Big Boss Interval Games

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Abstract

This paper introduces and studies a class of cooperative interval games suitable to model market situations with two corners where players face interval uncertainty regarding the outcome of cooperation. In one corner there is a powerful player called the big boss; the other corner contains players that need the big boss to benefit from cooperation. Various characterizations of big boss interval games are given. The interval core of a big boss interval game is explicitly described, bi-monotonic allocation schemes using interval core elements are introduced, and it is shown that each element of the interval core of a big boss interval game is extendable to such a scheme. Two value-type interval solution concepts are defined on the class of big boss interval games which generate for each such game the same interval core allocation which is extendable to a bi-monotonic interval allocation scheme.

Keywords: cooperative games, interval data, big boss games, bi-monotonic allocation schemes

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1 Introduction

This paper introduces and studies a class of cooperative interval games suitable to model market situations with two corners where players face interval uncertainty regarding the outcome of cooperation. In one corner there is a powerful player called the big boss; the other corner contains players that need the big boss to benefit from cooperation. Two-corner market situations where the outcome of cooperation is assumed to be known with certainty, that is each coalition value is a real number, led to classical big boss games (cf. Muto et al. (1988), Tijs (1990)). In a big boss game, the big boss has veto power (i.e. the worth of each coalition which does not include the big boss is zero) and the characteristic function of the game has a monotonicity property (i.e. joining the big boss is more beneficial when the size of coalitions grows larger) and a union property (expressed in terms of marginal contributions to the grand coalition of coalitions and individuals). Some of the most appealing examples of big boss games arise from production situations with one owner of production means versus laborers (Muto et al. (1988)), holding situations with one owner of storage capacities versus their potential users (Tijs et al. (2005)), information market situations with one owner of a patent license versus firms willing to implement it (Muto et al. (1989)), information collecting situations with one decision maker versus information providers (Branzei et al. (2001a, b), Tijs et al. (2006)). However, in most market situations with two corners and in many other economic situations, people or businesses considering cooperation can rather forecast lower and upper bounds for the outcome of their cooperation. To deal with reward sharing problems under interval uncertainty in two-corner market situations, big boss interval games and related solution concepts can be helpful. For example, let us consider a supply chain planning problem where one producer (let’s say a car manufacturer) orders material from several suppliers in order to meet some demand. The car manufacturer would be the big boss and he is naturally facing interval uncertainty regarding the supply demanded from different suppliers. Then, the outcome of cooperation between the manufacturer and different groups of suppliers is affected by interval uncertainty as well, i.e. coalition values are compact intervals. Thus, we have an interval game with a big boss and to solve the related reward sharing problem we need suitable sets of solutions.

The model of interval games has captured much attention within game theory (see Branzei et al. (2010a) to get some insight). We point out that
Kimms and Drechsel (2009) developed algorithms for computing interval core elements and applied them to lot sizing problems under uncertain demand whereas Bauso and Timmer (2009) introduced some dynamics in the model of cooperative interval games and examined how retailers approach joint replenishment when their demand is bounded by a minimum and a maximum value.

In this paper the class of classical big boss games is extended to the interval setting and related interval solutions are introduced and studied. Two original issues have allowed us to extend in a natural way some basic results from the theory of big boss games to the interval setting: the consideration of suitable assumptions regarding the corresponding length game, and the use of an appropriate (partial) subtraction operation. Our main interest is devoted to characterizations of big boss interval games and to the interval core of a big boss interval game and related bi-monotonic interval allocation schemes. Elements of the interval core of an interval game are interval payoff vectors that fully distribute the interval-type worth of the grand coalition among the players such that no coalition has an incentive to split off. We notice that the sum of intervals (Moore (1979)) and a preference relation between intervals allows us to easily check whether (or not) an interval payoff vector belongs to the interval core of an interval game.

Interval uncertainty affects our decision making activities on a daily basis, and in all fields people interact constantly. Sometimes the interaction is cooperative, other times the interaction is competitive. Noncooperative game theory has been also enriched in the last recent years with several models which provide decision making support in competitive situations under interval uncertainty. In particular, strict competition between two parties facing interval uncertainty on outcomes or costs has generated a productive line of research in recent years. We refer here to Collins and Hu (2008), Li (2008), Li and Nan (2009), Nayak and Pal (2009), Li (2010), Li et al. (2010), Nan et al. (2010).

The paper is organized as follows. We recall in Section 2 basic notions and facts from the theory of traditional cooperative games and cooperative interval games. In Section 3 big boss interval games are introduced, various characterizations of big boss interval games are given, and the interval core of a big boss interval game is explicitly described. In Section 4 bi-monotonic allocation schemes using interval core elements are introduced and it is shown that each element of the interval core of a big boss interval game is extendable to such a scheme. Furthermore, we introduce for big boss interval games a
compromise-like solution concept, the $T$-value, and the $AL$-value which is defined as the average of lexicographic maximum of the interval core of the game with respect to all orders on the player set. It turns out that on the class of big boss interval games the two interval values coincide and generate a bi-monotonic interval allocation scheme for each game in this class.

