Stability and control of feedback systems with time delays

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Abstract

Feedback systems are important in applications, for example, optical feedback lasers, phase-locked frequency synthesizers and wave equations with feedback stabilization at the boundary, and the problem regarding sensitivity and robustness of the feedback system with respect to time delays has attracted a lot of attention. In this paper we continue the discussion on sensitivity and robustness of the feedback stabilization of neutral differential delay equations with respect to variation in the delays and with respect to a time delay in the feedback loop. Our main result shows that robustness of the stabilization actually depends on the radius of the essential spectrum of the semigroup associated with the equation.

1 Introduction

In the implementation of any feedback control system, it is very likely that time delays will occur. Therefore, it is of importance to understand the sensitivity of the control system with respect to the introduction of small delays in the feedback loop. For some systems, small delays lead to destabilization while other systems are robust with respect to small time delays, e.g., (Logemann, Rebarber and Weiss 1996, Logemann and Rebarber 1996, Logemann and Townley 1996). In (Hale and Verduyn Lunel 2001) we made a first attempt for a unifying theory that explains the underlying mechanisms in terms of the radius of the essential

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spectrum of the semigroup associated with the problem and we refer to this paper for a historical overview and for further references.

In this paper we continue the investigation of robustness of the feedback stabilization of neutral differential delay equations with respect to variation in the delays and with respect to a delay in the feedback loop. We explain that the difference operator associated with the neutral part of the equation determines whether stabilization by feedback control is a finite or an infinite dimensional problem. If the control problem turns out to be infinite dimensional, then we show that feedback stabilization of the system is sensitive to a small time delay in the feedback loop.

Before we introduce the class of neutral functional differential equations that will be considered in this paper, we need some preparations. Let \( C = C([-h,0],\mathbb{R}^n) \) denote the Banach space of continuous functions provided with the supremum norm, i.e., \( \|\varphi\| := \max_{-h < \theta \leq 0} |\varphi(\theta)| \) for \( \varphi \in C \). From the Riesz representation theorem, it follows that a bounded linear mapping \( L : C \to \mathbb{R}^n \) can be represented by a matrix function of bounded variation, i.e.,

\[
L\varphi = \int_{-h}^0 d\eta(\theta)\varphi(\theta),
\]

where \( \eta(\theta), -h \leq \theta \leq 0 \), is an \( n \times n \)-matrix whose elements are of bounded variation, normalized so that \( \eta \) is continuous from the left on \(( -h, 0 )\) and \( \eta(0) = 0 \). For a function \( x : [-h,\infty) \to \mathbb{R}^n \), we define \( x_t \in C \) by \( x_t(\theta) = x(t + \theta), -h \leq \theta \leq 0 \) and \( t \geq 0 \).

An initial value problem for a linear neutral functional differential equation (NFDE) is given by the following relation

\[
\frac{d}{dt}Dx_t = Lx_t, \quad t \geq 0,
\]

where \( D \) and \( L \) are bounded linear maps from \( C \) into \( \mathbb{R}^n \). The initial data for (2) are given by \( x_0 = \varphi \), where \( \varphi \) belongs to \( C \).

A continuous function \( x : [-h,\infty) \to \mathbb{R}^n \) is called a solution of the initial value problem (2) if \( x_0 = \varphi \), the function \( t \mapsto Dx_t \) is continuously differentiable on \(( 0,\infty )\) with a continuous right hand derivative at \( t = 0 \) and (2) is satisfied for \( t \geq 0 \).

If the map \( D : C \to \mathbb{R}^n \) is atomic at zero, that is,

\[
D\varphi = \varphi(0) - \int_{-h}^0 d\mu(\theta)\varphi(\theta),
\]

where \( \mu \) is continuous at zero, then the initial value problem (2) has a unique solution on \([-h,\infty)\) (Hale and Verduyn Lunel 1993).

The difference equation associated with (2)–(3) is given by \( DX_t = 0 \), i.e.,

\[
x(t) = \int_{-h}^0 d\mu(\theta)x(t + \theta).
\]
For any $\varphi \in \mathcal{C}$ for which $D\varphi = 0$, there is a unique solution $x(t)$ of (4) which is continuous on $[-h, \infty)$ and satisfies $x_0 = \varphi$.

