A new method based on Haar wavelet for numerical solution of two-dimensional nonlinear integral equations

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Abstract
A new numerical method based on Haar wavelet is proposed for two-dimensional nonlinear Fredholm, Volterra and Volterra-Fredholm integral equations of first and second kind. The proposed method is an extension of the Haar wavelet method [1–3] from one-dimensional nonlinear integral equations (Fredholm and Volterra) to two-dimensional nonlinear integral equations (Fredholm, Volterra and Volterra-Fredholm). The main characteristic of the method is that, unlike several other methods, it does not involve numerical integration which results in an improved accuracy of the method. In order to show the effectiveness of the method, it is applied to several benchmark problems. The numerical results are compared with other methods existing in the recent literature.

Keywords: Haar wavelet, nonlinear Fredholm integral equations, nonlinear Volterra integral equations, nonlinear Volterra-Fredholm integral equations

1 Introduction
Several problems in engineering and physics can be modeled using two-dimensional integral equations of the second kind. Fredholm integral equations have applications in plasma physics [4], telegraph equations [5] and electrical engineering [6]. Due to limited scope of closed form solutions, efforts are needed to construct efficient and accurate numerical algorithms for simulating such models.

Some detailed treatments of numerical method for solving one-dimensional integral equation can be found in [7–9]. Recently, some numerical methods have been proposed for solution of two-dimensional integral equations by different researchers. These include polynomial interpolation methods [10], Gauss product quadrature rule [11], discrete Galerkin and iterated discrete Galerkin methods [12], triangular functions method [13,14], Legendre polynomial method [15], differential transform method [16], Nystrom method [17], meshless method [18] and Haar wavelet method [19]. Due challenges offered by nonlinearities in higher dimensions, more concerted efforts are needed to address the issues related to the numerical solution of integral equations in higher dimensions.

The present work is focused on proposing a generic framework for two main classes of integral equations, i.e., two-dimensional nonlinear Fredholm, Volterra and Volterra-Fredholm integral equations of first and second kind. This work should be considered as a logical continuation

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of the work [1–3]. In [1] a new numerical method based on Haar wavelet is introduced for solution of nonlinear one-dimensional Fredholm and Volterra integral equations. In [2] the Haar wavelet method [1] is extended to numerical solution of integro-differential equation. In [3] the Haar wavelet method [1, 2] is improved in terms of efficiency by introducing one-dimensional Haar wavelet approximation of the kernel function. Moreover in [3] the method is extended to numerical solution of nonlinear Fredholm and Volterra integro-differential equations of higher orders. In the present work we are extending the Haar wavelet method [1–3] to numerical solution of nonlinear two-dimensional Fredholm, Volterra and Volterra-Fredholm integral equations. The present method is fundamentally different from the other numerical methods based on Haar wavelet for the numerical solution of integral equations as it approximates kernel function using Haar wavelet. Unlike several other methods, the present method does not involve any intermediate technique for numerical integration of the kernel function and this is the major advantage of the method.

In this work we will consider the following three types of integral equation. These include two-dimensional nonlinear Fredholm integral equation of the second kind:

\[
  u(x, y) = f(x, y) + \int_0^1 \int_0^1 K(x, y, s, t, u(s, t)) \, ds \, dt, \quad (x, y) \in [0, 1] \times [0, 1].
\]  

(1)

two-dimensional nonlinear Volterra integral equation:

\[
  u(x, y) = f(x, y) + \int_0^y \int_0^x K(x, y, s, t, u(s, t)) \, ds \, dt, \quad (x, y) \in [0, 1] \times [0, 1].
\]  

(2)

two-dimensional nonlinear Volterra-Fredholm integral equation:

\[
  u(x, y) = f(x, y) + \int_0^y \int_0^1 K(x, y, s, t, u(s, t)) \, ds \, dt, \quad (x, y) \in [0, 1] \times [0, 1],
\]  

(3)

where the kernel \( K(x, y, s, t, u(s, t)) \) in the above integral equations is a known nonlinear function in \( u \) and \( f(x, y) \) is also a known function whereas \( u(x, y) \) is unknown function called the solution of two-dimensional integral equation.

