FINITISTIC n-SELF-COTILTING MODULES

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Abstract. We study a class of modules which can be characterized using a duality theorem, called finitistic n-self-cotilting. Such a module Q can be characterized using dual conditions of some generalizations for star modules: every module M which has a right resolution with n terms isomorphic to finite powers of Q (i.e. M is n-finitely Q-copresented) has a right resolution with $(n+1)$ terms, and the functor $\mathrm{Hom}_R(-,Q)$ preserves the exactness of all monomorphisms with their ranges finite powers of Q and cokernels n-finitely Q-copresented modules. In the general case, these modules are independent toward other kinds of modules which are characterized using some dualities (w-Π_f-quasi injective modules, costar modules, f-cotilting modules). Closure properties for the classes involved in the duality are studied. In the end of the paper connections with the cotilting theory are exhibited, in the case of finitely dimensional algebras over fields.

1. Introduction

In this paper R denotes a unital associative ring, Q is a right R-module, S is the endomorphism ring of Q, Mod-R is the category of all right R-modules, S-Mod denotes the category of all left S-modules and n will be always a positive integer.

Then Q is naturally a left S-module and we have a pair of adjoint contravariant functors

$$\Delta = \mathrm{Hom}_R(-,Q) : \text{Mod-}R \rightleftarrows S\text{-Mod} : \mathrm{Hom}_S(-,Q) = \Delta$$

with arrows of adjunction $\delta : 1 \to \Delta^2$. There exist two classes

$$\mathrm{Refl}(Q_R) = \{M \in \text{Mod-}R \mid \delta_M \text{ is an isomorphism}\}$$

and

$$\mathrm{Refl}({}_SQ) = \{A \in S\text{-Mod} \mid \delta_A \text{ is an isomorphism}\}$$

such that if $\Delta : \mathcal{C} \rightleftarrows \mathcal{D} : \Delta$ is a duality, where $\mathcal{C}$ is a class of right R-modules and $\mathcal{D}$ is a class of left S-modules, then $\mathcal{C} \subseteq \mathrm{Refl}(Q_R)$ and $\mathcal{D} \subseteq \mathrm{Refl}({}_SQ)$. A module which belongs to one of these classes is called *Q-reflexive*. If X is a right R-module or a left S-module such that the homomorphism δ_X is monic, then X is *Q-cogenerated*, i.e. it can be embedded in a power of Q as a right R-module, respectively as a left S-module. The class of all Q-cogenerated left S-modules is denoted by $\mathrm{Cog}({}_SQ)$.

The general problem is to establish connections between properties of Q as an R-module, properties of Q as an S-module, properties of rings R and S and properties of some classes $\mathcal{C} \subseteq \mathrm{Refl}(Q_R)$, $\mathcal{D} \subseteq \mathrm{Refl}({}_SQ)$ such that the pair $\Delta : \mathcal{C} \rightleftarrows \mathcal{D} : \Delta$ is a duality. An important result in this direction was obtained by Colby and Fuller in [11]. They introduced the notion of *costar* module as an R-module Q such that

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the functors Δ induce a duality between the class of all Q-cogenerated R-modules M such that the dual modules $\Delta(M)$ are finitely generated left S-modules and the class of all finitely generated Q-cogenerated left S-modules. They characterized this situation with some properties of Q as an R-module in [11, Theorem 2.7]. We recall that Q is costar if and only if every Q-cogenerated right R-module M with finitely generated dual $\Delta(M)$ is *semi-finitely-copresented* by Q (which means that there exists an exact sequence $0 \to M \to Q^X \to Q^Y$ with X a finite set) and Q is injective relative to all short exact sequences $0 \to L \to Q^X \to M \to 0$ in Mod-R where X is a finite set and M is a Q-cogenerated R-module (Q is *w-Π_f-quasi injective*). We also recall that Q is w-Π_f-quasi injective if and only if the functors Δ induce a duality between all semi-finitely Q-copresented right R-modules and all finitely generated Q-cogenerated left S-modules ([11, Proposition 2.6] and [20, 4.8]). A particular case of costar modules are *f-cotilting*-modules, introduced by Wisbauer in [20]: Q is f-cotilting if Δ induces a duality between all finitely Q-cogenerated modules and all finitely generated Q-cogenerated left S-modules, [20, 4.10]. We recall from [20, 3.12] that Q is f-cotilting if and only if every finitely Q-cogenerated right R-module is semi-finitely-copresented and Q is w-Π_f-quasi injective. Other similar results can be found in [2], [6], [8], [16].

We shall study dualities which are induced by modules which verify some conditions which have the same flavor as those which appear in the definition of costar modules [11], self-cotilting modules [20, 3.3], and in some generalizations for the notion of self-tilting modules introduced in [18] and [19], adding some finiteness conditions. These conditions are necessary to obtain dualities between full subcategories of some module categories. We recall that in the study of equivalences between subcategories of module categories it is important that the R-module Q under consideration is self-small since this condition implies that the right S-module $\mathrm{Hom}(Q, Q^{(I)})$ is free for every set I. The dual condition "$\mathrm{Hom}(Q^I, Q)$ is free" is valid only if I is a finite set or under some set theoretic conditions imposed to Q and I (see [13]).