The solution of (2) with initial data $x_0 = \varphi$ is denoted by $x(\cdot; \varphi)$. The state of the solution at time $t$ is the minimal amount of data needed in order to continue the solution of (2) forward in time starting at time $t$. In the present setting the state at time $t$ is precisely $x_t$ and we define the solution map $T(t) : \mathcal{C} \to \mathcal{C}$ to be the linear operator that maps the initial data onto the state at time $t$, i.e.,

$$ (T(t)\varphi)(\theta) = x_t(\theta; \varphi) = x(t + \theta; \varphi) \quad \text{for} \quad h \leq \theta \leq 0. $$

Here we have set up the problem with respect to the state space $\mathcal{C}$, but, depending on the applications, we can consider other state spaces as well. For example, the product space $M_p = \mathbb{R}^n \times L^p[-h,0]$, see (Burns, Herdman and Stech 1983).

In this case, we define a solution of (2) to be a locally integrable function such that $Dx_t$ is absolutely continuous on $[0, \infty)$ and (2) is satisfied for almost every $t \geq 0$. The solution of (2) with initial data $Dx_0 = c$ and $x_0 = \varphi$ for $(c, \varphi) \in M_p$, is denoted by $x(\cdot; c, \varphi)$. The solution map $S(t) : M_p \to M_p$ is given by

$$ S(t)(c, \varphi) = (Dx_t, x_t), \quad \text{where} \quad x_t = x_t(\cdot; c, \varphi). $$

The solution maps $T(t)$ and $S(t)$ are closely related. In fact, if we define the embedding $j : \mathcal{C} \to M_p$ by $j\varphi = (D\varphi, \varphi)$, then the solution map $T(t)$ corresponds to the restriction of $S(t)$ to the subspace $j\mathcal{C}$.

Notice that in the definition of a solution, we do only require differentiability properties on $Dx_t$ and not on the solution $x(\cdot; c, \varphi)$ itself. However, in applications we sometimes need solutions with more regularity. For example, if the feedback control depends on the derivative of the solution. This naturally leads to the state space $W^{1,p}[-h,0]$, the Sobolev spaces of all $L^1$-integrable functions that have a generalized derivative that belongs to $L^p$. It has been proved in (Henry 1974) that if $x_0 = \varphi$ with $\varphi \in W^{1,p}[-h,0]$, then (2) has a unique solution such that $x_t \in W^{1,p}[-h,0]$ for every $t \geq 0$. The corresponding solution map can again be obtained by the restriction of $S(t)$ to the image of $W^{1,p}[-h,0]$ under the mapping $\varphi \mapsto (D\varphi, \varphi)$.

The zero solution of (2) or (8) is stable if for any $\epsilon > 0$ there is a $\delta > 0$ such that $\|\varphi\| < \delta$ implies $\|x_t(\cdot; \varphi)\| \leq \epsilon$ for $t \geq 0$. The zero solution of (2) is asymptotically stable if $\|x_t(\cdot; \varphi)\| \to 0$ as $t \to \infty$ and exponentially stable if there exists an $\alpha > 0$ and a positive constant $M$ such that $\|x_t(\cdot; \varphi)\| \leq Me^{-\alpha t}\|\varphi\|$ for $t \geq 0$ (Hale and Verduyn Lunel 2002).

For some of our results, we need further restrictions on the operator $D$ and therefore we assume in this paper that $D$ is a difference operator given by

$$ D\varphi = \varphi(0) - \sum_{j=1}^{M} A_j \varphi(-r_j), $$

where the coefficients $A_1, A_2, \ldots, A_M$ are $n \times n$-matrices and the delays satisfy $0 < r_1 < r_2 < \cdots < r_M \leq h$. In this case, the difference equation associated
with (2)–(3) becomes

\[
x(t) = \sum_{j=1}^{M} A_j x(t - r_j), \quad t \geq 0.
\]  

(8)

Since we are interested in the sensitivity of stability to variations in the delays, we say that the origin is strongly stable (resp. strongly asymptotically stable) (resp. strongly exponentially stable) if it is stable (resp. asymptotically stable) (resp. exponentially stable) when subjected to small variations in the delays. For the difference equation (8), we have that (see Remark 2.1) strong exponential stability is equivalent to exponential stability for all variations in the delays. In this case we say that the operator \( D \) is strongly exponentially stable. We remark that previous work on this subject have referred to strong exponential stability of \( D \) as stability of the \( D \) operator (Hale and Verduyn Lunel 1993).

We now consider the problem of feedback stabilization of the neutral functional differential equation (2)

\[
\frac{d}{dt} \left[ x(t) - \sum_{j=1}^{M} A_j x(t - r_j) \right] = Lx_t + Bu(t),
\]

(9)

where \( B \) is an \( n \times m \)-matrix and \( u : [0, \infty) \rightarrow \mathbb{R}^m \) is a vector-valued control. So we consider the case that there are no input delays in the control.