The existence and uniqueness of the solution of the following general form of two-dimensional Volterra integral equations has been proved in the reference [20]:

\[
  u(x, y) = f(x, y) + \int_0^x K_2(x, y, s, u(s, t)) \, ds + \int_0^y K_1(x, y, t, u(s, t)) \, dt
  + \int_0^y \int_0^x K(x, y, s, t, u(s, t)) \, ds \, dt, \quad (x, y) \in [0, 1] \times [0, 1].
\]  

(4)

Since Eqs. (1), (2) and (3) are special cases of Eq. (4) therefore existence and uniqueness of Eqs. (1), (2) and (3) follows from [20].

## 2 Haar Wavelets

A wavelet family \( (\psi_{j,i}(x))_{j\in\mathbb{N},i\in\mathbb{Z}} \) is an orthonormal subfamily of the Hilbert space \( L^2(\mathbb{R}) \) with the property that all function in the wavelet family are generated from a fixed function \( \psi \) called mother wavelet through dilations and translations. The wavelet family satisfies the following relation:

\[
  \psi_{j,i}(x) = 2^{j/2}\psi(2^j x - i).
\]
The Haar wavelet family defined on the interval \([0, 1)\) consists of the following functions:

\[
h_1(x) = \begin{cases} 
1 & \text{for } x \in [0, 1) \\
0 & \text{elsewhere},
\end{cases}
\]

and

\[
h_i(x) = \begin{cases} 
1 & \text{for } x \in [\alpha, \beta) \\
-1 & \text{for } x \in [\beta, \gamma) \\
0 & \text{elsewhere},
\end{cases} \quad i = 2, 3, \ldots,
\]

where

\[
\alpha = \frac{k}{m}, \quad \beta = \frac{(k + 0.5)}{m}, \quad \gamma = \frac{(k + 1)}{m}; \\
m = 2^j, \quad j = 0, 1, \ldots, \quad k = 0, 1, \ldots, m - 1.
\]

The integer \(j\) indicates the level of the wavelet and \(k\) is the translation parameter. The relation between \(i, m\) and \(k\) is given by \(i = m + k + 1\). The function \(h_1(x)\) is called scaling function whereas \(h_2(x)\) is the mother wavelet for the Haar wavelet family.

Any square integrable function \(f(x)\) defined on \([0, 1)\) can be expressed as an infinite sum of Haar wavelets as follows:

\[
f(x) = \sum_{i=1}^{\infty} a_i h_i(x),
\]

where \(a_i\) are real constants.

For approximation purpose we consider a maximum value \(J\) of the integer \(j\), level of the Haar wavelet in the above definition. The integer \(J\) is then called maximum level of resolution. We also define the integer \(M = 2^J\). With these notations any square integrable function \(f(x)\) defined on \([0, 1)\) can be approximated as a finite sum of Haar wavelets as follows:

\[
f(x) \approx \sum_{i=1}^{2M} a_i h_i(x).
\]

The following notation is introduced:

\[
p_{i,1}(x) = \int_{0}^{x} h_i(x') \, dx'. \tag{5}
\]

The integral given in Eq. (5) can be evaluated using the definition of Haar wavelet and is given as follows:

\[
p_{i,1}(x) = \begin{cases} 
x - \alpha & \text{for } x \in [\alpha, \beta), \\
\gamma - x & \text{for } x \in [\beta, \gamma), \\
0 & \text{elsewhere},
\end{cases}
\]

\section{Numerical Method}

In this section, proposed numerical method will be discussed for three different types of integral equation. In the first and second subsection, we state some results for efficient evaluation of one- and two-dimensional Haar wavelet approximations. In the third, fourth and fifth subsections we
apply these results for finding numerical solutions of Fredholm, Volterra and Volterra-Fredholm integral equations.