2 Preliminaries

We start this section with basic definitions and results from the classical cooperative game theory which we use in Sections 3 and 4. For details we refer the reader to Owen (1995), Peleg and Sudhölter (2003) and Part I in Branzei et al. (2008a).

**Definition 2.1.** A cooperative game in coalitional form is an ordered pair $< N, v >$, where $N = \{1, 2, ..., n\}$ is the set of players, and $v : 2^N \rightarrow \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$.

Often, we also refer to such a game as a TU (transferable utility) game. We denote by $G^N$ the family of all classical cooperative games with player set $N$. In most situations to form the grand coalition $N$ is the best choice regarding cooperation, and then the main problem is how to distribute $v(N)$ among the players. Many solution concepts are helpful to solve this sharing problem.

**Definition 2.2.** (Muto et al. (1988), Tijs (1990)) A game $< N, v >$ is called a big boss game with $n$ as a big boss if the following conditions are satisfied:

(i) $v \in G^N$ is monotonic, i.e. $v(S) \leq v(T)$ if for each $S, T \in 2^N$ with $S \subset T$;

(ii) $v(S) = 0$ if $n \notin S \subseteq N$;

(iii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} (v(N) - v(N \setminus \{i\}))$ for all $S$ with $n \in S \subseteq N$.

Property (ii) expresses the veto power of the big boss: the worth of each coalition which does not include the big boss is zero. Property (i) says that joining the big boss is more beneficial when the size of coalitions grows larger, while (iii) expresses the fact that the force of non big boss players is in their union. We denote the set of all big boss games with $n$ as a big boss by $BBG^N$. 


DEFINITION 2.3. Let \( v \in G^N \) and \( n \in N \). This game is a total big boss game with \( n \) as a big boss if the following conditions are satisfied:

(i) \( v \in G^N \) is monotonic, i.e. \( v(S) \leq v(T) \) if for each \( S, T \in 2^N \) with \( S \subset T \);

(ii) \( v(S) = 0 \) if \( n \notin S \subset N \);

(iii) \( v(T) - v(S) \geq \sum_{i \in T \setminus S} (v(T) - v(T \setminus \{i\})) \) for all \( S, T \) with \( n \in S \subset T \).

We denote the set of all total big boss games with \( n \) as a big boss by \( TBBG^N \) and notice that the difference between a total big boss game and a big boss game is due to condition (iii). Thus, a game \( < N, v > \) is a total big boss game with big boss \( n \) if and only if \( < T, v > \) is a big boss game for each \( T \in 2^N \) with \( n \in T \), where \( < T, v > \) is the subgame of \( v \in G^N \) based on coalition \( T \in 2^N \setminus \{\emptyset\} \) defined by \( v_T(S) = v(S) \) for all \( S \in 2^T \). Clearly, \( TBBG^N \subset BBG^N \) and this inclusion may be strict. In the rest of the paper we only use total big boss games and the fact that \( TBBG^N \) is a cone in \( G^N \).

In the sequel we recall basic interval calculus (Moore (1979)) and useful notions and results regarding cooperative interval games (Alparslan Gök et al. (2008)).

Let \( I, J \in I(\mathbb{R}) \) with \( I = [I, \overline{I}] \) and let \( \alpha > 0 \). Then, \( I + J = [I, \overline{I}] + [J, \overline{J}] = [I + J, \overline{I} + \overline{J}] \) and \( \alpha I = \alpha [I, \overline{I}] = [\alpha I, \alpha \overline{I}] \). A preference relation between intervals allows us to compare intervals which are not nested. We say that \( I \) is weakly better than \( J \), which we denote by \( I \succeq J \), if and only if \( I \geq J \) and \( \overline{I} \geq \overline{J} \). We also use the reverse notation \( J \lessdot I \), if and only if \( J \leq I \) and \( \overline{J} \leq \overline{I} \).

The partial order relation \( \succeq \) allows us to define a partial subtraction operator \(^1\), which is very important in this paper. We define \( I - J \) (Alparslan Gök et al. (2009)), only if \( |I| \geq |J| \), where \( |I| = \overline{I} - \underline{I} \) and \( |J| = \overline{J} - \underline{J} \), by \( I - J = [I, \overline{I}] - [J, \overline{J}] = [\overline{I} - J, \overline{I} - \overline{J}] \). Clearly, this operator cannot be applied to nested intervals, but \( I - J \) is defined whenever \( I \) is geometrically situated to the right of \( J \) (in all the cases: \( I \cap J = \emptyset \); \( I = J \); \( I \) and \( J \) are overlapping intervals), and \( |I - J| = |I| - |J| \), which can be interpreted in terms of the spread of uncertainty. This subtraction operator is suitable for

\(^1\)Moore (1979) defined another subtraction operator as follows: \( I \odot J = [I, \overline{I}] - [J, \overline{J}] = [I - J, \overline{I} - \overline{J}] \). This operator is not suitable for interval-valued cooperative games because \( (I \odot J) + (J \odot K) \neq I \odot K \) for some \( I, J, K \in I(\mathbb{R}) \).
interval-valued cooperative games because \((I - J) + (J - K) = I - K\) for all \(I, J,\) and \(K \in I(\mathbb{R})\).

