On the covering radius of Reed–Muller codes

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Abstract
We present lower and upper bounds on the covering radius of Reed–Muller codes, yielding asymptotical improvements on known results. The lower bound is simply the sphere covering one (not very new). The upper bound is derived from a thorough use of a lemma, the ‘essence of Reed–Mullerity’. The idea is to find a ‘seed’ upper bound—a properly chosen combination of binomial coefficients—well fitted to the respective growths of $m$ (log of length) and $r$ (order), to initiate double induction on $m$ and $r$. Surprisingly enough, these two simple ingredients suffice to essentially fill the gaps between lower and upper bounds, a result stated in our theorem.

1. Introduction

We present lower and upper bounds on the covering radius of Reed–Muller codes, yielding asymptotical improvements on known results (see for example [4, 5]). The lower bound is simply the sphere covering one (not very new). The upper bound is derived from a thorough use of a lemma, the ‘essence of Reed–Mullerity’. The idea is to find a ‘seed’ upper bound—a properly chosen combination of binomial coefficients—well fitted to the respective growths of $m$ (log of length) and $r$ (order), to initiate double induction on $m$ and $r$. Surprisingly enough, these two simple ingredients suffice to essentially fill the gaps between lower and upper bounds, a result stated in our theorem.

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As is well known (see, e.g., [3]), \( R(r, m) \), the \( r \)th order Reed-Muller code has length \( n = 2^m \), dimension \( k = \sum_{i=0}^{r} \binom{m}{i} \) and minimum distance \( d = 2^{m-r} \). We denote by \( \rho(r, m) \) or simply \( \rho \) its covering radius (see [1]) and deal with asymptotical lower and upper estimates of it when \( m \to \infty \). The entropy function is \( H(x) := -x \log x - (1-x) \log(1-x) \), where \( \log \) is binary.

2. Lower bounds

We use the classical sphere-covering bound:

\[
k(r, m) + \log \sum_{i=0}^{\rho} \binom{2^m}{i} \geq 2^m. \tag{0}
\]

Case 1: \( m - r = o(m) \).

We write (0) as

\[
\log \sum_{i=0}^{\rho} \binom{2^m}{i} \geq \sum_{i=0}^{m-r-1} \binom{m}{i}.
\]

Now, combining (a) and (b), where

(a) \( \sum_{i=0}^{m-r-1} \binom{m}{i} \geq \frac{m^{m-r-1}}{(m-r-1)!} \);

(b) \( \log \sum_{i=0}^{\rho} \binom{2^m}{i} \leq \log \rho + m\rho - \log \rho = m\rho; \)

we get

\[
\rho(r, m) \geq \frac{m^{m-r-2}}{(m-r-1)!}. \tag{1}
\]

Case 2: \( r = o(m) \).

Here (0) gives

\[
\log \sum_{i=0}^{\rho} \binom{2^m}{i} \geq 2^m - \sum_{i=0}^{r} \binom{m}{i}.
\]

(a) \( 2^m - \sum_{i=0}^{r} \binom{m}{i} \geq 2^m - m^r/r!; \)

(b) \( \log \sum_{i=0}^{\rho} \binom{n}{i} \leq \log \rho + nH(\rho/n). \)

Setting \( x := 2^{m-1} - \rho \), \( y := x/n \) and taking into account \( H(\frac{1}{2} - y) = 1 - 2y^2 \) for \( y \to 0 \), we get

\[
\rho \geq 2^{m-1} - 2^{m/2} m^{r^2}((\ln 2)/2r!)^{1/2}. \tag{2}
\]
Case 3: \( r/m =: \alpha > \frac{1}{2} \).
In this case \( \rho \ll n \),

(a) \( \log \sum_{i=0}^{r} \binom{n}{i} \leq m\rho \);

(b) \( \sum_{i=0}^{m-r-1} \binom{m}{i} \geq \frac{1}{2\sqrt{2ma(1-\alpha)}} 2^{mH(1-\alpha)} \);

and

\[ \rho \approx (2m^3)^{-\frac{1}{2}} 2^{mH(\alpha)}. \]  

(3)

Case 4: \( \alpha < \frac{1}{2} \).
In this case \( x = 2^{m-1} - \rho \ll 2^m \).