For Haar wavelet approximation of a function \( f(x, y) \) of two real variables \( x \) and \( y \), we assume that the domain \( 0 \leq x, y \leq 1 \) is divided into a grid of size \( 2M \times 2N \) using the following collocation points:

\[
x_m = \frac{m - 0.5}{2M}, \quad m = 1, 2, \ldots, 2M,
\]

\[
y_n = \frac{n - 0.5}{2N}, \quad n = 1, 2, \ldots, 2N.
\]

### 3.1 One-dimensional Haar wavelet system

Any square integrable function \( f(x) \) can be approximated using Haar wavelets as follows:

\[
f(x) = \sum_{i=1}^{2M} a_i h_i(x).
\]

Substituting the collocation points given in Eq. (6), we obtain the following linear system of equations:

\[
f(x_m) = \sum_{i=1}^{2M} a_i h_i(x_m), \quad m = 1, 2, \ldots, 2M.
\]

This is a \( 2M \times 2M \) linear system of equations whose solution for the unknown coefficients \( a_i \) can be calculated using the following theorem.

**Theorem 1.** The solution of the system (8) is given as follows:

\[
a_1 = \frac{1}{2M} \sum_{m=1}^{2M} f(x_m),
\]

\[
a_i = \frac{1}{\rho} \left( \sum_{m=\alpha}^{\beta} f(x_m) - \sum_{m=\beta+1}^{\gamma} f(x_m) \right), \quad i = 2, 3, \ldots, 2M,
\]

where

\[
\alpha = \rho(\sigma - 1) + 1,
\]

\[
\beta = \rho(\sigma - 1) + \frac{\rho}{2},
\]

\[
\gamma = \rho\sigma,
\]

\[
\rho = \frac{2M}{\tau},
\]

\[
\sigma = i - \tau,
\]

\[
\tau = 2^{\left\lfloor \log_2(i-1) \right\rfloor}.
\]

**Proof.** See [1].
3.2 Two-dimensional Haar wavelet system

A real-valued function $F(x, y)$ of two real variables $x$ and $y$ can be approximated using two-dimensional Haar wavelets basis as:

$$F(x, y) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j} h_i(x) h_j(y). \quad (9)$$

In order to calculate the unknown coefficients $b_{i,j}$’s, the collocation points defined in Eqs. (6) and (7) are substituted in Eq. (9). Hence, we obtain the following $4MN \times 4MN$ linear system with unknowns $b_{i,j}$’s:

$$F(x_m, y_n) = \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j} h_i(x_m) h_j(y_n), m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N. \quad (10)$$

The solution of this system can be calculated using the following theorem.

**Theorem 2.** The solution of the system (10) is given below:

$$b_{1,1} = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} F(x_m, y_n),$$

$$b_{i,1} = \frac{1}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} F(x_m, y_n) - \sum_{p=\beta_1+1}^{\gamma_1 \alpha_1} \sum_{q=1}^{2N} F(x_m, y_n) \right), \quad i = 2, 3, \ldots, 2M,$$

$$b_{1,j} = \frac{1}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} F(x_m, y_n) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2 \alpha_2} F(x_m, y_n) \right), \quad j = 2, 3, \ldots, 2N,$$

$$b_{i,j} = \frac{1}{\rho_1 \rho_2} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} F(x_m, y_n) - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_2+1}^{\gamma_2 \alpha_2} F(x_m, y_n) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} F(x_m, y_n) + \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\beta_2+1}^{\gamma_2 \alpha_2} F(x_m, y_n) \right), \quad i = 2, 3, \ldots, 2M, \quad j = 2, 3, \ldots, 2N,$$

where

$$\alpha_1 = \rho_1 (\sigma_1 - 1) + 1,$$

$$\beta_1 = \rho_1 (\sigma_1 - 1) + \frac{\rho_1}{2},$$

$$\gamma_1 = \rho_1 \sigma_1,$$

$$\rho_1 = \frac{2M}{\tau_1},$$

$$\sigma_1 = i - \tau_1,$$

$$\tau_1 = 2^{\lfloor \log_2(i-1) \rfloor}, \quad (11)$$
and similarly,

\[
\alpha_2 = \rho_2(\sigma_2 - 1) + 1, \\
\beta_2 = \rho_2(\sigma_2 - 1) + \frac{\rho_2}{2}, \\
\gamma_2 = \rho_2\sigma_2, \\
\rho_2 = \frac{2N}{\tau_2}, \\
\sigma_2 = j - \tau_2, \\
\tau_2 = 2^{\lceil \log_2(j-1) \rceil}.
\] (12)