**Definition 2.4.** A cooperative interval game in coalitional form is an ordered pair \(<N, w>\) where \(N = \{1, 2, \ldots , n\}\) is the set of players, and \(w : 2^N \to I(\mathbb{R})\) is the characteristic function such that \(w(\emptyset) = [0, 0]\), where \(I(\mathbb{R})\) is the set of all closed intervals in \(\mathbb{R}\). For each \(S \in 2^N\), the worth set (or worth interval) \(w(S)\) of the coalition \(S\) in the interval game \(<N, w>\) is of the form \([\underline{w}(S), \overline{w}(S)]\), where \(\underline{w}(S)\) is the lower bound and \(\overline{w}(S)\) is the upper bound of \(w(S)\).

We denote by \(IG^N\) the family of all interval games with player set \(N\).

**Definition 2.5.** The interval core \(\mathcal{C}(N, w)\) of the interval game \(w\) is defined by

\[
\mathcal{C}(N, w) = \left\{ (I_1, \ldots , I_n) \in I(N, w) \mid \sum_{i \in S} I_i \succ w(S), \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\},
\]

where \(I(N, w)\) is the interval imputation set defined by

\[
I(N, w) = \left\{ (I_1, \ldots , I_n) \in I(\mathbb{R})^N \mid \sum_{i \in N} I_i = w(N), w(i) \preceq I_i, \text{ for all } i \in N \right\}.
\]

Here, \(\sum_{i \in N} I_i = w(N)\) is the efficiency condition and \(\sum_{i \in S} I_i \succ w(S), S \in 2^N \setminus \{\emptyset\}\), are the stability conditions of \((I_1, \ldots , I_n)\). Both interval imputations and interval core elements incorporate at some extent the uncertainty on the coalition values. They are useful before cooperation starts to inform the players about what they can expect if the grand coalition will form, and after the joint enterprise was carried out, together with the realization of the outcome of the grand coalition, to determine individual shares consistent with players’ expectations. We refer the reader to Branzei et al. (2010b) for some hints about using interval solutions in practical situations.

Given a game \(w \in IG^N\) and a coalition \(T \in 2^N \setminus \{\emptyset\}\) the interval subgame with player set \(T\) is the game \(w_T\) defined by \(w_T(S) = w(S)\) for all \(S \in 2^T\).

An interval game \(<N, w>\) is called \(\mathcal{I}\)-balanced if the interval core \(\mathcal{C}(N, w)\) is nonempty. An interval game \(<N, w>\) is called size monotonic if \(<N, |w|>\) is monotonic, i.e. \(|w|(S) \leq |w|(T)\) for all \(S, T \in 2^N\) with \(S \subset T\). For further use we denote by \(SMICG^N\) the class of size monotonic
interval games with player set $N$. For each $w \in SMIG^N$ and each $i \in N$, the *marginal contribution* of player $i$ in the game $w$ is defined by $M_i(N, w) = w(N) - w(N \setminus \{i\})$.

### 3 Big boss interval games

To handle market situations with two corners under interval uncertainty we introduce the model of big boss interval games.

Some classical $TU$-games associated with an interval game $w \in IG^N$ will play a key role namely the *border game* $< N, w >$ and the *length game* $< N, |w| >$, where $|w|(S) = w(S) - w(S)$ for each $S \in 2^N$.

**Definition 3.1.** A game $< N, w >$ is called a big boss interval game if its border game $< N, w >$ and the length game $< N, |w| >$ are classical total big boss games.

We denote by $BBIG^N$ the set of all big boss interval games with player set $N$ (without loss of generality we denote the big boss by $n$). The interval game in the next example is not a big boss interval game since the related length game is not a total big boss game.

**Example 3.1.** Let $< N, w >$ be a three-person interval game with $w(1) = w(2) = w(3) = w(1, 2) = [0, 0], w(2, 3) = [5, 6], w(1, 3) = [6, 6]$ and $w(N) = [9, 11]$. Here, $< N, w >$ is a total big boss game, but the length game $< N, |w| >$ is not because it does not satisfy the condition (iii) in Definition 2.3 (take $S = \{1\}$).