(a) \( \log \sum_{i=0}^{r} \binom{n}{i} \leq n - 2x^2/(n \ln 2) \);

(b) \( \sum_{i=0}^{m-r-1} \binom{m}{i} = 2^m - \sum_{i=0}^{r} \binom{m}{i} \geq 2^m - 2^{mH(\alpha)} \);

and

\[ \rho \approx 2^{m-1} - ((\ln 2)/2)^{1/2} 2^{m(1+H(\alpha))}. \]  

(4)

3. Upper bounds

The following result is simple but essential to our purpose.

Lemma. \( \rho(r, m) \leq \rho(r - 1, m - 1) + \rho(r, m - 1) \).

Proof. Reed–Muller codes are inductively defined (see [3]) by \( R(r, m) \) \(=\) \( \{(u, u + v): u \in R(r, m - 1), \ v \in R(r - 1, m - 1)\} \) where \((\cdot, \cdot)\) denotes concatenation. Let \( x = (x_1, x_2) \) be an arbitrary vector, then \( x \) can be ‘approximated’ by first choosing \( u \), a closest word to \( x_1 \) in \( R(r, m - 1) \), and then \( v \), a closest word to \( x_2 + u \) in \( R(r - 1, m - 1) \). Then

\[ d((x_1, x_2), (u, u + v)) = d(x_1, u) + d(x_2, u + v) \leq \rho(r, m - 1) + \rho(r - 1, m - 1). \]  

Case 1: \( m - r = \lambda = o(m) \).

We prove by induction on \( \lambda \),

\[ \rho(r, m) \leq \frac{m^{m-r-2}}{(m-r-2)!} + O(m^{m-r-3}). \]  

(5)
The inequality is valid for \( \lambda = 3 \) (see [2]). Let it be valid for \( m - r - 1 \). Then

\[
\rho(r, m) \leq \rho(r - 1, m - 1) + \rho(r, m - 1)
\]

\[
\leq \rho(r - 1, m - 1) + \frac{(m - 1)^{m-r-3}}{(m - r - 3)!} + O(m^{m-r-4})
\]

\[
\leq \sum_{i=1}^{r} \frac{(m-i)^{m-r-3}}{(m - r - 3)!} + \sum_{i=1}^{r} O(m^{m-r-4})
\]

\[
+ \rho(0, m - r) = \frac{m^{m-r-2}}{(m - r - 2)!} + O(m^{m-r-3}).
\]

**Case 2:** \( r = o(m) \).

We prove by induction that for \( r \geq 2 \)

\[
\rho(r, m) \leq 2^{m-1} - (\sqrt{2} + 1)^{r-1}2^{(m-3)/2} + O(m^{r-2}). \tag{6}
\]

Using results on upper bounds for \( \rho(1, m) \), we have

\[
\rho(2, m) < \rho(2, m - 1) + \rho(1, m - 1) \leq \rho(2, m - 1) + 2^{m-2} - 2^{(m-3)/2}
\]

\[
\leq \rho(2, 2) + \sum_{i=1}^{m-2} 2^i - \sum_{i=0}^{m-3} 2^{i/2} \leq 2^{m-1} - (\sqrt{2} + 1)2^{(m-2)/2}.
\]

Let (6) be valid for \( r - 1 \). Then

\[
\rho(r, m) \leq \rho(r, m - 1) + \rho(r - 1, m - 1)
\]

\[
\leq \rho(r, m - 1) + 2^{m-2} - (m-r)^r (\sqrt{2} + 1)^{r-2} + O(m^{r-3})
\]

\[
\leq \rho(r, r) + \sum_{i=r}^{m-2} 2^i - (\sqrt{2} + 1)^r + \sum_{i=r-3}^{m-3} 2^{i/2} + O(m^{r-2})
\]

\[
\leq 2^{m-1} - (2^{(m-2)/2} (\sqrt{2} + 1)^{r-1}).
\]