**Proof.** See [1].

Consider a function \(F(x, y, s, t)\) of four variables \(x, y, s\) and \(t\). Suppose \(F(x, y, s, t)\) is approximated using two-dimensional Haar wavelet as follows:

\[
F(x, y, s, t) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y)h_i(s)h_j(t)
\] (13)

Substituting the collocation points:

\[
s_p = \frac{p - 0.5}{2M}, \quad p = 1, 2, \ldots, 2M,
\]

and

\[
t_q = \frac{q - 0.5}{2M}, \quad q = 1, 2, \ldots, 2N,
\]

we obtain the following system of linear equations:

\[
F(x, y, s_p, t_q) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y)h_i(s_p)h_j(t_q), \quad p = 1, 2, \ldots, 2M, \quad q = 1, 2, \ldots, 2N. \quad (14)
\]

**Corollary 1.** The solution of the system (14) for any value of \(x, y \in [0, 1]\) is given as follows:

\[
b_{1,1}(x, y) = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} F(x, y, s_p, t_q),
\]

\[
b_{i,1}(x, y) = \frac{1}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{2M} \sum_{q=1}^{2N} F(x, y, s_p, t_q) - \sum_{p=\beta_1+1}^{2M} \sum_{q=1}^{2N} F(x, y, s_p, t_q) \right), \quad i = 2, 3, \ldots, 2M,
\]

\[
b_{1,j}(x, y) = \frac{1}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{2N} F(x, y, s_p, t_q) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{2N} F(x, y, s_p, t_q) \right), \quad j = 2, 3, \ldots, 2N,
\]

\[
b_{i,j}(x, y) = \frac{1}{\rho_1 \rho_2} \left( \sum_{p=\alpha_1}^{2M} \sum_{q=\alpha_2}^{2N} F(x, y, s_p, t_q) - \sum_{p=\alpha_1}^{2M} \sum_{q=\beta_2+1}^{2N} F(x, y, s_p, t_q) - \sum_{p=\beta_1+1}^{2M} \sum_{q=\alpha_2}^{2N} F(x, y, s_p, t_q) \right.
\]

\[
+ \left. \sum_{p=\beta_1+1}^{2M} \sum_{q=\beta_2+1}^{2N} F(x, y, s_p, t_q) \right), \quad i = 2, 3, \ldots, 2M, \quad j = 2, 3, \ldots, 2N,
\]

where \(\alpha_1, \beta_1, \gamma_1\) and \(\rho_1\) are defined as in Eq. (11) and \(\alpha_2, \beta_2, \gamma_2\) and \(\rho_2\) are defined as in Eq. (12).
Corollary 2. Suppose a function \( F(x, y) \) of two variables \( x \) and \( y \) is approximated using Haar wavelet approximation given in Eq. (9). Suppose further that \( F(x, y) \) is known at collocation points \((x_m, y_n), m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N\). Then the approximate value of the function \( F(x, y) \) at any other point of the domain can be calculated as follows:

\[
F(x, y) = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} F(x_m, y_n) h_1(x) h_1(y)
+ \sum_{i=1}^{2M} \frac{1}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{2M} \sum_{q=1}^{2N} F(x_m, y_n) - \sum_{p=\beta_1+1}^{2M} \sum_{q=1}^{2N} F(x_m, y_n) \right) h_i(x) h_1(y)
+ \sum_{j=1}^{2N} \frac{1}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{2N} F(x_m, y_n) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{2N} F(x_m, y_n) \right) h_1(x) h_j(y)
+ \sum_{i=1}^{2M} \sum_{j=1}^{2N} \frac{1}{\rho_1 \rho_2} \left( \sum_{p=\alpha_1}^{2M} \sum_{q=\alpha_2}^{2N} F(x_m, y_n) - \sum_{p=\beta_1+1}^{2M} \sum_{q=\beta_2+1}^{2N} F(x_m, y_n) \right) h_i(x) h_j(y),
\]

where \( \alpha_1, \beta_1, \gamma_1 \) and \( \rho_1 \) are defined as in Eq. (11) and \( \alpha_2, \beta_2, \gamma_2 \) and \( \rho_2 \) are defined as in Eq. (12).