In the next section we shall introduce the notion of finitistic n-self-cotilting module, we characterize these modules using a duality (Theorem 2.7). We show in Example 2.9 and Example 2.11 that the notion "finitistic n-self-cotilting module" is not connected, in the general case, to other notions which can be characterized using dualities (w-Π_f-quasi injective module, costar module or f-cotilting modules). The main aim of the third section is to enunciate closure properties for the classes which are involved in Theorem 2.7. In Proposition 3.2 it is proved that one of these classes is a resolving class. This establish a connection between our setting and cotilting theory as a consequence of [5, Theorem 2.2]. A new characterization for finitistic n-self-cotilting modules is proved in Theorem 3.4. This result is dual, modulo a finitistic condition, to Wei's result [18, Theorem 3.5]. In Example 3.6 it is proved that this finitistic condition is not superfluous. At the end of the paper we consider finitely generated finitistic n-self-cotilting modules over finite dimensional algebras over fields. In Proposition 4.2 we present some connections between n-cotilting modules and finitistic n-self-cotilting modules. From this proposition we deduce that for every integer $n > 0$ there exists a finitistic $(n + 1)$-self-cotilting module which is not a finitistic n-self-cotilting module.

2. Finitistic n-self-cotilting modules

We say that an R-module L is *n-finitely Q-copresented* whenever there exists a long exact sequence

$$0 \to L \xrightarrow{\alpha_0} Q^{X_0} \xrightarrow{\alpha_1} Q^{X_1} \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} Q^{X_{n-1}}$$

such that X_i are finite sets for all $i \in \{0, \dots, n-1\}$. The class of all n-finitely Q-copresented modules is denoted by n-cop(Q). Note that 1-cop(Q) is the class of all *finitely Q-cogenerated* modules, and it is denoted by cog(Q). We recall that 2-finitely Q-copresented modules are studied in [2], [3] and [6]. If S is a ring, a left S-module A is *n-finitely presented* if there exists an exact sequence

$$S^{Y_{n-1}} \to \dots \to S^{Y_0} \to A \to 0$$

such that Y_i are finite sets for all $i \in \{0, \dots, n-1\}$, and $\mathrm{FP}_n(S)$ denotes the class of all n-finitely presented modules left S-modules. The intersection $\bigcap_{n=1}^{\infty} \mathrm{FP}_n(S)$ is denoted by $\mathrm{FP}_\infty(S)$. Of course, $\mathrm{FP}_1(S)$ is the class of all finitely generated left S-modules, and $\mathrm{FP}_2(S)$ is in fact the class of all finitely presented left S-modules.

Let Q be a right R-module. As in [1], a short exact sequence of right R-modules

$$0 \to K \to L \to M \to 0$$

is called *Q-cobalanced* if the induced sequence

$$0 \to \Delta(M) \to \Delta(K) \to \Delta(L) \to 0$$

is exact. The right R-module Q is *n-w_f-quasi-injective* if every exact sequence in Mod-R

$$0 \to L \to Q^X \to M \to 0$$

with $M \in n$-cop(Q) and X a finite set is Q-cobalanced.

Definition 2.1. We say that a right R-module Q is a *finitistic n-self-cotilting* module if it is n-w_f-quasi-injective and n-cop$(Q) = (n+1)$-cop(Q).

Example 2.2. *Every torsion quasi-injective abelian group is a finitistic* 1*-self-cotilting $\mathbb{Z}$-module.*

Proof. A torsion abelian group Q is quasi-injective if and only if for every prime p the p-component Q_p of Q has the form $Q_p \cong \bigoplus_{I_p} \mathbb{Z}(p^{n_p})$ with $n_p \in \mathbb{N} \cup \{\infty\}$, where I_p is an index set. If Q is quasi-injective, it is not hard to deduce the equality 1-cop$(Q) =$ 2-cop(Q). Moreover, for every finite set X the $\mathbb{Z}$-module Q^X is also quasi-injective, hence every exact sequence $0 \to L \to Q^X \to M \to 0$ is Q-cobalanced. □

The following examples show that the conditions which define finitistic n-self-cotilting modules are independent. Moreover, there exists a finitistic 2-self-cotilting module which is not a finitistic 1-self-cotilting module.

Example 2.3. *The rational group $\mathbb{Q}$ is an injective $\mathbb{Z}$-module which is not a finitistic* 1*-self-cotilting $\mathbb{Z}$-modules, but it is a finitistic* 2*-self-cotilting $\mathbb{Z}$-module.*

Proof. Of course $\mathbb{Z} \in$ 1-cop$(\mathbb{Q})$, but for every monomorphism $\alpha : \mathbb{Z} \to \mathbb{Q}^X$ (where X is a finite set) the group Coker(α) is not torsion-free, hence it is not $\mathbb{Q}$-cogenerated. In fact, an abelian group is 2-finitely $\mathbb{Q}$-copresented if and only if it is isomorphic to a finite power of $\mathbb{Q}$. □

Example 2.4. *If we consider the $\mathbb{Z}$-module $Q = \mathbb{Z}(2) \oplus \mathbb{Z}(4)$ then* $1\text{-cop}(Q) = 2\text{-cop}(Q)$*, but Q is not 1-w_f-quasi-injective.*

Proof. If X is a finite set then every factor group M of Q^X is a finite 2-group which is bounded by 4, hence it is isomorphic to a finite direct sum of copies of $\mathbb{Z}(2)$ and $\mathbb{Z}(4)$. It follows that we can embed M in a finite power of $\mathbb{Z}(4)$, and we obtain $1\text{-cop}(Q) = 2\text{-cop}(Q)$. But Q is not quasi-injective, hence there exists an exact sequence $0 \to L \to Q \to M \to 0$ which is not Q-cobalanced and $M \in 1\text{-cop}(Q)$, hence Q is not 1-w_f-quasi-injective. $\square$

To characterize finitistic n-self-cotilting modules using a duality theorem we need the following left orthogonal classes:

$$^{\perp_{<n}}Q = \{A \in S\text{-Mod} \mid \mathrm{Ext}^i_S(A, Q) = 0 \text{ for all } 0 < i < n\}$$

and

$$^{\perp}Q = \{A \in S\text{-Mod} \mid \mathrm{Ext}^i_S(A, Q) = 0 \text{ for all } 0 < i < \omega\}.$$

We start with two useful lemmas.