It turns out that the eigenvectors and generalized eigenvectors of the solution map \( S(t) \) are smooth. Therefore, exponential stability of (2) is independent of the particular state space \( C, M_p \) or \( W^{1,p}[-h,0] \) under consideration (see also Section 2). In order to construct a feedback control to stabilize the system, it suffices to define the control on a dense set of initial data containing the eigenvectors and generalized eigenvectors of the solution map \( S(t) \).

If we close the loop with a proportional and a derivative feedback

\[
\frac{dx}{dt}(t - r_j) + Gx_t,
\]

(10)

where \( G : C \rightarrow \mathbb{R}^m \) is a bounded linear map, then the control problem (9) is called stabilizable (resp. strongly stabilizable) by a proportional and a derivative feedback control (10) if the zero solution of the closed-loop system is exponentially stable (resp. strongly exponentially stable). Although problems of feedback stabilization for retarded equations have been widely studied, little work has been done on feedback stabilization of neutral functional differential equations. The reason being the fact that feedback stabilization of retarded equations is a finite dimensional problem (Pandolfi 1975). While feedback stabilization for neutral equations with a \( D \)-operator that is not exponentially stable is an infinite dimensional problem (Salamon 1984). In general, we do not know under what conditions there exists a linear feedback so that the closed
loop system is exponentially stable; see (O’Connor and Tarn 1983, Salamon 1984) for some positive results in the case of point-delays. However, we do have necessary and sufficient conditions for the existence of a feedback control so that the resulting closed-loop neutral functional differential equation is strongly exponentially stable (Hale and Verduyn Lunel 2002). In this paper we start with the assumption that (9) is strongly stabilizable and address the question whether the stabilizability of (9) is robust with respect to a small delay in the feedback loop.

2 Spectral theory for neutral equations

The dynamical system approach to functional differential equations is to associate with (2) a semigroup $T(t) : \mathcal{C} \to \mathcal{C}$ defined by (5). In other words, the time evolution of the state $x_t$ is given by an abstract ordinary differential equation in the infinite dimensional state space $\mathcal{C}$. Given equation (2), this ordinary differential equation can be computed explicitly.

Let $v : [0, \infty) \to \mathcal{C}$ be a solution of

$$\frac{dv}{dt} = Av, \quad v(0) = \varphi, \quad \varphi \in \mathcal{C},$$

(11)

where $A : D(A) \to \mathcal{C}$ is an unbounded operator defined by

\[
\begin{align*}
A\varphi &= \frac{d\varphi}{d\theta} \\
D(A) &= \{\varphi \in \mathcal{C} | \frac{d\varphi}{d\theta} \in \mathcal{C} \text{ and } D \frac{d\varphi}{d\theta} = L\varphi\}.
\end{align*}
\]

(12)

Here the functionals $D$ and $L$ are, respectively, given by (3) and (1). The operator $A$ defined by (12) is the infinitesimal generator of the strongly continuous semigroup $T(t) : \mathcal{C} \to \mathcal{C}$ defined by translation along the solution, viz. (5).

Thus, the neutral functional differential equations (2) with time delays can be viewed as a transport equation (11) with nonlocal boundary conditions. Furthermore, the solutions of (2) are in one-to-one correspondence with the solutions of the infinite dimensional ordinary differential equation (11) and this correspondence is given by

$$v(t)(\theta) = x(t + \theta).$$

(13)

This observation originated with Krasovskii and has been crucial in the development of the qualitative theory of functional differential equations (Hale and Verduyn Lunel 1993).

The asymptotic behaviour of the semigroup $T(t)$ is determined by the spectrum of the time-one map $T(1)$. Recall that the spectrum $\sigma(T(1))$ of $T(1)$ is the set of complex numbers $\mu$ for which the operator $\mu I - T(1)$ is not invertible. A complex number $\mu$ is an eigenvalue of $T(1)$ if there is a nonzero $\varphi \in \mathcal{C}$ such that $T(1)\varphi = \mu \varphi$. If $\mu$ is an eigenvalue of $T(1)$, the kernel $N(\mu I - T(1))$ is called the eigenspace at $\mu$ and its dimension $d_\mu$ is called the geometric multiplicity of
The generalized eigenspace $M_{\mu}$ is the smallest closed subspace that contains all $N((\mu I - T(1))^j), j = 1, 2, \ldots$. The dimension of $M_{\mu}$ is called the algebraic multiplicity of $\mu$. The spectral radius of $T(1)$ is defined by

$$r_{\sigma}(T(1)) := \max\{|\mu| \in \mathbb{C} : \mu \in \sigma(T(1))\}. \quad (14)$$

The number $r_{\sigma}(T(1))$ determines the asymptotic behaviour of the semigroup $T(t)$. If $r_{\sigma}(T(1)) < 1$, then zero is exponentially stable. If $r_{\sigma}(T(1)) > 1$, then there are exponentially unbounded orbits and $T(t)$ is unstable.

