infty \approx 2.3 \)).

Finally, the relative dispersion of \( \langle k_{\text{core}} \rangle(t) \) is always small and approximately constant showing that the fluctuations among different networks are also small. In this
case however, Fig. 5 shows that $\langle k_{\text{core}} \rangle(t)$ increases very slowly, pointing to a mild yet continuing densification of the core, even after a long period of time. The fact that $\langle k_{\text{core}} \rangle$ is approximately the same for all networks seem to indicate that the fraction of connecting nodes (i.e., with $k \geq 2$) is a constant. In these networks, there is an obvious cost associated with large value of $k \geq 4$ and such a constant fraction could be due to some economical balance between the cost and the ease to navigate on these networks.

These results thus suggest that these large subway networks may converge to a long time limiting network largely independent of their historical and geographical differences. So far, we can characterize the ‘shape’ of this long time limiting network with values of $\beta_\infty \approx 40\%$, $\eta_\infty \approx 2.3$, and a core whose average degree is slowly increasing and is equal to $\langle k_{\text{core}} \rangle_\infty \approx 2.5$ for the year 2009.

**Balance between the core density and the branch structure**

Even if it seems that the values of various indicators converge with the size of the networks, we still have appreciable variations. For example $\eta(t)$ varies from $\approx 1$ to $\approx 3$ and exhibits a relatively constant and not negligible relative dispersion (Fig. 6). It is thus important to understand the remaining differences between these networks. To achieve this goal, we focus on the relation between $\eta(t)$ which characterizes the spatial extension of the branches relatively to the core, and the parameter $\alpha(t)$ which indicates how well connected the core is (Eq. 5).

We focus on the ‘final’ values of these parameters obtained for 2009 for the various networks and we obtain the plot shown in Fig. 7.

From this figure, we first see that $(\eta(t), \alpha(t))$ ranges from $(\approx 1.4, \approx 0.07)$ for NYC up to $(\approx 3.3, \approx 0.3)$ for Moscow which is indeed a highly ramified network with a very dense core.

Very roughly speaking, we first observe that when $\eta$ increases so does $\alpha$, or in other words when longer branches exists, the core is also denser. This property might simply reflect the idea that there is always a balance in the planner’s budget between constructing longer branches and adding stations or lines in the core.

At a finer level, we observe on this figure that two groups of networks with similar properties also emerge. The first group comprises Berlin, Shanghai, and Seoul...
which are remarkably close to each other: $\alpha(t)$ is of order $0.098 \pm 0.005$ and $\eta(t) \approx 2.84 \pm 0.1$. These networks seem a little unbalanced with respect to the others, with a less dense core as would be expected for this level of branches-to-core ratio $\eta(t)$. We can also conjecture that these networks result from a rapid evolution of branches and that a probable future evolution of these networks would involve a densification of their cores. We also see that NYC, Barcelona and Mexico are at the same level of $\alpha(t)$ but with smaller values of $\eta(t)$ showing a firmer equilibrium between core density and branches. In the ‘intermediate’ region, we observe a second group consisting of London, Madrid, and Paris with $\eta(t) \approx 2.0$ and $\alpha(t) \approx 0.12$.

**SPATIAL ORGANIZATION OF THE CORE AND BRANCHES**

Following earlier studies on the fractal aspects of subway networks [10], we can inspect the spatial subway organization by considering the number of stations $N(r)$ at a distance less than or equal to $r$, where the origin of distances is the barycenter of all stations. Interestingly, the barycenter of all stations is almost motionless, except in the case of NYC where the barycenter moves from Manhattan to the Queens and we will thus exclude from this study NYC. For the year 2009, the limiting shape made of a core and branches implies that there is a distance $r_C$ which determines the core. In practice, we can measure on the network the size $N_C$ of the core and we then define $r_C$ such that $N(r = r_C) = N_C$. For the various cities, we can easily compute the function $N = N(r)$ from which we can extract $r_C$ and we report the results in the Table III.

\[
\begin{array}{ccc}
\text{City} & N_C & r_C (\text{kms}) \\
Tokyo & 151 & 6.4 \\
Seoul & 243 & 11.6 \\
Mexico & 90 & 4.7 \\
Shanghai & 57 & 3.7 \\
Moscow & 39 & 5.9 \\
London & 142 & 7.3 \\
Paris & 186 & 4.2 \\
Madrid & 113 & 4.4 \\
Berlin & 68 & 5.5 \\
Barcelona & 57 & 3.5 \\
Osaka & 46 & 3.6 \\
\end{array}
\]

TABLE III: For each city, we compute the number of stations in the core and from the numerical calculation of $N(r)$ we can estimate $r_C$ the size of the core (in kms) from $N(r = r_C) = N_C$.

Next, we can rescale $r$ by $r_C$ and $N(r)$ by $N_C$ and we then obtain the results shown in the Fig. 8.