Lemma 2.5. *If $M \in \mathrm{Refl}(Q_R)$ such that $\Delta(M) \in {}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S)$, then $M \in (n+1)\text{-cop}(Q_R)$.*

Proof. Let

$$S^{X_n} \to S^{X_{n-1}} \to \cdots \to S^{X_0} \to \Delta(M) \to 0$$

be a free resolution such that the sets X_i are finite for all $i \in \{0, \ldots, n\}$. Because $\Delta(M) \in {}^{\perp_{<n}}Q$ we obtain the exact sequence

$$0 \to \Delta^2(M) \to \Delta(S^{X_0}) \to \cdots \to \Delta(S^{X_n})$$

and this completes the proof since $\Delta(S^{X_i}) \cong Q^{X_i}$ for all $i \in \{0, \ldots, n\}$, and $M \cong \Delta^2(M)$. $\square$

Lemma 2.6. *Suppose that $n\text{-cop}(Q_R) \subseteq \mathrm{Refl}(Q_R)$ and*

$$(\star)\ \Delta(n\text{-cop}(Q_R)) \subseteq {}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) \subseteq \mathrm{Refl}({}_SQ).$$

Then Q is n-w_f-quasi-injective.

The same conclusion is valid if we replace the condition $(\star)$ with

$$(\star')\quad \Delta(n\text{-cop}(Q_R)) \subseteq {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) \subseteq \mathrm{Refl}({}_SQ).$$

Proof. Let $0 \to M \xrightarrow{\alpha} Q^X \xrightarrow{\beta} N \to 0$ be an exact sequence such that X is a finite set and $N \in n\text{-cop}(Q_R)$. If $A = \mathrm{Im}(\Delta(\alpha))$ then it is enough to prove that A is reflexive as a consequence of [12, Lemma 4.2.4].

From the exact sequence $0 \to \Delta(N) \to \Delta(Q^X) \xrightarrow{\Delta(\alpha)} A \to 0$ we obtain that $A \in \mathrm{FP}_{(n+1)}(S)$ since $\Delta(N) \in \mathrm{FP}_{(n+1)}(S)$. Moreover $A \in \mathrm{Cog}({}_SQ)$ since it is a submodule of the Q-cogenerated module $\Delta(M)$, so, under the hypothesis $(\star)$, it is enough to prove the claim $A \in {}^{\perp_{<n}}Q$. Using the long exact sequence

$$\begin{aligned} 0 \to &\Delta(A) \to \Delta^2(Q^X) \xrightarrow{\Delta^2(\beta)} \Delta^2(N) \to \\ &\to \mathrm{Ext}^1_S(A, Q) \to \mathrm{Ext}^1_S(\Delta(Q^X), Q) \to \mathrm{Ext}^1_S(\Delta(N), Q) \to \ldots \end{aligned}$$

we observe that this claim is equivalent to the equality $\mathrm{Ext}^1_S(A, Q) = 0$ since $\mathrm{Ext}^i_S(\Delta(Q^X), Q) = \mathrm{Ext}^i_S(S^X, Q) = 0$ and $\mathrm{Ext}^i_S(\Delta(N), Q) = 0$ for all $0 < i < n$, by hypothesis. In fact it is enough to prove that $\Delta^2(\beta)$ is an epimorphism. In order to prove this, we consider the commutative diagram

$$\begin{tikzcd} Q^X \arrow[r, "\beta"] \arrow[d, "\delta_{Q^X}"'] & N \arrow[r] \arrow[d, "\delta_N"'] & 0 \\ \Delta^2(Q^X) \arrow[r, "\Delta^2(\beta)"] & \Delta^2(N) & \end{tikzcd}$$

in which the top sequence is exact and all vertical arrows are isomorphisms.

Under the hypothesis $(\star')$ the proof can be transferred *verbatim*. □

Theorem 2.7. *The following are equivalent for a right R-module Q, with the endomorphism ring S, and an integer $n > 0$.*

a) *Q is a finitistic n-self-cotilting module;*
b) *$\Delta : n\text{-cop}(Q) \rightleftarrows {}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) : \Delta$ is a duality.*
c) *$\Delta : n\text{-cop}(Q) \rightleftarrows {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) : \Delta$ is a duality.*

Under these conditions we have

$${}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) = {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ).$$

Proof. a)⇒b) Since we shall use this part of the proof to obtain other consequences as well, we split it in two steps.

Step I. We will prove that the functor $\Delta = \mathrm{Hom}_R(-, Q)$ is well defined, and $n\text{-cop}(Q) \subseteq \mathrm{Refl}(Q_R)$.

Let $M \in n\text{-cop}(Q)$. Then $M \in (n+1)\text{-cop}(Q)$, hence there exists an exact sequence

$$0 \to M \overset{\alpha}{\to} Q^X \overset{\beta}{\to} N \to 0$$

such that X is a finite set and $N \in n\text{-cop}(Q)$. Since Q is n-w_f-quasi-injective, the sequence

$$0 \to \Delta(N) \overset{\Delta(\beta)}{\to} \Delta(Q^X) \overset{\Delta(\alpha)}{\to} \Delta(M) \to 0$$

is exact, so the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\delta_M"'] & Q^X \arrow[r, "\beta"] \arrow[d, "\delta_{Q^X}"'] & N \arrow[r] \arrow[d, "\delta_N"'] & 0 \\ 0 \arrow[r] & \Delta^2(M) \arrow[r, "\Delta^2(\alpha)"] & \Delta^2(Q^X) \arrow[r, "\Delta^2(\beta)"] & \Delta^2(N) & \end{tikzcd}$$

is commutative and with exact sequences. From the Ker-Coker Lemma and the fact that M and N belong to $n\text{-cop}(Q)$ (hence δ_M and δ_N are monomorphisms), we obtain that M is reflexive. Therefore $n\text{-cop}(Q) \subseteq \mathrm{Refl}(Q)$.