Next we give an example with an economic flavour leading to a big boss interval game.

**Example 3.2.** Let us consider a production economy with one landlord and many peasants. Let $N = \{1, 2, \ldots, n\}$ be the player set, where $n$ is the landlord that can not produce anything alone, and $1, 2, \ldots, n-1$ are landless peasants. Let $f : [0, n-1] \to I(\mathbb{R})$ be the production function with interval values, where $f(s)$ is the interval reward $[f_1(s), f_2(s)] \supseteq [0, 0]$ if $s$ peasants are hired by the landlord, where $f(0) = [0, 0]$, $f_1$ and $f_2 - f_1$ are concave with $f_2 - f_1 \geq 0$. This situation corresponds to the big boss interval game $< N, w >$ whose characteristic function is given by

$$w(S) = \begin{cases} [0, 0], & n \notin S \\ f(|S| - 1), & n \in S. \end{cases}$$
Various characterizations of big boss interval games are given in Theorems 3.1, 3.2 and 3.3. All these theorems provide useful criteria to check whether (or not) an interval game is a big boss interval game. To prove Theorem 3.1 we need the next proposition.

**Proposition 3.1.** Let $w \in IG^N$ and its related games $|w|, \underline{w}, \overline{w} \in G^N$. Then, $w \in BBIG^N$ if and only if its length game $< N, |w| >$ and its border games $< N, \underline{w} >, < N, \overline{w} >$ are total big boss games.

**Proof.** The proof is straightforward. Note that $\overline{w} = w + |w|$ is a total big boss game because classical total big boss games form a cone. □

From Proposition 3.1 it follows that any subgame of a big boss interval game is also a big boss interval game.

**Theorem 3.1.** Let $w \in SMIG^N$. Then, the following two conditions are equivalent:

(i) $w \in BBIG^N$;

(ii) $< N, w >$ satisfies:

(a) **Veto power property:**

$$w(S) = [0,0] \text{ for each } S \in 2^N \text{ with } n \notin S;$$

(b) **Monotonicity property:**

$$w(S) \preceq w(T) \text{ for each } S, T \in 2^N \text{ with } n \in S \subset T;$$

(c) **Union property:**

$$w(T) - w(S) \succeq \sum_{i \in T \setminus S} (w(T) - w(T \setminus \{i\})) \text{ for all } S, T \text{ with } n \in S \subset T.$$

**Proof.** By Proposition 3.1, $w \in BBIG^N$ if and only if $< N, \underline{w} >, < N, \overline{w} >$ and $< N, \overline{w} >$ are classical total big boss games. Now, using Definition 2.3 we obtain that $w \in BBIG^N$ if and only if $< N, w >$ satisfies (a), (b) and (c). □

Now, we give a concavity property for big boss interval games with $n$ as a big boss which plays an important role in Theorem 3.2.
(d) $n$-concavity property:

$$w(S \cup \{i\}) - w(S) \succ w(T \cup \{i\}) - w(T),$$

for all $S, T \in 2^N$ with $n \in S \subset T \subset N \setminus \{i\}$.

**Theorem 3.2.** Let $w \in \mathcal{IG}^N$ satisfying properties (a) and (b) from Theorem 3.1. Then properties (c) and (d) are equivalent.

**Proof.**

(i) Suppose that (d) holds. Let $S, T$ be such that $n \in S \subset T$. Suppose $T \setminus S = \{i_1, \ldots, i_h\}$. Then,

$$w(T) - w(S) = w(S \cup \{i_1\}) - w(S) + \sum_{r=2}^h (w(S \cup \{i_1, \ldots, i_r\}) - w(S \cup \{i_1, \ldots, i_{r-1}\})) = \sum_{r=1}^h M_{i_r}(S \cup \{i_1, \ldots, i_r\}, w) \geq \sum_{r=1}^h M_{i_r}(T, w) = \sum_{i \in T \setminus S} M_i(T, w),$$

where “the inequality” follows from (d). So, (d) implies (c).

(ii) Suppose that (c) holds. Then,

$$w(U \cup \{j\}) - w(U \setminus \{i\}) \geq M_j(U \cup \{j\}, w) + M_i(U \cup \{j\}, w). \quad (1)$$

But,

$$w(U \cup \{j\}) - w(U \setminus \{i\}) = w(U \cup \{j\}) - w(U) + w(U) - w(U \setminus \{i\}) = M_j(U \cup \{j\}, w) + M_i(U, w). \quad (2)$$

