Robust Stabilization for a Class of Uncertain Discrete-time Switched Linear Systems

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1. Introduction

Switched systems are a class of hybrid systems consisting of several subsystems (modes of operation) and a switching rule indicating the active subsystem at each instant of time. In recent years, considerable efforts have been devoted to the study of switched system. The motivation of study comes from theoretical interest as well as practical applications. Switched systems have numerous applications in control of mechanical systems, the automotive industry, aircraft and air traffic control, switching power converters, and many other fields. The basic problems in stability and design of switched systems were given by (Liberzon & Morse, 1999). For recent progress and perspectives in the field of switched systems, see the survey papers (DeCarlo et al., 2000; Sun & Ge, 2005).

The stability analysis and stabilization of switching systems have been studied by a number of researchers (Branicky, 1998; Zhai et al., 1998; Margaliot & Liberzon, 2006; Akar et al., 2006). Feedback stabilization strategies for switched systems may be broadly classified into two categories in (DeCarlo et al., 2000). One problem is to design appropriate feedback control laws to make the closed-loop systems stable for any switching signal given in an admissible set. If the switching signal is a design variable, then the problem of stabilization is to design both switching rules and feedback control laws to stabilize the switched systems. For the first problem, there exist many results. In (Daafouz et al., 2002), the switched Lyapunov function method and LMI based conditions were developed for stability analysis and feedback control design of switched linear systems under arbitrary switching signal. There are some extensions of (Daafouz et al., 2002) for different control problem (Xie et al., 2003; Ji et al., 2003). The pole assignment method was used to develop an observer-based controller to stabilizing the switched system with infinite switching times (Li et al., 2003).

It should be noted that there are relatively little study on the second problem, especially for uncertain switched systems. Ji had considered the robust $H_\infty$ control and quadratic stabilization of uncertain discrete-time switched linear systems via designing feedback control law and constructing switching rule based on common Lyapunov function approach (Ji et al., 2005). The similar results were given for the robust guaranteed cost control problem of uncertain discrete-time switched linear systems (Zhang & Duan, 2007). Based on multiple Lyapunov functions approach, the robust $H_\infty$ control problem of uncertain continuous-time switched linear systems via designing switching rule and state feedback was studied (Ji et al., 2004). Compared with the switching rule based on common Lyapunov function approach (Ji et al., 2005; Zhang & Duan, 2007), the one based on multiple Lyapunov
functions approach (Ji et al., 2004) is much simpler and more practical, but discrete-time case was not considered.
Motivated by the study in (Ji et al., 2005; Zhang & Duan, 2007; Ji et al., 2004), based on the multiple Lyapunov functions approach, the robust control for a class of discrete-time switched systems with norm-bounded time-varying uncertainties in both the state matrices and input matrices is investigated. It is shown that a state-depended switching rule and switched state feedback controller can be designed to stabilize the uncertain switched linear systems if a matrix inequality based condition is feasible and this condition can be dealt with as linear matrix inequalities (LMIs) if the associated scalar parameters are selected in advance. Furthermore, the parameterized representation of state feedback controller and constructing method of switching rule are presented. All the results can be considered as extensions of the existing results for both switched and non-switched systems.

2. Problem formulation

Firstly, we consider a class of uncertain discrete-time switched linear systems described by

\[
\begin{align*}
    x(k+1) &= (A_{\sigma(k)} + \Delta A_{\sigma(k)})x(k) + (B_{\sigma(k)} + \Delta B_{\sigma(k)})u(k) \\
    y(k) &= C_{\sigma(k)}x(k)
\end{align*}
\]

where \( x(k) \in \mathbb{R}^n \) is the state, \( u(k) \in \mathbb{R}^m \) is the control input, \( y(k) \in \mathbb{R}^q \) is the measurement output and \( \sigma(k) \in \Xi = \{1, 2, \cdots, N\} \) is a discrete switching signal to be designed. Moreover, \( \sigma(k) = i \) means that the \( i \)-th subsystem \((A_i, B_i, C_i)\) is activated at time \( k \) (For notational simplicity, we may not explicitly mention the time-dependence of the switching signal below, that is, \( \sigma(k) \) will be denoted as \( \sigma \) in some cases). Here \( A_i, B_i \) and \( C_i \) are constant matrices of compatible dimensions which describe the nominal subsystems. The uncertain matrices \( \Delta A_i \) and \( \Delta B_i \) are time-varying and are assumed to be of the forms as follows.