3.3 Two-dimensional Fredholm integral equation

Consider the two-dimensional Fredholm integral equation (1). Assume that the kernel function \( K \) is approximated using two-dimensional Haar wavelet as follows:

\[
K(x, y, s, t, u(s, t)) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{ij}(x, y) h_i(s) h_j(t).
\]

Substituting this approximation of the kernel function in Eq. (1) we obtain the following equation:

\[
u(x, y) = f(x, y) + \int_0^1 \int_0^1 \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{ij}(x, y) h_i(s) h_j(t) \, ds \, dt.
\]

(16)

Using the properties of Haar wavelet Eq. (16) reduces to the following form:

\[
u(x, y) = f(x, y) + b_{1,1}(x, y).
\]

Substituting the collocation points given in Eq. (6) and Eq. (7), the following system of equations is obtained:

\[
u(x_m, y_n) = f(x_m, y_n) + b_{1,1}(x_m, y_n), \quad m = 1, 2, \ldots, 2M, \quad n = 1, 2, \ldots, 2N.
\]

Finally substituting the expressions for \( b_{1,1} \) using Corollary 1 we obtain the following system of nonlinear equations:

\[
u(x_m, y_n) = f(x_m, y_n) + \frac{1}{2M \times 2M} \sum_{p=1}^{2M} \sum_{q=1}^{2N} K(x_m, y_n, s_p, t_q, u(s_p, t_q)),
\]

(17)

\[
m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N.
\]
Now Eq. (17) represents $4MN \times 4MN$ system of nonlinear equations with the following unknowns:

$$u(x_m, y_n), \quad m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N.$$ 

This system can be solved using either Newton’s method or Broyden’s method. The solution of the system (17) gives values of the unknown function $u(x, y)$ at the following collocation points:

$$(x_m, y_n), \quad m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N.$$ 

The value of the unknown function $u(x, y)$ at any point other than the collocation points can be calculated using Corollary 2.

For calculating Jacobian the partial derivative of Eq. (17) with respect to unknown $u(x_i, y_j)$ is given as follows:

$$\frac{\partial u(x_m, y_n)}{\partial u(x_i, y_j)} = \frac{1}{2M \times 2M} K(x_m, y_n, s_i, t_j, u(s_i, t_j)). \quad i, j, m, n = 1, 2, \ldots, 2M.$$ 

### 3.4 Two-dimensional Volterra integral equation

Consider the two-dimensional Volterra integral equation (2). As in the case of two-dimensional Fredholm integral equation, the kernel function $K(x, y, s, t, u(s, t))$ is approximated using two-dimensional Haar wavelet approximation given in Eq. (15). With this approximation Eq. (2) can be written as follows:

$$u(x, y) = f(x, y) + \int_0^y \int_0^x \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y) h_i(s) h_j(t) \, ds \, dt. \quad (18)$$

Eq. (18) can be written in a more compact form using the notation introduced in Eq. (5) and is given as follows:

$$u(x, y) = f(x, y) + \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y) p_{i,1}(x)p_{j,1}(y).$$

Substituting the collocation points given in (6) and (7), we obtain the following system of equations:

$$u(x_m, y_n) = f(x_m, y_n) + \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x_m, y_n) p_{i,1}(x_m)p_{j,1}(y_n), \quad m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N.$$
Now $b_{ij}, i = 1, 2, \ldots, 2M, j = 1, 2, \ldots, 2N$ can be replaced with their expressions given in Corollary 1 and the following system of equations is obtained:

$$u(x_m, y_n) = f(x_m, y_n) + \frac{p_{1,1}(x_m)p_{1,1}(y_n)}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) +$$

$$\sum_{i=2}^{2M} \frac{p_{i,1}(x_m)p_{1,1}(y_n)}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2N} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) \right) +$$

$$\sum_{j=2}^{2N} \frac{p_{1,1}(x_m)p_{j,1}(y_n)}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) - \sum_{p=\beta_2 + 1}^{\gamma_2} \sum_{q=\alpha_2}^{\beta_2} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) \right) +$$

$$\sum_{i=2}^{2M} \sum_{j=2}^{2N} \frac{p_{i,1}(x_m)p_{j,1}(y_n)}{\rho_1 \times \rho_2} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) \right),$$

$$m = 1, 2, \ldots, 2M, n = 1, 2, \ldots, 2N.$$  \hspace{1cm} (19)