Next we define the essential spectrum $\sigma_e(T(1))$ of $T(1)$. We will show that the behaviour of the essential spectrum plays a fundamental role whether or not feedback stabilization is sensitive to small variations in the delays. The essential spectrum $\sigma_e(T(1))$ of $T(1)$ is that part of the spectrum of $T(1)$ that cannot be removed by a compact perturbation

$$\sigma_e(T(1)) = \bigcap_{K \text{ compact}} \sigma(T(1) + K). \quad (15)$$

The essential spectral radius is defined to be

$$r_e(T(1)) := \max\{|\mu| \in \mathbb{C} : \mu \in \sigma_e(T(1))\}. \quad (16)$$

If the semigroup $T(t)$ depends continuously on parameters, then the eigenvalues of $T(1)$ can be chosen to be continuous functions of the parameters. However, the spectral radius $r_{\sigma}(T(1))$ does not necessarily vary continuously with respect to the parameters, see Kato (1980: Theorem IV.3.1). In order to be continuous, further conditions are needed, see the discussion in (Hale and Verduyn Lunel 2000).

A fundamental role in the study of stability and the preservation of stability under perturbations in the parameters is played by the essential spectrum $\sigma_e(T(1))$ of $T(1)$. It is known that, if $r_{\sigma}(T(1)) < 1$ then the stability or instability of 0 is determined by eigenvalues of $T(1)$. Furthermore, if 0 is unstable, then it is due to only a finite number of eigenvalues and the instability occurs in a finite dimensional subspace of $\mathbb{C}$. If the latter situation occurs, then it is natural to expect that the stabilization of the system could be accomplished using a bounded finite rank control and the stabilization is insensitive to small changes in the parameters.

If $r_{\sigma}(T(1)) > 1$, then the instability of the system is of such a nature that, in general, it cannot be controlled by a finite rank control which is robust with respect to small changes in the parameter (Hale and Verduyn Lunel 2001, 2002).

Spectral mapping theorems relate the spectrum of $T(1)$ and the infinitesimal generator $A$ in (12), e.g., (Hale and Verduyn Lunel 1993). It is not difficult to show that the operator $A$ in (12) has compact resolvent and, therefore, the spectrum $\sigma(A)$ of $A$ consists only of eigenvalues of finite multiplicity. To be precise, the complex number $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is the root of the determinant of the characteristic matrix $\Delta(z)$, where

$$\Delta(z) = z\Delta_0(z) - \int_{-h}^{0} e^{z\eta} dy(\eta), \quad (17)$$

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where
\[ \Delta_0(z) = I - \int_{-h}^{0} e^{z\theta} d\mu(\theta). \] (18)

In this representation, we have used the representations (1) and (3) for \( L \) and \( D \).

It is known that there is a close connection between the spectral properties of the operator \( A \) and the characteristic matrix \( \Delta(z) \) given by (17). In particular, the spectral properties of the generator \( A \) are independent of the state space \( C, M_p, \) or \( W^{1,p}[-h,0] \) (Kaashoek and Verduyn Lunel 1992).

From a result due to (Henry 1974), we have for the semigroup \( T(t) \) associated with the neutral functional differential equation (2) that
\[ r_{\sigma}(T(t)) = e^{\beta t}, \]
where \( \beta \) is defined by
\[ \beta = \sup \{ \Re \lambda \mid \det \Delta(\lambda) = 0 \}, \] (19)
and
\[ r_{\epsilon}(T_{D,L}(t)) = e^{\alpha t}, \]
where \( \alpha \) is defined by
\[ \alpha = \sup \{ \Re \lambda \mid \det \Delta_0(\lambda) = 0 \} \] (20)

Here \( \Delta(z) \) and \( \Delta_0(z) \) are, respectively, given by (17) and (18), see (Hale and Verduyn Lunel 1993). Thus the eigenvalues of the difference equation (8) play a fundamental role in the asymptotic behaviour of the solutions of the neutral equation (2).