\[
\begin{align*}
    \Delta A_i &= M_{ai}F_{ai}(k)N_{ai} \quad \Delta B_i = M_{bi}F_{bi}(k)N_{bi}
\end{align*}
\]

where \( M_{ai}, N_{ai}, M_{bi}, N_{bi} \) are given constant matrices of compatible dimensions which characterize the structures of the uncertainties, and the time-varying matrices \( F_{ai}(k) \) and \( F_{bi}(k) \) satisfy

\[
F_{ai}^T(k)F_{ai}(k) \leq I, F_{bi}^T(k)F_{bi}(k) \leq I \quad \forall i \in \Xi
\]

where \( I \) is an identity matrix.

We assume that no subsystem can be stabilized individually (otherwise the switching problem will be trivial by always choosing the stabilized subsystem as the active subsystem). The problem being addressed can be formulated as follows:

For the uncertain switched linear systems (1), we aim to design the switched state feedback controller

\[
u(k) = K_\sigma x(k)
\]

and the state-depended switching rule \( \sigma(x(k)) \) to guarantee the corresponding closed-loop switched system
\[ x(k+1) = [A_\sigma + \Delta A_\sigma + (B_\sigma + \Delta B_\sigma)K_\sigma]x(k) \] (5)

is asymptotically stable for all admissible uncertainties under the constructed switching rule.

3. Main results

In order to derive the main result, we give the two main lemmas as follows.

Lemma 1: (Boyd, 1994) Given any constant \( \varepsilon \) and any matrices \( M, N \) with compatible dimensions, then the matrix inequality

\[
MFN + N^T M^T M < \varepsilon MM^T + \varepsilon^{-1} N^T N
\]

holds for the arbitrary norm-bounded time-varying uncertainty \( F \) satisfying \( F^T F \leq I \).

Lemma 2: (Boyd, 1994) (Schur complement lemma) Let \( S, P, Q \) be given matrices such that \( Q = Q^T, P = P^T \), then

\[
\begin{bmatrix}
P & S^T \\
S & Q
\end{bmatrix} < 0 \iff Q < 0, P - S^T Q^{-1} S < 0.
\]

A sufficient condition for existence of such controller and switching rule is given by the following theorem.

**Theorem 1:** The closed-loop system (5) is asymptotically stable when \( \Delta A_i = \Delta B_i = 0 \) if there exist symmetric positive definite matrices \( X_i \in \mathbb{R}^{n \times n} \), matrices \( G_i \in \mathbb{R}^{n \times n} \), \( Y_i \in \mathbb{R}^{n \times n} \), scalars \( \varepsilon_i > 0 \ (i \in \Xi) \) and scalars \( \lambda_{ij} < 0 \ (i, j \in \Xi, \lambda_{ii} = -1) \) such that

\[
\begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} (G_i^T + G_i - X_i) & * & * & \ldots & * \\
A_i G_i + B_i Y_i & -X_i & * & \ldots & * \\
G_i & 0 & \lambda_{11}^{-1} X_1 & * & \ldots & * \\
G_i & 0 & 0 & \lambda_{22}^{-1} X_2 & \ldots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & * \\
G_i & 0 & 0 & 0 & 0 & \lambda_{nN}^{-1} X_N
\end{bmatrix} < 0 \ \forall i \in \Xi
\] (6)

is satisfied (\( * \) denotes the corresponding transposed block matrix due to symmetry), then the state feedback gain matrices can be given by (4) with

\[
K_i = Y_i G_i^{-1}
\] (7)

and the corresponding switching rule is given by

\[
\sigma(x(k)) = \arg\min_{i \in \Xi} \{x^T(k)X_i^{-1}x(k)\}
\] (8)

Proof. Assume that there exist \( G_i, X_i, Y_i, \varepsilon_i \) and \( \lambda_{ij} \) such that inequality (6) is satisfied. By the symmetric positive definiteness of matrices \( X_i \), we get

\[
(G_i - X_i)^T X_i^{-1} (G_i - X_i) \geq 0
\] (8)


which is equal to

\[ G_i^T X_i^{-1} G_i \geq G_i^T + G_i - X_i \]