It follows that δ_N is an isomorphism. This implies that $\Delta^2(\beta)$ is an epimorphism and it follows that $\mathrm{Ext}^1_S(\Delta(M), Q) = 0$ for all $M \in n\text{-cop}(Q)$. Applying this result to N together with the isomorphism $\mathrm{Ext}^1_S(\Delta(N), Q) \cong \mathrm{Ext}^2_S(\Delta(M), Q)$, we obtain $\mathrm{Ext}^2_S(\Delta(M), Q) = 0$ for all $M \in n\text{-cop}(Q)$ and inductively, using the same technique, it follows that $\mathrm{Ext}^i_S(\Delta(M), Q) = 0$ for all $i > 0$. Therefore $\Delta(M) \in {}^{\perp_{<n}}Q$ for all $M \in n\text{-cop}(Q)$.

Since $n\text{-cop}(Q) = (n+1)\text{-cop}(Q)$, there exists a long exact sequence

$$0 \to M \overset{\alpha_0}{\to} Q^{X_0} \overset{\alpha_1}{\to} Q^{X_1} \overset{\alpha_2}{\to} \dots \overset{\alpha_n}{\to} Q^{X_n} \overset{\alpha_{n+1}}{\to} Q^{X_{n+1}}$$

such that X_k are finite sets and $\mathrm{Im}(\alpha_k) \in n\text{-cop}(Q)$ for all $k \in \{0, \dots, n+1\}$. For every $k \in \{0, \dots, n\}$ we consider the exact sequence

$$0 \to \mathrm{Im}(\alpha_k) \to Q^{X_k} \to \mathrm{Im}(\alpha_{k+1}) \to 0$$

and we observe that it is Q-cobalanced since Q is n-w_f-quasi-injective. Moreover $M \cong \mathrm{Im}(\alpha_0)$, hence we obtain a long exact sequence

$$\Delta(Q^{X_n}) \stackrel{\Delta(\alpha_n)}{\to} \Delta(Q^{X_{n-1}}) \stackrel{\Delta(\alpha_{n-1})}{\to} \dots \stackrel{\Delta(\alpha_1)}{\to} \Delta(Q^{X_0}) \stackrel{\Delta(\alpha_0)}{\to} \Delta(M) \to 0$$

which shows that $\Delta(M) \in \mathrm{FP}_{(n+1)}(S)$. Consequently, the functor

$$\Delta : n\text{-cop}(Q) \to {}^{\perp_{<n}} Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$$

is well defined.

Step II. We prove that $\Delta = \mathrm{Hom}_S(-, Q)$ is well defined and we have the inclusion ${}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) \subseteq \mathrm{Refl}({}_SQ)$.

First we suppose $n > 1$. Let $A \in {}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$. It follows that there exists an exact sequence $0 \to B \to S^Y \to A \to 0$ with Y a finite set and $B \in \mathrm{FP}_n(S)$. Moreover $\mathrm{Ext}^i_S(B, Q) = 0$ for all $i \in \{1, \dots, n-2\}$ since $\mathrm{Ext}^i_S(A, Q) = 0$ for all $i \in \{1, \dots, n-1\}$. If

$$S^{Y_{n-1}} \to \dots \to S^{Y_0} \to B \to 0$$

is an exact sequence such that Y_k are finite sets for all $k \in \{0, \dots, n-1\}$ then it induces the long exact sequence

$$0 \to \Delta(B) \to \Delta(S^{Y_0}) \to \dots \to \Delta(S^{Y_{n-1}})$$

which shows that $\Delta(B) \in n\text{-cop}(Q)$. The sequence

$$(\sharp)\ 0 \to \Delta(A) \to \Delta(S^Y) \to \Delta(B) \to 0$$

is exact since $\mathrm{Ext}^1_S(A, Q) = 0$ and we obtain $\Delta(A) \in n\text{-cop}(Q)$, hence the functor

$$\Delta : {}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) \to n\text{-cop}(Q)$$

is well defined. Moreover the short exact sequence $(\sharp)$ is Q-cobalanced since Q is n-w_f-quasi-injective. Therefore the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \longrightarrow & S^Y & \longrightarrow & A & \longrightarrow & 0 \\
 & & \delta_B \big\downarrow & & \delta_{S^Y} \big\downarrow & & \delta_A \big\downarrow & & \\
0 & \longrightarrow & \Delta^2(B) & \longrightarrow & \Delta^2(S^Y) & \longrightarrow & \Delta^2(A) & \longrightarrow & 0
\end{array}$$

has exact rows; it follows that δ_A is an epimorphism. But A is Q-cogenerated by the hypothesis, hence $A \in \mathrm{Refl}({}_SQ)$ and the proof is complete.