Let us compare our bound with one from [4],

\[
\rho(r, m) \leq 2^{m-1} - (2^{r/2} - 2^{r/3}2^{m/2})^{1/2}
\]

For \( r = \text{const} \), this gives \( \rho(r, m) \approx 2^{m-1} - (2^{r/2} - 2^{r/3}2^{m/2})^{1/2} \). We have in this range \( \rho(r, m) \approx 2^{m-1} - \frac{1}{2}(2.42)^{r-1} \). For example, for \( r = 2 \), [4] and (6) yield respectively

\[
\rho(2, m) \approx 2^{m-1} - 0.74 \ldots 2^{m/2},
\]

\[
\rho(2, m) \approx 2^{m-1} - 1.21 \ldots 2^{m/2}.
\]

**Case 3:** \( \alpha = r/m = \text{const} > \frac{1}{2} \).

We prove by induction that for \( r \leq m - 3 \)

\[
\rho(r, m) \leq \sum_{i=1}^{m-r-2} \binom{m + 2}{i}.
\]

The inequality holds for \( r = m - 3 \), as well as it is valid for \( r = 0 \). Let it be valid
for \((m - r - 1)\) and all \(m\), and for \((m - r)\) and all lengths less than \(2^m\). Then,

\[
\rho(r, m) \leq \sum_{i=1}^{m-r-2} \binom{m+1}{i} + \sum_{i=1}^{m-r-3} \binom{m+1}{i} = \sum_{i=1}^{m-r-2} \binom{m+2}{i} - 1 < \sum_{i=1}^{m-r-2} \binom{m+2}{i}.
\]

Standard arguments give

\[
\rho(r, m) \leq 2^{mH(1-\alpha)}. \tag{7}
\]

Case 4: \(\alpha = r/m = \text{const} < \frac{1}{2}\).

We prove by induction that for \(r \geq 1\)

\[
\rho(r, m) \leq 2^{m-1} - \binom{m}{r}. \tag{10}
\]

The inequality holds for \(r = 1\) and all \(m\), and for \(r = m\) and all \(r\). Let it be valid for \(r - 1\) and all \(m\), and for \(r\) and \(m - 1\). Then,

\[
\rho(r, m) \leq 2^{m-2} - \binom{m-1}{r} + 2^{m-2} - \binom{m-1}{r-1} = 2^{m-1} - \binom{m}{r}.
\]

This gives asymptotically

\[
\rho(r, m) \leq 2^{m-1} - 2^{mH(\alpha)}(2m\alpha(1 - \alpha))^{-\frac{1}{2}}. \tag{8}
\]

For \(0 \leq r/m \leq (2 + \sqrt{2})^{-1}\), the following inequality holds:

\[
\rho(r, m) \leq 2^{m-1} - (\sqrt{2} + 1)^{-\frac{1}{2}(m-2)^2} + r \binom{m}{r}. \tag{9}
\]

The proof by induction is quite technical and is presented as an appendix. Denoting \(\rho(r, m) = 2^{m-1} - 2^{mH(\alpha)}\) and combining (3), (9) and (8) gives the following.

\[
\frac{1}{2}(1 + H(\alpha)) \geq f(\alpha) \geq \begin{cases} \frac{1}{2} + \alpha \log 2(\sqrt{2} + 1), & 0 \leq \alpha \leq 1 - \sqrt{2}/2 = 0.293, \\ H(\alpha), & 1 - \sqrt{2}/2 \leq \alpha \leq \frac{1}{2}. \end{cases}
\]

This corresponds to the dotted area in Fig. 1.

![Fig. 1. Bounds for \(r/m < 0.5\).](image-url)
When \( r/m \) grows from 0 to 1, the rate \( R = k/m \) of \( R(r, m) \) grows from 0 to 1. Asymptotically the behavior is discontinuous, with

\[
R = 0 \text{ for } r/m < \frac{1}{2},
\]

\[
R = 1 \text{ for } r/m > \frac{1}{2},
\]

and

\[
R = \frac{1}{2} \text{ for } r/m = \frac{1}{2}.
\]

We shall now treat this last case.