From (1) and (2) we obtain

$$M_i(U, w) \succ M_i(U \cup \{j\}, w) \quad (3)$$

for all $U \subset N$ and $i, j \in N \setminus \{n\}$ such that $\{i, n\} \subset U \subset N \setminus \{j\}$. Now, take $S, T \subset N$ with $\{i, n\} \subset S \subset T$ and suppose that $T \setminus S = \{i_1, \ldots, i_h\}$. If we apply (3) $h$ times then we have $M_i(S, w) \succ M_i(S \cup \{i_1\}, w) \succ M_i(S \cup \{i_1, i_2\}, w) \succ \ldots \succ M_i(T, w)$. So, (c) implies (d). \qed
Theorem 3.3 is based on an explicit description of the interval core of a big boss interval game given in Proposition 3.2 and on two additive maps on the class $BBIG^N$. We define the set $\mathcal{K}(T, w)$ for each subgame $<T, w>$ of $<N, w>$ by

$$\mathcal{K}(T, w) = \{ (I_1, \ldots, I_n) \in \mathcal{I}(T, w) \mid [0, 0] \preceq I_i \preceq M_i(T, w) \text{ for each } i \in T \setminus \{n\} \}.$$ 

**Proposition 3.2.** Let $w \in BBIG^N$. Then,

$$C(T, w) = K(T, w).$$

**Proof.** It is sufficient to show $C(T, w) = K(T, w)$ for $T = N$.

(i) Suppose that $I = (I_1, \ldots, I_n) \in C(N, w)$. Then, $w(N) = \sum_{i \in N} I_i$ and $\sum_{j \in N \setminus \{i\}} I_j \succeq w(N \setminus \{i\})$, for all $i \in N \setminus \{n\}$. Further,

$$I_i = \sum_{j \in N} I_j - \sum_{j \in N \setminus \{i\}} I_j = w(N) - \sum_{j \in N \setminus \{i\}} I_j \preceq w(N) - w(N \setminus \{i\}) = M_i(N, w),$$

where the second equality follows from efficiency and “the inequality” follows from stability. Clearly, $I_i \succeq [0, 0] = w(i)$ for $i \in N \setminus \{n\}$. So, $I \in K(N, w)$. Therefore $C(N, w) \subset K(N, w)$ holds.

(ii) Suppose that $I = (I_1, \ldots, I_n) \in K(N, w)$. Then, for a coalition $S$ which does not contain $n$, one finds that \( \sum_{i \in S} I_i \succeq [0, 0] = w(S) \). To prove that $\sum_{i \in S} I_i \succeq w(S)$ for each $S$ such that $n \in S$ we first show that $w(N) - w(S) \succeq \sum_{i \in N \setminus S} M_i(N, w)$. Let $N \setminus S = \{i_1, \ldots, i_k\}$. Then, in a similar way as in the proof of Theorem 3.2 (i) with $N$ in the role of $T$ we have

$$w(N) - w(S) = w(S \cup \{i_1\}) - w(S)$$

$$+ \sum_{s=2}^k (w(S \cup \{i_1, \ldots, i_s\}) - w(S \cup \{i_1, \ldots, i_{s-1}\}))$$

$$= \sum_{s=1}^k M_{i_s}(S \cup \{i_1, \ldots, i_s\}, w)$$

$$\succeq \sum_{s=1}^k M_{i_s}(N, w) = \sum_{i \in N \setminus S} M_i(N, w).$$
where “the inequality” follows from the \( n \)-concavity property. Then, using the definition of \( \mathcal{K}(N, w) \) we have

\[
 w(S) \preceq w(N) - \sum_{i \in N \setminus S} M_i(N, w) \preceq w(N) - \sum_{i \in N \setminus S} I_i = \sum_{i \in S} I_i.
\]
So, \( I \in C(N, w) \). Therefore \( K(N, w) \subset C(N, w) \) holds. 

Two additive maps \( B : BBIG^N \to I(\mathbb{R})^N \) and \( U : BBIG^N \to I(\mathbb{R})^N \) associate with each big boss interval game two special interval “points”.

**Definition 3.2.** Let \( < T, w > \) be a big boss subgame of \( < N, w > \) with \( n \) as a big boss. The big boss interval point \( B(T, w) \) is defined by

\[
B_j(T, w) = \begin{cases} 
[0, 0], & j \in T \setminus \{n\} \\
w(T), & j = n,
\end{cases}
\]

and the union interval point \( U(T, w) \) is defined by

\[
U_j(T, w) = \begin{cases} 
M_j(T, w), & j \in T \setminus \{n\} \\
w(T) - \sum_{i \in T \setminus \{n\}} M_i(T, w), & j = n.
\end{cases}
\]

**Theorem 3.3.** Let \( w \in IG^N \) be such that property (a) in Theorem 3.1 holds. Then, \( w \in BBIG^N \) if and only if for each \( T \subset N \) with \( n \in T \) the big boss interval point \( B(T, w) \) and the union interval point \( U(T, w) \) belong to the interval core of \( < T, w > \).

**Proof.** If \( w \in BBIG^N \) then by Proposition 3.2 it is clear that \( B(T, w) \) and \( U(T, w) \in C(T, w) \) for each \( T \subset N \) with \( n \in T \).