If $n = 1$ we need other arguments since in this case we have not $\mathrm{Ext}^1_S(A, Q) = 0$. The proof is almost the same. The only change is to replace the sequence $(\sharp)$ with a sequence

$$(\sharp')\ 0 \to \Delta(A) \to \Delta(S^Y) \to K \to 0$$

with $K \leq \Delta(B)$. Since $1\text{-cop}(Q)$ is closed with respect submodules, we obtain $K \in 1\text{-cop}(Q)$, hence $(\sharp')$ is Q-cobalanced.

b)$\Rightarrow$a) The equality $n\text{-cop}(Q) = (n+1)\text{-cop}(Q)$ is a consequence of Lemma 2.5, and from Lemma 2.6 with the hypothesis $(\star)$ we deduce that Q is an n-w_f-quasi-injective module. The same argument is valid for c)$\Rightarrow$a), using this time Lemma 2.6 with the hypothesis $(\star')$.

To prove a)⇒c) we observe that in Step I we proved in fact that

$$\Delta(n\text{-cop}(Q)) \subseteq {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$$

hence we have the equality

$${}^{\perp_{<n}}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ) = {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$$

as a consequence of b). □

If $n = 1$ then the class ${}^{\perp_{<1}}{}_SQ$ coincides with the class of all left S-modules, 1-cop(Q) is the class of all finitely Q-cogenerated modules, and $\mathrm{FP}_2(S)$ is the class of all finitely presented S-modules. Therefore we can reformulate Theorem 2.7 in the following way:

Theorem 2.8. *The following conditions are equivalent for a right module Q with the endomorphism ring S:*

a) *Q is a finitistic 1-self-cotilting module;*
b) $\Delta : \mathrm{cog}(Q) \rightleftarrows \mathrm{FP}_2(S) \cap \mathrm{Cog}({}_SQ) : \Delta$ *is a duality.*

In these conditions $\mathrm{FP}_2(S) \cap \mathrm{Cog}({}_SQ) \subseteq {}^{\perp}Q$.

In the following we show that in the general case finitistic 1-self-cotilting modules are not related to other kinds of modules (w-Π_f-quasi-injective module, costar modules or f-cotilting modules) which are involved in some dualities theorems (see [11] and [20, Section 4]). In the next example we use an idea from [1].

Example 2.9. *There exists a finitistic 1-self-cotilting $\mathbb{Z}$-module which is not a w-Π_f-quasi-injective module, hence it is not costar nor f-cotilting.*

Proof. Let $Q = \bigoplus_{\aleph_1} \mathbb{Z}$ be an (uncountable) free abelian group. Then for every positive integer n the $\mathbb{Z}$-module Q^n is free and it is isomorphic to Q. Therefore every 1-finitely Q-copresented $\mathbb{Z}$-module is free hence it is isomorphic to a direct summand of Q. Then 1-cop(Q) = 2-cop(Q). Moreover, every exact sequence $0 \to L \to Q^n \to M \to 0$ with M an 1-finitely Q-copresented splits since M is free, hence every such a sequence is Q-cobalanced. It follows that Q is a finitistic 1-self-cotilting module.

To prove that Q is not a w-Π_f-quasi-injective $\mathbb{Z}$-module we consider a locally free group G of cardinality $\aleph_1$ which is not a free group (such a group is constructed in [15]). Then G is a Q-cogenerated $\mathbb{Z}$-module. Moreover, there exists an exact sequence

$$(\star)\ 0 \to L \to Q \to G \to 0.$$

If we suppose that $(\star)$ is Q-cobalanced it follows that it is a splitting sequence because L is isomorphic to a direct summand of Q, and this implies that $(\star)$ is L-cobalanced. Then G is a free module, a contradiction. Hence $(\star)$ is not Q-cobalanced.

The $\mathbb{Z}$-module Q is not a costar $\mathbb{Z}$-module as a consequence of [11, Theorem 2.7 (e)], and it is not f-cotilting by [20, 3.12]. □

Remark 2.10. The cardinal number $\aleph_1$ is the least cardinal m with the property that $Q = \bigoplus_m \mathbb{Z}$ is a finitistic 1-self-cotilting module which is not a costar module. In fact, using the same proof as in the previous example we can prove that $Q = \bigoplus_m \mathbb{Z}$ is a finitistic 1-self-cotilting $\mathbb{Z}$-module for every cardinal m and it is not a costar module if $m \geq \aleph_1$. If $m \leq \aleph_0$ then we deduce that $Q = \bigoplus_m \mathbb{Z}$ is a costar module using [14, Theorem 19.2].

Example 2.11. *There exists an injective cogenerator (hence a costar module or an f-cotilting module) for* Mod-$\mathbb{Z}$ *which is not a finitistic 1-self-cotilting $\mathbb{Z}$-module.*

Proof. We consider the $\mathbb{Z}$-module

$$Q = (\bigoplus_{\aleph_0} \mathbb{Q}) \bigoplus (\bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)),$$

which is an injective cogenerator for Mod-$\mathbb{Z}$. But Q is not a finitistic 1-self-cotilting module since the group $L = \bigoplus_{\aleph_0} \mathbb{Z} \in 1\text{-cop}(Q) \setminus 2\text{-cop}(Q)$. This is a consequence of the fact that for every monomorphism $\alpha : L \to Q^n$ and for every prime p the p-socle $\mathrm{Coker}(\alpha)[p]$ has infinite cardinality, hence $\mathrm{Coker}(\alpha)$ cannot be embedded in a finite power of Q.

Using again [11, Theorem 2.7] and [20, 3.12] we deduce that Q is a costar modules and an f-cotilting module. □

From [20, 4.10] we deduce the following

Corollary 2.12. *If Q is a right R-module such that its endomorphism ring S is left noetherian then Q is a finitistic 1-self-cotilting module if and only if it is an f-cotilting module.*

3. Closure properties

In this section we study some closure properties for the class n-cop(Q) and for the class ${}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$, where Q is a finitistic n-self-cotilting module.