Eq. (19) represents $4MN \times 4MN$ system which as in the case of two-dimensional Fredholm integral equation can be solved using either Newton’s method or Broyden’s method. The solution of this system gives values of $u(x, y)$ at the collocation points. The values of $u(x, y)$ at points other than collocation points can be calculated using Corollary 2.

For calculating Jacobian the partial derivative of Eq. (19) with respect to unknown $u(x_i, y_j)$ is given as follows:

$$\frac{\partial u(x_m, y_n)}{\partial u(x_i, y_j)} = \frac{p_{1,1}(x_m)p_{1,1}(y_n)}{2M \times 2N} K(x_m, y_n, s_i, t_j, u(s_i, t_j))$$

$$+ \sum_{i=2}^{2M} \frac{p_{i,1}(x_m)p_{1,1}(y_n)}{\rho_1 \times 2N} K(x_m, y_n, s_i, t_j, u(s_i, t_j)) \left( I_{[\alpha_1, \beta_1]}(i) - I_{[\beta_1+1, \gamma_1]}(i) \right)$$

$$+ \sum_{j=2}^{2N} \frac{p_{1,1}(x_m)p_{j,1}(y_n)}{2M \times \rho_2} K(x_m, y_n, s_i, t_j, u(s_i, t_j)) \left( I_{[\alpha_2, \beta_2]}(j) - I_{[\beta_2+1, \gamma_2]}(j) \right)$$

$$+ \sum_{i=2}^{2M} \sum_{j=2}^{2N} \frac{p_{i,1}(x_m)p_{j,1}(y_n)}{\rho_1 \times \rho_2} K(x_m, y_n, s_i, t_j, u(s_i, t_j)) \left( I_{[\alpha_1, \beta_1]}(i) I_{[\alpha_2, \beta_2]}(j) - I_{[\alpha_1, \beta_1]}(i) I_{[\beta_2+1, \gamma_2]}(j) \right)$$

$$- I_{[\beta_1+1, \gamma_1]}(i) I_{[\alpha_2, \beta_2]}(j) + I_{[\beta_1+1, \gamma_1]}(i) I_{[\beta_2+1, \gamma_2]}(j).$$

$I$ denotes the characteristic function defined as follows:

$$I_{[a, b]}(x) = \begin{cases} 
1 & \text{if } a \leq x \leq b \\
0 & \text{otherwise.}
\end{cases}$$
3.5 Two-dimensional Volterra-Fredholm integral equation

We will consider two-dimensional Volterra-Fredholm integral equation (3). We proceed similar to the previous two cases and obtain the following system of nonlinear equations:

\[ u(x_m, y_n) = f(x_m, y_n) + \frac{h_1(x_m)p_{1,1}(y_n)}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) + \sum_{j=2}^{2N} \frac{h_1(x_m)p_{j,1}(y_n)}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} K(x_m, y_n, s_p, t_q, u(s_p, t_q)) \right), \]

where \( \alpha_2, \beta_2, \gamma_2 \) are defined as follows:

\[ \alpha_2, \beta_2, \gamma_2 \text{ are defined as follows:} \]

The partial derivative in this case is given as follows:

\[ \frac{\partial u(x_m, y_n)}{\partial u(x_i, y_j)} = \frac{h_1(x_m)p_{1,1}(y_n)}{2M \times 2N} K(x_m, y_n, s_i, t_j, u(s_i, t_j)) + \sum_{j=2}^{2N} \frac{h_1(x_m)p_{j,1}(y_n)}{2M \times \rho_2} K(x_m, y_n, s_i, t_j, u(s_i, t_j)) \left( I_{[\alpha_2,\beta_2]}(j) - I_{[\beta_2+1,\gamma_2]}(j) \right). \]