It follows from (6) and \( \lambda_{ij} < 0 \) that

\[
\begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} G_i^T X_i^{-1} G_i & * & * \\
A_i G_i + B_i Y_i & -X_i & * \\
\Gamma_i & 0 & \Phi_i
\end{bmatrix} < 0
\]

where \( \Gamma_i = [G_i, G_i, \cdots G_i]^T \), \( \Phi_i = \text{diag}\{1 / \lambda_{i1} X_1, 1 / \lambda_{i2} X_2, \cdots, 1 / \lambda_{i(i-1)} X_{i-1}, 1 / \lambda_{i(i+1)} X_{i+1}, \cdots, 1 / \lambda_{iN} X_N\} \)

Pre- and post-multiplying both sides of inequality (9) by \( \text{diag}[G_i^{-1}, I, I]^T \) and \( \text{diag}[G_i^{-1}, I, I] \), we get

\[
\begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} X_i^{-1} & * & * \\
A_i + B_i K_i & -X_i & * \\
\Pi_i & 0 & \Phi_i
\end{bmatrix} < 0
\]

where \( \Pi_i = [I, I, \cdots I]^T \).

By virtue of the properties of the Schur complement lemma, inequality (10) is equal to

\[
\begin{bmatrix}
-X_i^{-1} + \sum_{j \in \Xi, j \neq i} \lambda_{ij} (X_i^{-1} - X_j^{-1}) & * \\
A_i + B_i K_i & -X_i
\end{bmatrix} < 0
\]

Letting \( P_i = X_i^{-1} \) and applying Schur complement lemma again yields

\[ (A_i + B_i K_i)^T P_i (A_i + B_i K_i) - P_i + \sum_{j \in \Xi, j \neq i} \lambda_{ij} (P_i - P_j) < 0 \]

Since \( P_i = X_i^{-1} (\forall i \in E) \), the switching rule (8) can be rewritten as

\[ \sigma(x(k)) = \arg \min_{i \in E} [x^T(k) P_i x(k)]. \]

By (13), \( \sigma(k) = i \) implies that

\[ x^T(k) (P_i - P_j) x(k) \leq 0, \quad \forall j \in \Xi, j \neq i. \]

Multiply the above inequalities by negative scalars \( \lambda_{ij} \) for each \( j \in \Xi, j \neq i \) and sum to get

\[ x^T(k) \left[ \sum_{j \in \Xi, j \neq i} \lambda_{ij} (P_i - P_j) \right] x(k) \geq 0 \]
Associated with the switching rule (13), we take the multiple Lyapunov functions $V(x(k))$ as

$$V_{\sigma(k)}(x(k)) = x^T(k)P_{\sigma(k)}x(k) \quad (16)$$

then the difference of $V(x(k))$ along the solution of the closed-loop switched system (5) is

$$\Delta V = V(x(k+1)) - V(x(k)) = x^T(k+1)P_{\sigma(k+1)}x(k+1) - x^T(k)P_{\sigma(k)}x(k)$$

At non-switching instant, without loss of generality, letting $\sigma(k+1) = \sigma(k) = i(i \in \Xi)$, and applying switching rule (13) and inequality (15), we get

$$\Delta V = x^T(k+1)P_i x(k+1) - x^T(k)P_i x(k) \leq x^T(k+1)P_i x(k+1) - x^T(k)P_i x(k) \leq 0 \quad (17)$$

It follows from (12) and (15) that $\Delta V < 0$ holds.

At switching instant, without loss of generality, let $\sigma(k+1) = j, \sigma(k) = i(i, j \in \Xi, i \neq j)$ to get

$$\Delta V = x^T(k+1)P_j x(k+1) - x^T(k)P_j x(k) \leq x^T(k+1)P_j x(k+1) - x^T(k)P_j x(k) \leq 0 \quad (18)$$

It follows from (17) and (18) that $\Delta V < 0$ holds. In virtue of multiple Lyapunov functions technique (Branicky, 1998), the closed-loop system (5) is asymptotically. This concludes the proof.

**Remark 1:** If the scalars $\lambda_{ij}$ are selected in advance, the matrices inequalities (19) can be converted into LMIs with respect to other unknown matrices variables, which can be checked with efficient and reliable numerical algorithms available.