Lemma 3.1. *Let Q be a finitistic n-self-cotilting module. Then ${}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q) \subseteq \mathrm{FP}_\infty(S)$.*

Proof. If Q is a finitistic n-self-cotilting module then Q is finitistic $(n+k)$-self-cotilting for every integer $k > 0$. Using Theorem 2.7 and the equalities n-cop$(Q) = (n+k)$-cop(Q) we obtain

$${}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q) = {}^\perp Q \cap \mathrm{FP}_{(n+k+1)}(S) \cap \mathrm{Cog}({}_S Q)$$

for all $k > 0$. The conclusion is now obvious. □

We recall that a class $\mathcal{C}$ of finitely generated modules is *resolving* if $\mathcal{C}$ contains all finitely generated projective modules, and it is closed with respect to direct summands, extensions, and kernels of epimorphisms.

Proposition 3.2. *The class ${}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$ is a resolving class whenever Q is a finitistic n-self-cotilting module.*

Proof. Obviously, ${}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$ contains all finitely generated projective left S-modules. Using [9, Exercise VIII 4.2] we obtain that $\mathrm{FP}_{(n+1)}(S)$ is closed under direct summands, hence ${}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$ has the same property.

Let $0 \to A \to B \to C \to 0$ be an exact sequence with $A, C \in {}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$. It is easy to see that $B \in {}^\perp Q \cap \mathrm{Cog}({}_S Q)$. Moreover, $B \in \mathrm{FP}_\infty(S)$ as a consequence of $A, C \in \mathrm{FP}_\infty(S)$ and [10, Proposition V.2.2].

By [5, Lemma 1.1] it is enough to prove that if $0 \to A \to P \to B \to 0$ is an exact sequence of left S-modules such that P is finitely generated projective module and $B \in {}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$ then $A \in {}^\perp Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_S Q)$. But this is obvious (for $A \in \mathrm{FP}_{(n+1)}(S)$ we use [9, Proposition VIII. 4.3] together with $B \in \mathrm{FP}_{(n+2)}(S)$). □

Since $\mathrm{Refl}(Q_R)$ and ${}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$ are closed under direct summands we can enunciate a first result concerning closure properties of $n\text{-cop}(Q)$.

Proposition 3.3. *If Q is a finitistic n-self-cotilting module, then $n\text{-cop}(Q)$ is closed under direct summands.*

The following theorem dualizes results obtained in [18] for n-star modules. However, since in [18, Theorem 3.5] finiteness conditions are not imposed, in the following characterization we need an extra condition.

Theorem 3.4. *The following conditions are equivalent for a right R-module Q and an integer $n > 0$.*

a) *Q is a finitistic n-self-cotilting module;*

b) i) *An exact sequence $0 \to L \to Q^X \to N \to 0$ with $L \in n\text{-cop}(Q)$ and X a finite set is Q-cobalanced if and only if $N \in n\text{-cop}(Q)$,*

ii) *$\Delta(n\text{-cop}(Q)) \subseteq \mathrm{FP}_1(S)$.*

Proof. a)⇒b) Let

$$(\star)\ 0 \to L \to Q^X \to N \to 0$$

be an exact sequence of right R-modules such that $L \in n\text{-cop}(Q)$ and X is a finite set. If $N \in n\text{-cop}(Q)$ then $(\star)$ is Q-cobalanced by definition.

Now suppose that $(\star)$ is Q-cobalanced. In the exact sequence

$$(\Delta(\star))\quad 0 \to \Delta(N) \to \Delta(Q^X) \to \Delta(L) \to 0$$

all modules belong to ${}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$ as a consequence of Theorem 2.7 and Proposition 3.2. Then $\Delta(\star)$ is Q-cobalanced and using the obvious commutative diagram obtained applying Δ to $\Delta(\star)$ we obtain that N is Q-reflexive. Moreover $\Delta^2(N) \in n\text{-cop}(Q)$, hence $N \in n\text{-cop}(Q)$.

Therefore i) is valid and ii) is a consequence of Theorem 2.7.

b)⇒a) It is enough to prove $n\text{-cop}(Q) = (n+1)\text{-cop}(Q)$. If $N \in n\text{-cop}(Q)$ then the S-module $\Delta(N)$ is finitely generated. As a consequence of [12, Lemma 4.2.3], there exists a Q-cobalanced exact sequence

$$0 \to N \to Q^X \to M \to 0$$

with X a finite set. Then $M \in n\text{-cop}(Q)$, hence $N \in (n+1)\text{-cop}(Q)$. □

Remark 3.5. The proof for a)⇒b) can be obtained using a dual form of the proof for [18, Theorem 3.5] and Proposition 3.3.

Example 3.6. *a) There exists a module which verifies the condition i) in Theorem 3.4, but it does not satisfy ii).*

b) There exists a module which verifies the condition ii) in Theorem 3.4, but it does not satisfy i).

Proof. a) We consider the $\mathbb{Z}$-module $Q = \langle \frac{1}{p} \mid p \in \mathbb{P} \rangle \leq \mathbb{Q}$, where $\mathbb{P}$ is the set of all primes. Suppose that

$$(\star)\ 0 \to L \to Q^X \to M \to 0$$

is an exact sequence with X a finite set. We recall from [14, Lemma 86.8] that M is a torsion free abelian group (equivalently, the sequence $(\star)$ is pure) if and only if the sequence ($\star$) splits, hence $M \in 1\text{-cop}(Q)$ if and only if it is a torsion free group, and in this situation the sequence ($\star$) is Q-cobalanced as a splitting exact sequence. Moreover, $1\text{-cop}(Q) \subseteq \mathrm{Refl}(Q)$ as a consequence of Warfield duality, [17] (see also [7, Theorem 1.15]).