If \( r_{\epsilon}(T(1)) < 1 \), then there are only finitely many independent unstable solutions. On the other hand, \( r_{\epsilon}(T(1)) \geq 1 \) implies that there is an infinite dimensional linear subspace \( U \) of \( C \) such that \( U \) is invariant under \( T(1) \) and any solution with initial data in \( U \) has norm approaching \( \infty \) as \( t \to \infty \); that is, the zero solution is not only unstable, but it is unstable for every element of an infinite dimensional subspace of \( C \). If the unstable subspace of zero were finite dimensional, then \( r_{\epsilon}(T(1)) \leq 1 \) since the unstable part could be removed by a compact perturbation. If \( r_{\epsilon}(T(1)) = 1 \), then we may or may not have any unbounded solutions.

In order to use (19) and (20) efficiently in applications, we have to study how the values \( \alpha \) and \( \beta \) depend on the matrices \( A_j \) and the delays \( r_1, r_2, \ldots, r_M \) in case \( D \) is given by (7). Similarly as for neutral functional differential equations, we can associate with the difference equation (8) a semigroup of bounded linear operators by translation along the solution. If we define the closed subspace \( C_D \) of \( C \) by
\[ C_D = \{ \varphi \in C \mid \varphi(0) = \sum_{j=1}^{M} A_j \varphi(-r_j) \}, \] (21)
then translation along the solutions of (8), given by,
\[ T_D(t) \varphi = x_t(\cdot; \varphi) \] (22)
defines a strongly continuous semigroup on \( C_D \). An application of (19) with \( L = 0 \) yields that \( r_{\sigma}(T_D(1)) = r_{\epsilon}(T_D(1)) = e^{\alpha t} \), where \( \alpha \) is defined by (20).

Thus, if a difference equation is not exponentially stable, then \( r_{\epsilon}(T_D(1)) \geq 1 \) and there is the possibility that there are infinitely many independent unstable
solutions. This is always the case if \( r_e(T_D(1)) > 1 \). We next show that this property leads to rather strong conditions in order to have robustness of stability with respect to variations in the delays.

The following theorem is a slight generalization of Hale and Verduyn Lunel (1993: Theorem 9.6.1), see (Hale and Verduyn Lunel 2002) for a complete proof and corollaries.

**Theorem 2.1** Let \( T_D(t) \) denote the semigroup associated with (8). Define

\[
\gamma_0 = \max\left\{ r(\sum_{j=1}^{M} e^{i\theta_j} A_j) \mid \theta_j \in [0, 2\pi], 1 \leq j \leq M \right\} =: r(\sum_{j=1}^{M} e^{i\theta_j} A_j).
\]

If the components of \( r = (r_1, r_2, \ldots, r_M) \) in the difference operator are rationally independent, then \( T_D(t) \) is exponentially stable if and only if \( \gamma_0 < 1 \). Furthermore, if \( \gamma_0 > 1 \), then \( T_D(t) \) is exponentially unstable.

**Remark 2.1** It actually follows from the proof of Theorem 2.1 that if \( T_D(t) \) denotes the semigroup associated with (8) and \( \gamma_0 < 1 \), then \( T_D(t) \) is exponentially stable without any assumptions on the delays. For scalar equations, Theorem 2.1 actually gives more detailed information (Hale and Verduyn Lunel 2002).

As an application of Theorem 2.1, we observe that exponential stability of the neutral functional differential equation (2) is preserved under small perturbations in the delays if and only if \( \gamma_0 < 1 \), where \( \gamma_0 \) is defined by (23). This implies that if the difference equation (8) is locally stable in the delays, then (8) is globally stable in the delays!

### 3 Strong stabilization of neutral equations

Consider the following neutral differential difference equation

\[
\frac{d}{dt} [x(t) - \sum_{j=1}^{M} A_j x(t - r_j)] = \sum_{j=1}^{M} C_j x(t - r_j) + Bu(t),
\]

where \( x : [-h, \infty) \to \mathbb{R}^n \) is a continuous function, \( h = \max_{1 \leq k \leq M} r_k \), each \( A_k \) is an \( n \times n \)-matrix, \( u : [-h, \infty) \to \mathbb{R}^m \) is continuous and \( B \) is a rectangular \( n \times m \)-matrix. We shall assume that linear feedback control consists of proportional and derivative feedback

\[
u(t) = \sum_{j=1}^{M} F_j \dot{x}(t - r_j) + \sum_{j=1}^{M} G_j x(t - r_j).
\]

Let \( T(t) \) denote the \( C_0 \)-semigroup associated with (24) with \( B = 0 \), let \( T_D(t) \) denote the semigroup associated with the difference equation corresponding to (24) with \( B = 0 \):

\[
x(t) - \sum_{k=1}^{M} A_k x(t - r_k) = 0
\]
(see (21) and (22)). Furthermore let $T_c(t)$ and $T_{D,c}(t)$ denote the $C_0$-semigroup associated with the closed-loop equation (24) and (26), respectively, with $u$ given by (25).