**Theorem 2:** The closed-loop system (5) is asymptotically stable for all admissible uncertainties if there exist symmetric positive definite matrices $X_i \in \mathbb{R}^{n \times n}$, matrices $G_i \in \mathbb{R}^{m \times n}$, $Y_i \in \mathbb{R}^{m \times n}$, scalars $\epsilon_i > 0 \ (i \in \Xi)$ and scalars $\lambda_{ij} < 0 \ (i, j \in \Xi, \lambda_{ii} = -1)$ such that

$$\begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} (G_j^T + G_i - X_i) & * & * & * & * & \cdots & * \\
A_i G_i + B_i Y_i & \Theta_i & * & * & * & \cdots & * \\
N_{ai} G_i & 0 & -\epsilon_i I & * & * & \cdots & * \\
N_{bi} Y_i & 0 & 0 & -\epsilon_i I & * & \cdots & * \\
G_i & 0 & 0 & 0 & \lambda_{ij}^{-1} X_1 & * & \cdots & * \\
G_i & 0 & 0 & 0 & 0 & \lambda_{ij}^{-1} X_2 & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & * \\
G_i & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_{in}^{-1} X_N
\end{bmatrix} < 0 \ \forall i \in \Xi \quad (19)$$

is satisfied, where

$$\Theta_i = -X_i + \epsilon_i [M_{ai} M_{ai}^T + M_{bi} M_{bi}^T],$$

then the state feedback gain matrices can be given by (4) with

$$K_i = Y_i G_i^{-1} \quad (20)$$
and the corresponding switching rule is given by

\[ \sigma(x(k)) = \arg \min_{i \in \Xi} \{ x^T(k)X_i^{-1}x(k) \} \] (21)

Proof. By theorem 1, the closed-loop system (5) is asymptotically stable for all admissible uncertainties if there exist \( G_i, X_i, Y_i \) and \( \lambda_{ij} \) such that

\[
\begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} (G_i^T + G_i - X_i) & * & * \\
A_i G_i + B_i Y_i & \Theta_i & * \\
\Gamma_i & 0 & \Phi_i
\end{bmatrix} < 0
\] (22)

where \( \Gamma_i = [G_i, G_i, \ldots, G_i]^T \),

\[
\Phi_i = \text{diag}\{1 / \lambda_1 X_1, 1 / \lambda_2 X_2, \ldots, 1 / \lambda_{(i-1)} X_{i-1}, 1 / \lambda_{(i+1)} X_{i+1}, \ldots, 1 / \lambda_{iN} X_{iN}\},
\]

which can be rewritten as

\[
\tilde{A}_i + \tilde{M}_i \tilde{F}_i(k) \tilde{N}_i + \tilde{N}_i^T \tilde{F}_i^T(k) \tilde{M}_i^T < 0
\]

where

\[
\tilde{A}_i = \begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} (G_i^T + G_i - X_i) & * & * \\
A_i G_i + B_i Y_i & \Theta_i & * \\
\Gamma_i & 0 & \Phi_i
\end{bmatrix},
\]

\[
\tilde{M}_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
\tilde{F}_i(k) = \text{diag}(F_{ii}(k), F_{bi}(k)), \quad \tilde{N}_i = \begin{bmatrix} N_{ai} G_i & 0 & 0 \\ N_{bi} K_i & 0 & 0 \end{bmatrix}
\]

It follows from Lemma 1 and \( \tilde{F}_i^T(t) \tilde{F}_i^T(t) \leq I \) that

\[
\tilde{A}_i + \tilde{M}_i \tilde{F}_i(t) + \tilde{N}_i \tilde{N}_i < 0
\] (23)

By virtue of the properties of the Schur complement lemma, inequality (19) can be rewritten as

\[
\begin{bmatrix}
\sum_{j \in \Xi} \lambda_{ij} (G_i^T + G_i - X_i) & * & * & * \\
A_i G_i + B_i Y_i & \Theta_i & * & * \\
\Gamma_i & 0 & \Phi_i & * \\
N_{ai} G_i & 0 & 0 & -\varepsilon_i I \\
N_{bi} Y_i & 0 & 0 & 0 & -\varepsilon_i I
\end{bmatrix} < 0 \quad \forall i \in \Xi
\] (24)

It is obvious that inequality (24) is equal to inequality (19), which finished the proof.