Suppose that ($\star$) is Q-cobalanced. Then we can embed it in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & Q^X & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_{Q^X}} & & \downarrow{\scriptstyle \delta_M} & & \\ 0 & \longrightarrow & \Delta^2(L) & \longrightarrow & \Delta^2(Q^X) & \longrightarrow & \Delta^2(M) & & \end{array}$$

which has exact sequences. Applying the 3×3-Lemma we obtain that δ_M is monic, hence M is a torsion-free abelian group. This is possible if and only if ($\star$) splits, hence $M \in 1\text{-cop}(Q)$. Then Q verifies the condition i) in the previous theorem for $n = 1$. But $S = \mathrm{End}(Q) \cong \mathbb{Z}$, hence $\Delta(\mathbb{Z}) \cong Q$ is not a finitely generated S-module.

b) It is not hard to see that for $n = 1$, the $\mathbb{Z}$-module $\mathbb{Q}$ verifies ii) but not i). □

The following propositions are dual to results presented in [18, Section 3].

Proposition 3.7. *Let Q be a finitistic n-self-cotilting module.*

a) *n-cop(Q) is closed under kernels of epimorphisms.*
b) *Suppose that ($\star$) $0 \to L \to M \to N \to 0$ is an exact sequence of right R-modules which is Q-cobalanced. If two of the terms of ($\star$) belong to n-cop(Q) then the third term is in n-cop(Q).*
c) *Every exact sequence $0 \to L \to M \to N \to 0$ with $L, M, N \in n\text{-cop}(Q)$ is Q-cobalanced.*

Proof. a) The proof is a dualization for the proof of [18, Lemma 3.2]. We give some details for reader's convenience.

If $0 \to L \to M \xrightarrow{\pi} N \to 0$ is an exact sequence such that $M, N \in n\text{-cop}(Q)$ then we can include it in a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\ 0 & \longrightarrow & Q^X & \longrightarrow & Q^X \oplus Q^Y & \longrightarrow & Q^Y & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where $0 \to M \xrightarrow{\alpha} Q^X \to M_1 \to 0$ and $0 \to N \xrightarrow{\beta} Q^Y \to N' \to 0$ are exact sequences such that X, Y are finite sets, $M_1, N' \in n\text{-cop}(Q)$ (hence these exact sequences are Q-cobalanced), and $\gamma : M \to Q^X \oplus Q^Y$ is defined by $\gamma(m) = (\alpha(m), \beta\pi(m))$. For every R-homomorphism $f : M \to Q$ there exists a homomorphism $f' : Q^X \to M$

such that $f = f'\alpha$. If $\overline{f} : Q^X \oplus Q^Y \to M$, $\overline{f}(x, y) = f'(x)$ then $f = \overline{f}\gamma$, hence the middle column is Q-cobalanced. It follows that $M' \in n\text{-cop}(Q)$. Since $N' \in n\text{-cop}(Q)$ the conclusion is obtained repeating the above proof.

b) If $0 \to L \to M \to N \to 0$ is a Q-cobalanced exact sequence then we have the exact sequence $0 \to \Delta(N) \to \Delta(M) \to \Delta(L) \to 0$ and the commutative diagram

$$(\sharp)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta^2(L) & \longrightarrow & \Delta^2(M) & \longrightarrow & \Delta^2(N) & & \end{array},$$

which has exact rows.

If $L, M \in n\text{-cop}(Q)$ then $\Delta(L), \Delta(M) \in {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$ and using Proposition 3.2 we deduce $\Delta(N) \in {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$, hence $\Delta^2(N) \in n\text{-cop}(Q)$. Since $\Delta(L) \in {}^{\perp}Q \cap \mathrm{FP}_{(n+1)}(S) \cap \mathrm{Cog}({}_SQ)$, the bottom sequence in diagram $(\sharp)$ remains exact if we complete it with $\to 0$ on the right side. Then N is Q-reflexive, hence $N \in n\text{-cop}(Q)$.

A similar proof is valid for the hypothesis $L, N \in n\text{-cop}(Q)$.

If $M, N \in n\text{-cop}(Q)$, the conclusion is a consequence of a).

c) As in the proof of a) we obtain a commutative diagram with exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q^X & \longrightarrow & Q^X \oplus Q^Y & \longrightarrow & Q^Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with $L', M', N' \in n\text{-cop}(Q)$. Therefore all columns and the middle row are Q-cobalanced exact sequences. Applying the functor Δ we obtain the commutative square

$$\begin{array}{ccc} \Delta(M) & \longrightarrow & \Delta(L) \\ \uparrow & & \uparrow \\ \Delta(Q^X \oplus Q^Y) & \longrightarrow & \Delta(Q^X) \end{array}$$

in which all vertical arrows and the bottom arrow are epimorphisms and the conclusion follows. $\square$

Example 3.8. *The $\mathbb{Z}$-module $\mathbb{Z}$ is a finitistic 1-self-cotilting module, but in the exact sequence $0 \to 2\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}(2) \to 0$, the first two terms belong to* $1\text{-cop}(\mathbb{Z})$, *but $\mathbb{Z}(2) \notin$* $1\text{-cop}(\mathbb{Z})$. *A similar example can be obtained using the $\mathbb{Z}$-module from Example 3.6, a), which is a finitistic 2-self-cotilting module.*

Corollary 3.9. *Let Q be a finitistic n-self-cotilting module and $0 \to L \to M \to N \to 0$ an exact sequence with $L, N \in n\text{-cop}(Q)$. If M is an epimorphic image of an n-finitely Q-copresented module then $M \in n\text{-cop}(Q)$.*