It is clear that proportional feedback alone cannot change the radius of the essential spectrum of the semigroup $T(t)$. We also need derivative feedback in order to change $r_e(T(1)) = r_e(T_{D}(t))$, see (20). If $r_e(T(1)) \geq 1$, then the solutions of (24) in $C$ are not uniformly exponentially stable.

In the sequel, we shall assume that system (24) is strongly stabilizable, i.e., if we close the loop by a proportional linear feedback $u(t)$ given by (25), then the closed-loop problem (24)–(25) is strongly exponentially stable. Because we are interested in strong stabilizability, we can, without loss of generality, assume that the delays $r_k$ in (26) are rationally independent.

From the practical point of view the control $u(t)$ should also be insensitive to a small time delay at which it is implemented; that is, the family of closed-loop difference equations with

$$
\frac{d}{dt} \left[ x(t) - \sum_{j=1}^{M} A_j x(t-r_j) \right] = \sum_{j=1}^{M} C_j x(t-r_j) + Bu(t),
$$

where $\epsilon > 0$ is small, should be strongly stabilizable by the same linear feedback (25). Our main result shows that this actually holds if and only if the radius of the essential spectrum of $T(t)$ is less than or equal to one.

We are now ready to present our main result in this paper.

**Theorem 3.1** Let $T(t)$ denote the $C_0$-semigroup associated with (24).

(i) If $r_e(T(1)) < 1$ and there exists a linear feedback $u(t) = \sum_{j=1}^{M} G_j x(t-r_j)$ such that the closed loop system

$$
\frac{d}{dt} \left[ x(t) - \sum_{j=1}^{M} A_j x(t-r_j) \right] = \sum_{j=1}^{M} C_j x(t-r_j) + Bu(t)
$$

is strongly exponentially stable, then the feedback stabilization is robust with respect to a time delay in the feedback law, i.e., there exists $\epsilon > 0$ such that the system

$$
\frac{d}{dt} \left[ x(t) - \sum_{j=1}^{M} A_j x(t-r_j) \right] = \sum_{j=1}^{M} C_j x(t-r_j) + Bu(t-\epsilon)
$$

remains strongly exponentially stable.

(ii) If $r_e(T(1)) \geq 1$ and there exists a linear feedback control

$$
u(t) = \sum_{j=1}^{M} F_j \dot{x}(t-r_j) + \sum_{j=1}^{M} G_j x(t-r_j)
$$


such that the closed loop system

\[
\frac{d}{dt} [x(t) - \sum_{j=1}^{M} A_j x(t - r_j)] = \sum_{j=1}^{M} C_j x(t - r_j) + Bu(t) \tag{29}
\]

is strongly exponentially stable, there exists a dense set \( E \subset \mathbb{R} \) such that for every \( \epsilon \in E \), the closed-loop control system is not robust with respect to a time delay in the feedback law, i.e.,

\[
\frac{d}{dt} [x(t) - \sum_{j=1}^{M} A_j x(t - r_j)] = \sum_{j=1}^{M} C_j x(t - r_j) + Bu(t - \epsilon)
\]

is not uniformly exponentially stable. In fact, exponentially unstable if \( r_e(T(1)) > 1 \).

Before we will prove the theorem we make some remarks and prove an auxiliary lemma. Theorem 3.1 extends earlier results in the literature. In (Logemann and Townley, 1996) a similar result for stabilizable systems was presented using a completely different method of proof. In (Hale and Verduyn Lunel, 2001) a weaker result was proved by directly using Theorem 2.1. In this result, the perturbed delays in the feedback loop had to stay rationally independent which is not necessarily true in the present theorem. Necessary and sufficient conditions for strong stabilization were derived in (Hale and Verduyn Lunel 2002).

In the proof of Theorem 3.1, we need the following lemma.