4 Numerical Experiments

Some numerical examples have been solved to illustrate the efficiency of the given method. In all Test Problems we have used Broyden’s method. The initial guess for the Broyden’s method was taken to be zero and the iterations were terminated when the convergent criterion \( 10^{-10} \) was satisfied. We have also calculated the experimental rate of convergence \( R_c(N) \) which is defined as follows:

\[ R_c(N) = \frac{\log[E_c(N)/E_c(N-1)]}{\log 2} \]  

**Test Problem 1.** Consider the following nonlinear Fredholm integral equation [21, 22]:

\[ u(x, y) = f(x, y) + \int_0^1 \int_0^1 (s \sin t + 1)u^3(s, t)dsdt, \]

where

\[ f(x, y) = x \cos y + \frac{1}{20} \left( \cos^4 1 - 1 \right) - \frac{1}{12} \sin 1 \cos^2 1 + 2. \]

The exact solution of the problem is \( u(x, y) = x \cos y \). The comparison of the maximum absolute errors for the present method versus Haar wavelet method [21] is shown in Table 1. The better performance of the present method is evident from the figure. The figure also shows that the present method is applied to a very finer grid of size \( 64 \times 64 \). In Fig. 1, three-dimensional graphs of the approximate solution for various values of \( 2M \) are shown. It can easily be seen from the figure that as the number of collocation points increases the approximate solution converges rapidly towards the exact solution.

**Test Problem 2.** Let us consider the following nonlinear Fredholm integral equation of second kind [17, 22]:

\[ u(x, y) = -\log \left( 1 + \frac{xy}{1+y^2} \right) + \frac{x}{16(1+y)} + \int_0^1 \int_0^1 \frac{x(1-s^2)}{(1+y)(1+r^2)} \left( 1 - \exp(-u(s,t)) \right) dsdt. \]
Table 1: Comparison of maximum absolute errors with the Haar wavelet method [21] for Test Problem 1

<table>
<thead>
<tr>
<th>J</th>
<th>M</th>
<th>2M</th>
<th>Present Method</th>
<th>Rate of convergence ($R_c(N)$)</th>
<th>Haar wavelet method [21]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>9.6×10^{-2}</td>
<td>1.1587</td>
<td>5.2×10^{-2}</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4.3×10^{-2}</td>
<td>1.3388</td>
<td>2.1×10^{-2}</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
<td>1.7×10^{-2}</td>
<td>1.6545</td>
<td>6.8×10^{-3}</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>16</td>
<td>5.4×10^{-3}</td>
<td>1.8480</td>
<td>1.9×10^{-3}</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>32</td>
<td>1.5×10^{-3}</td>
<td>1.9809</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>64</td>
<td>3.8×10^{-4}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2M = 8

2M = 16

2M = 32

2M = 64

Figure 1: Plots of approximate solution for different value of 2M for Test Problem 1.
Table 2: Comparison of maximum absolute errors with the Nystrom method [17] for Test Problem 2

<table>
<thead>
<tr>
<th>J</th>
<th>M</th>
<th>2M</th>
<th>Present method</th>
<th>Rate of convergence ((R_e(N)))</th>
<th>Nystrom method [17]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>(8.7 \times 10^{-3})</td>
<td>(7.6 \times 10^{-2})</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>(2.6 \times 10^{-3})</td>
<td>(1.4 \times 10^{-2})</td>
<td>(1.7425)</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>16</td>
<td>(7.3 \times 10^{-4})</td>
<td>(1.9419)</td>
<td>(8.3 \times 10^{-4})</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>32</td>
<td>(1.9 \times 10^{-4})</td>
<td>(1.9260)</td>
<td>(1.9434)</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>64</td>
<td>(5.0 \times 10^{-5})</td>
<td>(1.9434)</td>
<td>(8.3 \times 10^{-4})</td>
</tr>
</tbody>
</table>

Table 3: Comparison of maximum absolute errors with the Haar wavelet method [21] for Test Problem 3