Let the scalars \( \lambda_{ij} = 0 \) and \( X_i = X_j = X \), it is easily to obtain the condition for robust stability of the closed-loop system (5) under arbitrary switching as follows.
Corollary 1: The closed-loop system (5) is asymptotically stable for all admissible uncertainties under arbitrary switching if there exist a symmetric positive definite matrix $X_i \in \mathbb{R}^{n \times n}$, matrices $G_i \in \mathbb{R}^{m \times n}$, $Y_i \in \mathbb{R}^{n \times n}$, scalars $\epsilon_i > 0$ and such that

$$
\begin{bmatrix}
-G_i^T G_i + X_i & * & * \\
A_i G_i + B_i Y_i & \Theta_i & * \\
N_{ia} G_i & 0 & -\epsilon_i I \\
N_{bi} Y_i & 0 & 0 & -\epsilon_i I
\end{bmatrix} < 0 \quad \forall i \in \Xi
$$

(25)

is satisfied, where $\Theta_i = -X_i + \epsilon_i [M_{ia} M_{ia}^T + M_{bi} M_{bi}^T]$, then the state feedback gain matrices can be given by (4) with

$$
K_i = Y_i G_i^{-1}
$$

(26)

4. Example

Consider the uncertain discrete-time switched linear system (1) with $N = 2$. The system matrices are given by

$$
A_1 = \begin{bmatrix} 1.5 & 1.5 \\ 0 & -1.2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, M_{a1} = \begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}, N_{a1} = \begin{bmatrix} 0.4 & 0.2 \end{bmatrix}, M_{b1} = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}, N_{b1} = [0.2],
$$

$$
A_2 = \begin{bmatrix} 1.2 & 0 \\ 0.6 & 1.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, M_{a2} = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}, N_{a2} = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix}, M_{b2} = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}, N_{b2} = [0.1].
$$

Obviously, the two subsystems are unstable, and it is easy to verify that neither subsystem can be individually stabilized via state feedback for all admissible uncertainties. Thus it is necessary to design both switching rule and feedback control laws to stabilize the uncertain switched system. Letting $\lambda_{12} = -10$ and $\lambda_{21} = -10$, the inequality (19) in Theorem 1 is converted into LMIs. Using the LMI control toolbox in MATLAB, we get

$$
X_1 = \begin{bmatrix} 41.3398 & -8.7000 \\ -8.7000 & 86.6915 \end{bmatrix}, X_2 = \begin{bmatrix} 38.1986 & -8.6432 \\ -8.6432 & 93.8897 \end{bmatrix}
$$

$$
G_1 = \begin{bmatrix} 41.3415 & -8.6656 \\ -8.7540 & 86.4219 \end{bmatrix}, Y_1 = \begin{bmatrix} -51.2846 \\ -26.5670 \end{bmatrix},
$$

$$
G_2 = \begin{bmatrix} 38.1665 & -8.6003 \\ -8.6186 & 93.6219 \end{bmatrix}, Y_2 = \begin{bmatrix} -44.3564 \\ 54.4478 \end{bmatrix},
$$

$$
\epsilon_1 = 56.6320, \epsilon_1 = 24.3598
$$

With $K_i = Y_i G_i^{-1}$, the switched state feedback controllers are

$$
K_1 = \begin{bmatrix} -1.4841 & -1.1505 \end{bmatrix}, K_2 = \begin{bmatrix} -1.0527 & 0.4849 \end{bmatrix}
$$
It is obvious that neither of the designed controllers stabilizes the associated subsystem. Letting that the initial state is $x_0 = [-3, 2]$ and the time-varying uncertain $E_{ia}(k) = E_{ib}(k) = f(k)$ $(i = 1, 2)$ as shown in Figure 1 is random number between -1 and 1, the simulation results as shown in Figure 2, 3 and 4 are obtained, which show that the given uncertain switched system is stabilized under the switched state feedback controller together with the designed switching rule.

![Fig. 1. The time-varying uncertainty f(k)](image1)

![Fig. 2. The state response of the closed-loop system](image2)

5. Conclusion

This paper focused on the robust control of switched systems with norm-bounded time-varying uncertainties with the help of multiple Lyapunov functions approach and
matrix inequality technique. By the introduction of additional matrices, a new condition expressed in terms of matrices inequalities for the existence of a state-based switching strategy and state feedback control law is derived. If some scalars parameters are selected in advance, the conditions can be dealt with as LMIs for which there exists efficient numerical software available. All the results can be easily extended to other control problems ($H_2, H_\infty$ control, etc.).

![Switching Signal](image1)

**Fig. 3.** The switching signal

![State Trajectory](image2)

**Fig. 4.** The state trajectory of the closed-loop system

### 6. Acknowledgment

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7. References