**Lemma 3.1** If \( G \) and \( H \) are square matrices, then

\[
\sup \{ |\sigma(G + e^{i\theta}H)| : 0 \leq \theta \leq 2\pi \} \geq \sigma(G). \tag{30}
\]

**Proof.** To arrive at a contradiction we assume that (30) is false and

\[
\sup \{ |\sigma(G + e^{i\theta}H)| : 0 \leq \theta \leq 2\pi \} < \sigma(G).
\]

From Gelfand’s formula for the spectral radius of a matrix \( M \)

\[
\sigma(M) = \lim_{n \to \infty} \|M^n\|^{\frac{1}{n}},
\]

we derive that there exists a \( \gamma_0 < \sigma(G) \) such that for any \( \delta \) sufficiently small, there exists an \( n_0 \) such that,

\[
\|(G + e^{i\delta}H)^n\| \leq (\gamma_0 + \delta)^n \quad \text{for all } n \geq n_0 \text{ and } \theta \in [0, 2\pi]. \tag{31}
\]

If we choose \( n \) odd and \( \theta_0 = 2\pi/n \), then we have the following identities

\[
\sum_{j=1}^{n} (G \pm e^{i\theta_0}H)^n = nG^n \pm nH^n. \tag{32}
\]
Using (31) and (32) we obtain the following estimates

\[ n\|G^n - H^n\| \leq \sum_{j=1}^{n} \|(G + e^{j\theta_0 + \pi \cdot i} H)^n\| \leq n(\gamma_0 + \delta)^n \]

\[ n\|G^n + H^n\| \leq \sum_{j=1}^{n} \|(G + e^{i\theta_0} H)^n\| \leq n(\gamma_0 + \delta)^n. \]

Next we combine these estimates to obtain the following estimate

\[ 2\|G^n\| = \|G^n - H^n + G^n + H^n\| \leq \|G^n - H^n\| + \|G^n + H^n\| \leq 2(\gamma_0 + \delta)^n. \]

Thus for every \( \delta \) there exists a \( n_0 \) such that for every \( n \geq n_0 \)

\[ \|G^n\|^\frac{1}{n} \leq \gamma_0 + \delta, \]

but this yields \( r_\sigma(G) \leq \gamma_0 \). A contradiction to the assumption that \( \gamma_0 < r_\sigma(G) \).

This completes the proof of the lemma. \( \square \)

**Proof of Theorem 3.1.** If the radius of the essential spectrum of \( T(1) \) is less than one, then the unstable subspace of \( T(t) \) is finite dimensional. So the strong stabilization of \( T(t) \) reduces to a finite pole shifting problem (Pandolfi 1975); and this problem is robust with respect to a small time delay in the feedback loop, see Hale and Verduyn Lunel (2001: Example 1.1).

If \( r_\sigma(T(1)) \geq 1 \) and the delays \( r_k \) in the difference operator are rationally independent, then it follows from Theorem 2.1 that there exist \( \theta^*_k, k = 1, \ldots, M \), such that the spectral radius of the matrix

\[ G = \sum_{k=1}^{M} A_k e^{i\theta_k^*} \]

is greater or equal to one, i.e., \( r_\sigma(G) \geq 1 \). Furthermore, if \( r_\sigma(T(1)) > 1 \), then \( r_\sigma(G) > 1 \).

If the control \( u(t) \) is not applied instantaneously but with a time delay \( \epsilon \), then the closed-loop system becomes (27). Choose a set \( E \subset \mathbb{R} \) such that for \( \epsilon \in E \) the reals \( r_1, r_2, \ldots, r_M \) and \( \epsilon \) are rationally independent. We claim that for every \( \epsilon \in E \), the corresponding closed-loop control difference equation

\[ x(t) - \sum_{k=1}^{M} A_k x(t - r_k) = B u(t - \epsilon) \quad (33) \]

with \( u(t) = \sum_{k=1}^{M} F_k x(t - r_k) \) is not uniformly exponentially stable (exponentially unstable). To prove this claim we first show that

\[ \sup \left\{ r_\sigma\left( \sum_{k=1}^{M} A_k e^{i\theta_k} + \sum_{k=1}^{M} B F_k e^{i\psi_k} \right) : \theta_k, \psi_k \in [0, 2\pi], k = 1, 2, \ldots, M \right\} \geq r_\sigma(G). \quad (34) \]
To prove (34) we restrict ourselves to the following specific choices of $\theta_k$ and $\psi_k$. Take $\theta_k = \theta_k^*$, $\psi_k = \theta + \theta_k^*$ and define

$$H = \sum_{k=1}^{M} BF_k e^{i\theta_k^*}.$$ 

Therefore to prove (34), it suffices to show that

$$\sup \{ r_\sigma (G + e^{i\theta} H) : 0 \leq \theta \leq 2\pi \} \geq r_\sigma (G).$$