Conversely, assume that for each \( T \subset N \) with \( n \in T \) the points \( B(T, w) \) and \( U(T, w) \) belong to the interval core. Since by hypothesis \( < N, w > \) satisfies (a) from Theorem 3.1, we only need to show that (b) and (c) from Theorem 3.1 hold.

First, take \( T \) such that \( n \in T \). Since \( B(T, w) \in C(T, w) \), we have for each \( S \subset T \) such that \( n \in S \),

\[
w(S) \leq \sum_{i \in S} B_i(T, w) = B_n(T, w) + \sum_{i \in S \setminus \{n\}} B_i(T, w) = w(T) + [0, 0] = w(T).
\]

So, (b) is satisfied.

Second, take \( S \) such that \( n \in S \subset T \). Since \( U(T, w) \in C(T, w) \) we have,

\[
w(S) \leq \sum_{i \in S} U_i(T, w) = U_n(T, w) + \sum_{i \in S \setminus \{n\}} U_i(T, w) = (w(T) - \sum_{i \in T \setminus \{n\}} M_i(T, w)) + \sum_{i \in S \setminus \{n\}} M_i(T, w) = w(T) - \sum_{i \in T \setminus S} M_i(T, w).
\]

So, (c) from Theorem 3.1 is satisfied. \qed
From the above theorem we learn that big boss interval games are totally \( \mathcal{I} \)-balanced games.

## 4 Bi-monotonic interval allocation schemes

We introduce here the notion of bi-monotonic interval allocation scheme for big boss interval games. We denote by \( P_n \) the set \( \{ S \subset N | n \in S \} \) of all coalitions containing the big boss.

**Definition 4.1.** Let \( w \in BBIG^N \). A scheme \( B = (B_iS)_{i \in S, S \in P_n} \) is a bi-monotonic interval allocation scheme (in short bi-mias) if it has the following two properties:

1. Stability: \( (B_iS)_{i \in S} \) is an interval core element of the subgame \( < S, w > \) for each coalition \( S \in P_n \);
2. Bi-monotonicity: For all \( S, T \in P_n \) with \( S \subset T \), \( B_iS \geq B_iT \) for all \( i \in S \setminus \{ n \} \) and \( B_nS \leq B_nT \).

Property (i) says that in a bi-mias the interval payoff vector for each subgame with \( n \) as a big boss belongs to the interval core of that subgame, whereas property (ii) says that the big boss is weakly better off in larger coalitions, while the other players are weakly worse off.

Special bi-mias for a big boss interval game are obtained by applying the two additive maps \( B : BBIG^N \rightarrow I(\mathbb{R})^N \) and \( U : BBIG^N \rightarrow I(\mathbb{R})^N \) to the game itself and all its proper interval subgames, as the next example illustrates.

**Example 4.1.** Consider the interval game in Example 3.2. Let \( < T, w > \) be a subgame of it with \( n \) as a big boss and let \( B(T, w) \) and \( U(T, w) \) be the big boss interval point and union interval point, respectively. For each \( i \neq n \) and for each \( S \subset T \), \( B_i(S, w) = B_i(T, w) = \left[ 0, 0 \right] \); for \( i = n \) and for each \( S \subset T \), \( B_n(S, w) = f(|S| - 1) \preceq f(|T| - 1) = B_n(T, w) \). For each \( i \neq n \) and for each \( S \subset T \),

\[
U_i(S, w) = M_i(S, w) = w(S) - w(S\setminus\{i\}) \preceq w(T) - w(T\setminus\{i\}) = M_i(T, w) = U_i(T, w);
\]

for \( i = n \) and for each \( S \subset T \),

\[
U_n(S, w) = w(S) - \sum_{i \in S\setminus\{n\}} M_i(S, w) \preceq w(T) - \sum_{i \in T\setminus\{n\}} M_i(T, w) = U_n(T, w).
\]
DEFINITION 4.2. Let $w \in \text{BBIG}^N$ with $n$ as a big boss. An interval imputation $I = (I_1, \ldots, I_n) \in \mathcal{I}(w)$ is said to be bi-mias extendable if there exists a bi-mias $B = (B_{iS})_{i \in S, S \in \wp_n}$ such that $B_{iN} = I_i$ for each $i \in N$.

The next theorem, inspired by Voorneveld et al. (2003), shows that each element of the interval core of a big boss interval game is extendable to a bi-mias.

**Theorem 4.1.** Let $w \in \text{BBIG}^N$ with $n$ as a big boss and let $I \in \mathcal{C}(N, w)$. Then $I$ is bi-mias extendable.