This figure displays several interesting features. First, the short distance regime $r < r_C$ is well described by a behavior of the form $N(r) \sim \rho_C \pi r^2$ consistent with a uniform density $\rho_C$ of core stations. For larger distances, we observe another regime characterized by a different variation with $r$ with different exponents. A similar result was obtained earlier [10] where the authors observed for Paris that $N(r \gg r_C) \sim r^{0.5}$, a result that was at that time difficult to understand in the framework of fractal geometry.

Here we show that these regimes can be easily understood in terms of the core and branches model, with the additional factor that the spacing between consecutive stations is increasing with $r$. Within this picture, $N(r)$ is given by

\[
N(r) \sim \begin{cases} 
\rho_C \pi r^2 & \text{for } r \ll r_C \\
\rho_C \pi r_C^2 + N_B \int_{r_C}^{r} \frac{dr}{\Delta(r)} & \text{for } r \gg r_C 
\end{cases}
\]

where $N_B$ is the number of branches and $\Delta(r)$ is the

\[\int_{r_C}^{r} \frac{dr}{\Delta(r)}\]

for $r \gg r_C$.

FIG. 8: (a) Rescaled number of stations at distance $r$ from the barycenter as a function of the rescaled variable $r/r_C$ where $r_C$ is the size of the core defined as $N(r = r_C) = N_C$ (shown here in loglog). The dotted line represents a power law $r^2$ and serves as a guide to the eye. (b) Case of Moscow where the two regimes ($r \ll r_C$ and $r \gg r_C$) with their different exponents are visible (the dotted lines serve here as a guide to the eye).
average spacing between stations on branches at distance $r$ from the barycenter. For example, if the interstation spacing on the branches is constant, we obtain a large $r$ behavior of the form $N(r) \sim N_0 r^2$ which is observed in the case of Moscow (see Fig. 8b). More generally, the large distance behavior will be of the form

$$N(r \gg r_C) \sim r^{1-\tau}$$

where $\tau$ denotes the exponent governing the interspacing decay $\Delta(r) \sim r^\tau$. For most networks, $\Delta(r)$ is very noisy but rough fits are in agreement with the large $r$ behavior observed in Eq. (7). For Moscow we observe a behavior consistent with $\Delta(r) \simeq$ constant while for the other networks, we observe an increasing trend but an accurate estimate of $\tau$ is difficult to obtain, given the small variation range of $r$ (for example, a fit over the one decade available gives for Moscow $\tau \approx 0.05$ with $R^2 = 0.34$ and for Paris $\tau \approx 0.5$ with $R^2 = 0.74$ – in agreement with the result obtained in [10]). Despite the difficulty of obtaining accurate quantitative results, we observe a very good qualitative agreement between the behavior observed for $N(r)$ and $\Delta(r)$ which confirms our picture of a long time limit network shape made of a core and radial branches.

**DISCUSSION**

In summary, we have observed a number of similarities between different subway systems for the world’s largest cities, despite their geographical and historical differences.

First, we have shown that the largest subway networks in the world exhibit a similar temporal decrease of most fluctuations and converge to a similar structure. We identified and characterized the shape of this long time limiting graph, which is a structure made of core and branches, and appears to be relatively independent of the peculiar historical idiosyncrasies associated with the evolution of these particular cities. For large networks, we generally observe a universal fraction of branches of about 40% for most networks, and a ratio for the spatial extensions of branches to the core of about 2.3. The core of these different city networks has approximately the same average degree which is slowly increasing with time (equal to $\approx 2.5$ for the year 2009). In addition, this picture of a core with branches and an increasing spacing between consecutive stations on these branches is confirmed by spatial measurements such as the number of stations at a given distance $r$ and provides a natural interpretation to these measures.

These networks however have not yet entirely reached this long time limit, and still display dynamical fluctuations. Indeed, for some of the networks — such as Moscow, Seoul, and Shanghai — we observe larger differences with respect to the average, limiting network. In the case of Moscow, its core appears over-developed compared to its branches. This network has resulted basically from a well-defined design and it is expected that it does not follow the same rules that govern networks evolving over a longer period which often appear to evolve in a slightly more self-organized manner. In the case of Seoul and Shanghai, it seems that their relatively young age could explain why they have not yet reached the long time limit. We can note here that the least expensive way for these almost mature networks to reach the well-balanced long time limit is by constructing the minimum number of stations and lines and this then suggests that Seoul and Shanghai need to increase their core density hence the value of $\alpha(t)$ (by adding inter-branches links for example) and Moscow needs to increase the number of its branches. It will be interesting to observe their future evolution.

The evolution of networks in general and urban networks in particular represents an exciting unexplored problem which mixes spatial and topological properties in unusual and often counterintuitive ways. We have elaborated here indicators based upon previous approaches. Other data such as population density, land use activity distribution, and traffic flows are likely to bring relevant information to this problem and would undoubtedly enrich our study. We believe however that the present approach represents an important step in our understanding and is crucial for the modeling of the evolution of urban networks. In particular, the existence of unique long-time limit topological and spatial features is a universal signature that fundamental mechanisms, independent of historical and geographical differences, govern the evolution of these transportation networks.

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