This inequality is a general fact for matrices and follows from Lemma 3.1. Thus there exists $\theta = \bar{\theta}$ such that

$$r_\sigma \left( \sum_{k=1}^{M} (A_k + BF_k e^{i\theta_k^*}) e^{i\theta_k^*} \right) \geq r_\sigma (G) \geq 1. \quad (35)$$

From (35), it follows that there exists a $\rho_0 \geq 0$ ($\rho_0 > 0$ if $r_\sigma (G) > 1$) such that

$$r_\sigma \left( \sum_{k=1}^{M} (A_k + BF_k e^{-\rho_\sigma x + i\theta} e^{-\rho_\sigma r_k + i\theta_k^*} \right) = 1.$$ 

By definition of the spectral radius, this implies that there exists a $\nu \in [0, 2\pi]$ such that

$$\det \left( I - e^{-i\nu} \sum_{k=1}^{M} (A_k + BF_k e^{-\rho_\sigma x + i\theta} e^{-\rho_\sigma r_k + i\theta_k^*} \right) = 0. \quad (36)$$

In order to prove that (33) is not uniformly exponentially stable, it suffices to construct a zero in the right half plane $\{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \}$ of the characteristic equation $\det \Delta(\lambda) = 0$, where

$$\Delta(\lambda) = I - \sum_{k=1}^{M} (A_k + BF_k e^{-\lambda}) e^{-r_k \lambda} \quad (37)$$

denotes the characteristic matrix associated with the closed-loop control system (33) with $u(t) = \sum_{k=1}^{M} F_k x(t - r_j)$. The construction follows from Kronecker’s theorem and basic properties of almost periodic functions.

In fact, we use the following result for almost periodic functions that are analytic in a strip Levin (1972: 268). For given real numbers $a \leq b$, there exists a number $N$ such that, for all real $t$, the function $\det \Delta(\lambda)$, where $\Delta(\lambda)$ is defined in (37), has no more than $N$ zeros in the box

$$\{ \lambda \in \mathbb{C} : a \leq \text{Re} \lambda \leq b, \ t \leq \text{Im} \lambda \leq t + 1 \}. $$

Furthermore, for every $\delta > 0$, there exists an $m = m(\delta)$ such that if $\lambda$ is at distance at least $\delta$ from every zero of $\det \Delta(\lambda)$, then

$$| \det \Delta(\lambda) | \geq m(\delta). \quad (38)$$
Since the $r_j, j = 1, 2, \ldots, M$ and $\epsilon$ are rationally independent, it follows from Kronecker theorem (Corduneanu, 1968) that, for every $\delta > 0$, we can find a $\tau$ such that

$$|\tau r_j - \nu + \theta_j^*| < \delta \mod 2\pi \quad \text{and} \quad |\tau \epsilon + \bar{\theta}| < \delta \mod 2\pi,$$

for $j = 1, 2, \ldots, M$. So, for a given sequence $\delta_n \downarrow 0$, there exists a sequence $\tau_n$ such that

$$\lim_{n \to \infty} e^{i\tau_n r_j} = e^{i(\nu - \theta_j^*)} \quad \text{and} \quad \lim_{n \to \infty} e^{i\tau_n \epsilon} = e^{-i\bar{\theta}},$$

where $j = 1, 2, \ldots, M$. As a consequence, it follows from (36) that

$$\lim_{n \to \infty} \det \Delta(\rho_0 + i\tau_n) = 0,$$

where $\Delta$ is given by (37).

On the other hand, since $|e^{i\tau_n}| = 1$, there exists a subsequence $\tau_{n_k}$ of $\tau_n$ such that $e^{i\tau_{n_k}} \to e^{i\tau^*}$ as $k \to \infty$. By continuity, it follows that, for $j = 1, 2, \ldots, M$, $e^{i\tau_{n_k} r_j} \to e^{i\tau^* r_j}$ as $k \to \infty$. Thus

$$\lim_{n \to \infty} \det \Delta(\rho_0 + i\tau_n) = \det \Delta(\rho_0 + i\tau^*).$$

Together with (38), this implies that $\det \Delta(\rho_0 + i\tau^*) = 0$. Thus for every $\epsilon \in E$, we have constructed a zero $\rho_0 + i\tau^*$ of the characteristic equation of (33) with $\rho_0 \geq 0$ if $r_\sigma(G) = 1$ and $\rho_0 > 0$ if $r_\sigma(G) > 1$. This shows that $r_\sigma(T_D(1)) \geq 1$ and hence $r_\sigma(T(1)) \geq 1$ (greater than one if $\rho_0 > 0$). This completes the proof of Theorem 3.1.

References