A lower bound on \( \rho \) is readily obtained through the sphere covering bound, yielding

\[
\rho \geq n H^{-1} \left( \frac{1}{3} \right).
\]  
\hspace{11.25cm} (10)

For the upper bound, we use a result from [4]: If \( h \) is a positive integer such that \( m - 2 \geq r \geq (h - 1)(m - 1)/h \), then

\[
\rho(r, m) \leq 2^{m-h} - 2^{r+2-h} + 2.
\]

For \( h = 2 \), this gives

\[
\rho(r, m) \leq 2^{m-2} - 2^{m/2} + 2 \leq \left( \frac{4}{3} \right) n.
\]

Summarizing (1)–(11), we have the following.

**Theorem.** For \( m \) large enough, if \( o(m) = m - r \geq 3 \), then

\[
\frac{m^{m-r-2}}{(m-r-1)!} \leq \rho(r, m) \leq \frac{m^{m-r-2}}{(m-r-2)!}.
\]

i.e., \( \rho(r, m) = \Theta(m^{m-r-2}) \). If \( r \geq 2 \), \( r = o(m) \), then

\[
2^{m-1} - 2^{m/2} m^{r/2} ((\ln 2)/2r!)^{1/2} \leq \rho(r, m) \leq 2^{m-1} - 2^{m/2} (\sqrt{2} + 1)^{r-1}.
\]

If \( r/m = \text{const} \), \( r/m \geq \frac{1}{3} \), then

\[
(2m^{3})^{-1} 2^{mH(1-r/m)} \leq \rho(r, m) \leq 2^{mH(1-r/m)}.
\]

If \((2 + \sqrt{2})^{-1} \leq r/m \leq \frac{1}{3} \), then

\[
2^{m-1} - ((\ln 2)/2)^{1/2} 2^{m(1+H(r/m))} \leq \rho(r, m) \leq 2^{m-1} - 2^{mH(r/m)}.
\]

If \( 0 \leq r/m = (2 + \sqrt{2})^{-1} \), then

\[
2^{m-1} - ((\ln 2)/2)^{1/2} 2^{m(1+H(r/m))} \leq \rho(r, m) \leq 2^{m-1} - (\sqrt{2} + 1)^{-1} 2^{m/2};
\]

If \( r/m = \frac{1}{3} \), then

\[
n/4 \geq \rho(m/2, m) \geq H^{-1} \left( \frac{1}{3} \right) n \approx 0.11 n,
\]

i.e., \( \rho(m/2, m) = \Theta(n) \).
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Appendix: Proof of formula (9)

We will show by induction that for \( m \geq (\sqrt{2} + 2)r \) the following inequality holds:

\[
\rho(r, m) \leq 2^{m-1} - \frac{1}{(\sqrt{2} - 1)^{r-1}}2^{(m-2)r} + \binom{m}{r}.
\]

Easy fact:

\[
H\left(\frac{1}{\sqrt{2} + 2}\right) = \frac{1}{2} + \frac{1}{\sqrt{2} + 2} \log(\sqrt{2} + 1).
\]

Hence,

\[
\left(\frac{(\sqrt{2} + 2)^r}{r}\right) \geq \frac{2^{(\sqrt{2} + 2)r}H\left(\frac{1}{\sqrt{2} + 2}\right)}{\sqrt{8(\sqrt{2} + 2)r} \cdot \frac{1}{\sqrt{2} + 2} \cdot \frac{\sqrt{2} - 1}{\sqrt{2} + 2}}
\]

\[
= \frac{1}{\sqrt{8r}} 2^{(\sqrt{2} + 2)r(\frac{1}{2} + (\sqrt{2} + 2)^{-1}) \log(\sqrt{2} + 1)}
\]

\[
= \frac{1}{\sqrt{8r}} 2^{(\sqrt{2} + 2)r/2 + r \log(\sqrt{2} + 1)}.
\]