**Proof.** Since $I \in \mathcal{C}(N, w)$, by (4), we can find for each $i \in N \setminus \{n\}$ an $\alpha_i \in [0, 1]$ such that $I_i = \alpha_i M_i(N, w)$, and then $I_n = w(N) - \sum_{i \in N \setminus \{n\}} \alpha_i M_i(N, w)$. We will show that $(B_{iS})_{i \in S, S \in \wp_n}$, defined by $B_{iS} = \alpha_i M_i(S, w)$ for each $S$ and $i$ such that $i \in S \setminus \{n\}$, and $B_{nS} = w(S) - \sum_{i \in S \setminus \{n\}} \alpha_i M_i(S, w)$ is a bi-mias. Take $S, T \in \wp_n$ with $S \subset T$ and $i \in S \setminus \{n\}$. We have to prove that $B_{iS} \succeq B_{iT}$ and $B_{nS} \preceq B_{nT}$. First, $B_{iS} = \alpha_i M_i(S, w) \succeq \alpha_i M_i(T, w) = B_{iT}$, where “the inequality” follows from (d). Second,

$$B_{nT} = w(T) - \sum_{i \in T \setminus \{n\}} \alpha_i M_i(T, w)$$

$$\succeq (w(S) + \sum_{i \in T \setminus S} M_i(T, w)) - \sum_{i \in T \setminus \{n\}} \alpha_i M_i(T, w)$$

$$= (w(S) - \sum_{i \in S \setminus \{n\}} \alpha_i M_i(T, w)) + \sum_{i \in T \setminus S} (1 - \alpha_i) M_i(T, w)$$

$$\succeq (w(S) - \sum_{i \in S \setminus \{n\}} \alpha_i M_i(S, w)) + \sum_{i \in T \setminus S} (1 - \alpha_i) M_i(T, w)$$

$$= B_{nS} + \sum_{i \in T \setminus S} (1 - \alpha_i) M_i(T, w) \succeq B_{nS},$$

where the first inequality follows from (c), the second follows from (d), and the third follows from $\alpha_i \leq 1$ and the nonincreasing of the interval marginal contribution vectors. So, $B_{nT} \succeq B_{nS}$. \hfill $\Box$

In the sequel, we introduce two value-type interval solutions for big boss interval games called the $T$-value and the $\mathcal{AL}$-value.

**Definition 4.3.** Let $w \in \text{BBIG}^N$. The $T$-value of $w$ is defined by

$$T(N, w) = \frac{1}{2}(\mathcal{U}(N, w) + \mathcal{B}(N, w)).$$
Example 4.2. Consider the following holding situation with interval data. Players 1 and 2 have each one container which they want to store, and player 3 is the owner of a holding house which has capacity for one container. If player 1 is allowed to store his/her container then the benefit belongs to $[10, 30]$ and if player 2 is allowed to store his/her container then the benefit belongs to $[50, 70]$. The situation is described by the interval game $< N, w >$ with $N = \{1, 2, 3\}$, $w(S) = [0, 0]$ if $3 \notin S$, $w(\emptyset) = w(3) = [0, 0]$, $w(1, 3) = [10, 30]$ and $w(N) = w(2, 3) = [50, 70]$. This game is a big boss interval game with player 3 as big boss because properties (a), (b) and (c) in Theorem 3.1 are satisfied. The $T$-value, in case of full cooperation, generates the interval allocation $T(N, w) = ([0, 0], [20, 20], [30, 50])$, which indicates sharp shares for players 1 and 2 equal to 0 and 20, respectively. The payoff for player 3 depends, in this case, only on the realization $R$ of $w(N)$. Assuming that $R = 60$ player 3 will receive a payoff equal to 40. However, in general the actual individual shares depend not only on $R$, but also on the interval payoff vector agreed upon before starting cooperation, in our case $T(N, w)$. Finally, the total $T$-value generates a bi-mias represented by the following matrix:

$$
\begin{bmatrix}
1 & 2 & 3 \\
1 & [0, 0] & [20, 20] & [30, 50] \\
\{1, 3\} & [5, 15] & * & [5, 15] \\
\{2, 3\} & * & [25, 35] & [25, 35] \\
\{3\} & * & * & [0, 0]
\end{bmatrix}
$$

Such a bi-mias extension of the interval core element $T(N, w)$ might be helpful in the decision making process regarding which coalitions should form and how to distribute the collective gains among the participants.

Definition 4.4. The interval average lexicographic value $\mathcal{AL}$-value is defined by

$$
\mathcal{AL}(N, w) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} L^\sigma(N, w),
$$

where $\Pi(N)$ is the set of permutations $\sigma : N \rightarrow N$ and the lexicographic interval vector $L^\sigma(N, w)$ is defined by

$$
L^\sigma_{\sigma(i)}(N, w) = \begin{cases} 
M_{\sigma(i)}(N, w), & i < k \\
[0, 0], & i > k \\
w(N) - \sum_{j=1}^{k-1} M_j(N, w), & i = k
\end{cases}
$$
if $\sigma(k) = n$, where $\sigma(k)$ stands for the position of the player $k$.