<table>
<thead>
<tr>
<th>J</th>
<th>M</th>
<th>2M</th>
<th>Present method</th>
<th>Rate of convergence ((R_e(N)))</th>
<th>Haar wavelet method [21]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>(1.1 \times 10^{-2})</td>
<td>(1.9 \times 10^{-2})</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>(4.2 \times 10^{-3})</td>
<td>(1.3890)</td>
<td>(6.1 \times 10^{-3})</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>8</td>
<td>(1.3 \times 10^{-3})</td>
<td>(1.6919)</td>
<td>(1.9 \times 10^{-3})</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>16</td>
<td>(3.6 \times 10^{-4})</td>
<td>(1.8524)</td>
<td>(1.7 \times 10^{-3})</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>32</td>
<td>(9.65 \times 10^{-5})</td>
<td>(1.8994)</td>
<td>(4.6 \times 10^{-4})</td>
</tr>
</tbody>
</table>

The exact solution of the problem is

\[
u(x, y) = \log \left(1 + \frac{xy}{1 + y^2}\right).
\]

Table 2 shows the comparison of the maximum absolute errors of the present method with Nystrom method [17]. It is obvious from the table that the present method is far more accurate than the Nystrom method [17].

**Test Problem 3.** Consider the following nonlinear two-dimensional Volterra integral equation of second kind [21]:

\[
u(x, y) = f(x, y) + \int_{0}^{y} \int_{0}^{x} (x + y - t - s)u^2(s, t)dsdt,
\]

where

\[
f(x, y) = x + y - \frac{1}{12}xy(x^3 + 4x^2y + 4xy^2 + y^3).
\]

The exact solution of the problem is \(u(x, y) = x + y\). In Table 3 we have shown the comparison of maximum absolute errors of the present method with the Haar wavelet method [21]. Once again a better performance of the present method is evident from the table.
Table 4: Comparison of maximum absolute errors with the Posteriori error estimate method [24] for Test Problem 4

<table>
<thead>
<tr>
<th>J</th>
<th>M</th>
<th>2M</th>
<th>Present Method</th>
<th>Rate of convergence ((R_c(N)))</th>
<th>Posteriori error estimate [24]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5.2×10^{-3}</td>
<td></td>
<td>1.4×10^{-2}</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1.6×10^{-3}</td>
<td>1.7004</td>
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<td>2</td>
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<td>8</td>
<td>4.4×10^{-4}</td>
<td>1.8625</td>
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<tr>
<td>3</td>
<td>8</td>
<td>16</td>
<td>1.2×10^{-4}</td>
<td>1.8745</td>
<td>3.7×10^{-4}</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>32</td>
<td>3.0×10^{-5}</td>
<td>2.0000</td>
<td>4.55×10^{-5}</td>
</tr>
<tr>
<td>5</td>
<td>32</td>
<td>64</td>
<td>7.5×10^{-6}</td>
<td>1.9809</td>
<td></td>
</tr>
</tbody>
</table>

**Test Problem 4.** Consider the two-dimensional nonlinear Volterra-Fredholm integral equation [23,24]:

\[
u(x,y) = f(x,y) + \int_{0}^{y} \int_{0}^{1} \frac{x(1-s^2)}{(1+y)(1+t^2)} (1 - \exp(-u(s,t))) ds dt,
\]

where

\[
f(x,y) = -\log \left(1 + \frac{xy}{1+y^2}\right) + \frac{xy^2}{8(1+y)(1+y^2)}.
\]

The exact solution of the problem is

\[
u(x,y) = -\log \left(1 + \frac{xy}{1+y^2}\right).
\]

The comparison of maximum absolute errors of the present method with the Posteriori error estimate method [24] is shown in Table 4. In Fig. 2 three-dimensional graphs of the approximate solution for various values of \(2M\) are shown. The figure clearly shows the convergence and accuracy of the method.
Figure 2: Plots of approximate solution for different value of $2M$ for Test Problem 4.
5  Conclusion

Haar wavelet collocation method is applied in order to find numerical solution of nonlinear two-dimensional Fredholm, Volterra and Volterra-Fredholm integral equations. The method is tested on several benchmark problems from the literature. The numerical results are compared with a few existing methods reported recently in the literature. The numerical evidence shows superiority of the new method in terms of fast convergence and better accuracy.

References