\[
\frac{1}{(\sqrt{2} - 1)^{r-1}} 2^{(\sqrt{2} + 2)r/2 - 1} = 2^{r-1} \log(\sqrt{2} + 1) + r(\sqrt{2} + 2)^{2-1} - \log(\sqrt{2} + 1)
\]

\[
= 2^{r-1} \log(\sqrt{2} + 1) + r(\sqrt{2} + 2)^{2-1} - \log(\sqrt{2} + 1)
\]

\[
= \frac{1}{(\sqrt{2} + 1)^r} 2^{r \log(\sqrt{2} + 1) + r(\sqrt{2} + 2)^{2-1} - \log(\sqrt{2} + 1)}.
\]
Furthermore, 

\[
\left( \frac{\left[ (\sqrt{2} + 2)r \right]}{r} \right) \geq \left( \frac{(\sqrt{2} + 2)r}{r} \right) \frac{r}{\left( \frac{(\sqrt{2} + 2)r}{r} \right)}
\]

\[
\geq \left( \frac{(\sqrt{2} + 2)r}{r} \right) \frac{(\sqrt{2} + 2)r - 1}{\left( \frac{(\sqrt{2} + 2)r}{r} \right)}
\]

\[
= \left( \frac{(\sqrt{2} + 2)r}{r} \right) \frac{(\sqrt{2} + 2)r + 1}{\left( \frac{(\sqrt{2} + 2)r}{r} \right)} > \frac{1}{\sqrt{2}} \left( \frac{(\sqrt{2} + 2)r}{r} \right),
\]

\[
\frac{1}{(\sqrt{2} - 1)^{r^2 - 1}} 2^{(\sqrt{2} + 2)r^2 - 1} < \frac{1}{\sqrt{2}(\sqrt{2} + 1)^{r^2}} 2^{r \log(\sqrt{2} + 1) + r(\sqrt{2} + 2)/2}.
\]

Thus, 

\[
\rho(r, \left[ (\sqrt{2} + 2)r \right]) \leq 2^{m-1} - \frac{1}{\sqrt{2}(\sqrt{2} + 1)} 2^{r \log(\sqrt{2} + 1) + r(\sqrt{2} + 2)/2}
\]

\[+ \frac{r}{4\sqrt{r}} 2^{r \log(\sqrt{2} + 1) + r(\sqrt{2} + 2)/2}
\]

and for \(r^{1/4} \geq 1/(\sqrt{2}(\sqrt{2} + 1))\), or for \(r \geq 2\), we have \(\rho(r, \left[ (\sqrt{2} + 2)r \right]) \leq 2^{m-1} + \varepsilon\), \(\varepsilon > 0\), which is always valid since \(\rho(r, m) < 2^{m-1}\) for \(m \geq r\), and \(r \geq 1\). Furthermore, \(\left[ (\sqrt{2} + 2)(r + 1) \right] - \left[ (\sqrt{2} + 2)r \right] \geq 3\).

The inequality (9) holds for \(r = 2\) and all \(m\). Let it be valid for some \(r - 1\) and all \(m \geq \left[ (\sqrt{2} + 2)(r - 1) \right]\). We will prove it for \(r\) and \(m \geq \left[ (\sqrt{2} + 2)r \right]\). It is O.K. for \(m = \left[ (\sqrt{2} + 2)r \right]\), thus for \(m > \left[ (\sqrt{2} + 2)r \right]\) we have (note that \(\left[ (\sqrt{2} + 2)(r - 1) \right]\))

\[
\rho(r, m) \leq 2^{m-2} \frac{1}{(\sqrt{2} - 1)^{r-1}} 2^{(m-3)/2} + r\binom{m-1}{r}
\]

\[+ 2^{m-2} - \frac{1}{(\sqrt{2} - 1)^{r-2}} 2^{(m-3)/2} + (r - 1)\binom{m-1}{r-1}
\]

\[\leq 2^{m-1} - \frac{1}{(\sqrt{2} - 1)^{r-1}} 2^{(m-3)/2} + r\binom{m}{r}.\]

\[\Box\]

References

On the covering radius of Reed–Muller codes

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