In Theorem 4.2 we show that these values generate the same interval core allocation for each big boss interval game.

**Theorem 4.2.** Let $w \in BBIG^N$ with $n$ as a big boss. Then,

$$T(N, w) = A\mathcal{L}(N, w) \in \mathcal{C}(N, w)$$

and the (total) $A\mathcal{L}$-value generates a bi-mias for $w \in BBIG^N$.

**Proof.** Applying (5) we obtain

$$A\mathcal{L}(N, w) = \left(\frac{1}{2}M_1(N, w), \ldots, \frac{1}{2}M_{n-1}(N, w), w(N) - \frac{1}{2} \sum_{i=1}^{n-1} M_i(N, w)\right).$$

Now, using Definitions 3.2 and 4.3 we obtain by a simple calculus that $T(N, w) = A\mathcal{L}(N, w)$. Now we apply Theorem 4.1.

The bi-mias in Example 3.2 can be interpreted, according to Theorem 4.2, in terms of lexicographic maximization within the interval core of the game and the interval cores of the subgames where the big boss $n$ is a player.

## 5 Concluding remarks

In this paper we introduce and study the class of big boss interval games. Various characterizations of big boss interval games are given in Theorems 3.1, 3.2 and 3.3. We refer the reader to Section 4 in Branzei et al. (2008b) where other characterizations of big boss interval games can be found.

Proposition 3.2 provides a characterization of the interval core of a big boss interval game. We also introduce the notion of bi-monotonic interval allocation schemes and show in Theorem 4.1 that each element of the interval core of a big boss interval game is extendable to such a scheme. Two value-type interval solution concepts, namely the $T$-value and the $A\mathcal{L}$-value, are defined on the class of big boss interval games. They generate for each such game the same interval core allocation which is extendable to a bi-monotonic interval allocation scheme. We notice that the $T$-value and the $A\mathcal{L}$-value can be seen as extensions of the $\tau$-value (Tijs (1981)) and the $AL$-value (Tijs (2005)) for classical big boss games to the interval setting, and Theorem 4.1 extends for
big boss interval games the known result that on the class of big boss games the \( \tau \)-value and the \( AL \)-value coincide (Tijs (2005)).

In our opinion, two directions for further research on big boss interval games look promising. One is to look at existing results for classical big boss games and try to extend them to the interval setting. Another one is to consider another type of uncertainty on coalition values than interval uncertainty. Various types of uncertainty regarding the coalition values have been considered in the game theory literature. We refer here to stochastic uncertainty (Charnes and Granot (1973), Suijs et al. (1999) and Timmer et al. (2005)) and fuzzy uncertainty (Mares (2001)).

References


decision-making”, International Game Theory Review, Vol. 3, No. 1
(2001a) 1-12.

and bi-monotonic allocation schemes”, Mathematical Methods of Operations
Research, Vol. 54 (2001b) 303-313.

nucleolus solutions to chance-constrained games”, Proceedings of the Computer
Science and Statistics Seventh Symposium at Iowa State University
(1973) 323-332.


[12] Kimms A and Drechsel J., “Cost sharing under uncertainty: an algo-
rithmic approach to cooperative interval-valued games”, BuR - Business

[13] Li D.F., “Lexicographic method for matrix games with payoffs of trian-
gular fuzzy numbers”, International Journal of Uncertainty, Fuzziness

[14] Li D.F., “Notes on ”Linear programming technique to solve two person
matrix games with interval pay-offs””, Asia-Pacific Journal of Operational


[16] Li D.F., Nan J.X. and Zhang M.J., “Interval programming models for
matrix games with interval payoffs”, Optimization Methods and Software
[17] Mares M., “Fuzzy Cooperative Games: Cooperation with Vague Expec-

[18] Moore R., “Methods and Applications of Interval Analysis”, SIAM Stud-


games with payoffs of triangular intuitionistic fuzzy numbers”, Inter-
national Journal of Computational Intelligence Systems, Vol. 3, No. 3

[22] Nayak, P.K. and Pal M., “Linear programming technique to solve two
person matrix games with interval pay-offs”, Asia-Pacific Journal of


with stochastic payoffs”, European Journal of Operational Research, Vol. 113

[26] Tijs S., “Bounds for the core and the τ-value.” In: Moeschlin O., Pal-
laschke D. (eds.), Game Theory and Mathematical Economics, North

[27] Tijs S., “Big boss games, clan games and information market games.”
In: Ichiishi T., Neyman A., Tauman Y. (eds.), Game Theory and Ap-

[28] Tijs S., “The first steps with Alexia, the average lexicographic value”,
Tilburg University, Center for Economic Research, The Netherlands,
CentER DP 123 (2005).