Proof. It is enough to prove that the exact sequence $0 \to L \to M \to N \to 0$ is Q-cobalanced. Let $\alpha : L \to Q$ be a homomorphism. If $K \in n\text{-cop}(Q)$ and $K \to M$ is an epimorphism then we can construct a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L' & \longrightarrow & K & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

in which the left square is a pushout. Observe that $L' \in n\text{-cop}(Q)$ by Theorem 3.7 a), hence the top row is Q-cobalanced. Hence, as in the proof of [20, 2.4], the homomorphism $L' \to L \xrightarrow{\alpha} Q$ can be extended to K, and using the pushout properties we deduce that we can extend α to a homomorphism $M \to Q$. □

We remark that, in general, the class $n\text{-cop}(Q)$ is not closed under extensions.

Example 3.10. *The group $Q = \mathbb{Z}(2)$ is a finitistic 1-self-cotilting module, but $1\text{-cop}(Q)$ is not closed under extensions.*

In the following proposition we characterize finitistic n-self-cotilting modules Q such that the class $n\text{-cop}(Q)$ is a resolving class. The proof is similar to the proofs of [18, Proposition 3.6] and [20, 3.3]. We sketch it for the reader's convenience.

Proposition 3.11. *The following are equivalent for a right R-module Q and an integer $n > 0$:*

a) *Q is a finitistic n-self-cotilting module such that the class $n\text{-cop}(Q)$ is closed under extensions.*
b) *$n\text{-cop}(Q) = (n+1)\text{-cop}(Q) \subseteq \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M,Q) = 0\}$.*
c) *$\Delta(n\text{-cop}(Q)) \subseteq \text{FP}_1(S)$ and $n\text{-cop}(Q) = \{M \in \mathcal{C} \mid \text{Ext}^1_R(M,Q) = 0\}$, where $\mathcal{C}$ is the class of all cokernels of monomorphisms between n-finitely Q-copresented modules.*

Proof. a)⇒b) Let $M \in n\text{-cop}(Q)$. If $(\star)$ $0 \to Q \to L \to M \to 0$ is an exact sequence then its terms belong to $n\text{-cop}(Q)$, hence it is Q-cobalanced. But this implies that $(\star)$ splits.

b)⇒a) If $(\star)$ $0 \to L \to Q^X \to N \to 0$ is an exact sequence with X a finite set and $N \in n\text{-cop}(Q)$, then $\text{Ext}^1_R(N,Q) = 0\}$, hence $(\star)$ is Q-cobalanced. Then Q is a finitistic n-self-cotilting module. Moreover, under the hypothesis b) every exact sequence $0 \to L \to M \to N \to 0$ with $N \in n\text{-cop}(Q)$ stays exact under Δ, hence $n\text{-cop}(Q)$ is closed under extensions as a consequence of Proposition 3.7 b).

a, b)⇒c) The inclusion $n\text{-cop}(Q) \subseteq \{M \in \mathcal{C} \mid \text{Ext}^1_R(M,Q) = 0\}$ is obvious. Let $M \in \mathcal{C}$ such that $\text{Ext}^1_R(M,Q) = 0$, and $0 \to K \to L \to M \to 0$ an exact sequence with $K, L \in n\text{-cop}(Q)$. If $0 \to K \to Q^X \to N$ with X a finite set and $N \in n\text{-cop}(Q)$

is a Q-copresentation of K then, using a pushout, we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & Q^X & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & & & \\ & & N & = & N & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Observe that $P \in n\text{-cop}(Q)$, since $n\text{-cop}(Q)$ is closed with respect extensions, and $L, N \in n\text{-cop}(Q)$. Moreover, the exact sequence $0 \to Q^X \to P \to M \to 0$ splits, hence $M \in n\text{-cop}(Q)$.

c)⇒b) It is enough to prove the equality $n\text{-cop}(Q) = (n+1)\text{-cop}(Q)$. Let $M \in n\text{-cop}(Q)$. Since $\Delta(M)$ is finitely generated as a left S-module, there exists a Q-cobalanced exact sequence $0 \to M \to Q^X \to N \to 0$, with X a finite set. Then $\mathrm{Ext}^1_R(N, Q) = 0$, since $\mathrm{Ext}^1_R(Q, Q) = 0$. Moreover $N \in \mathcal{C}$, hence $N \in n\text{-cop}(Q)$ and $M \in (n+1)\text{-cop}(Q)$. □

We close this section with a new closure property which generalizes Example 2.2.

Proposition 3.12. *Let Q be a finitistic n-self-cotilting module. The following are equivalent:*

a) *$n\text{-cop}(Q)$ is closed under factor modules.*
b) *Q is a quasi-injective finitistic 1-self-cotilting module.*

Proof. a)⇒b) Since the class $n\text{-cop}(Q)$ is closed under kernels of epimorphisms, we deduce that $n\text{-cop}(Q) = 1\text{-cop}(Q)$, hence Q is a finitistic 1-self-cotilting module. Moreover, every exact sequence $0 \to L \to Q \to M \to 0$ is Q-cobalanced since Q is 1-w_f-quasi-injective.

b)⇒a) We recall from [21, 17.11] that a module is quasi-injective if and only if it is a fully invariant submodule of its injective hull. It follows that Q^X is quasi-injective for all finite sets X.

Suppose that $M \in 1\text{-cop}(Q)$, $\iota : L \to M$ is a monomorphism, and $\alpha : L \to Q$ is a homomorphism. If $\rho : M \to Q^X$ is a monomorphism, where X is a finite set, then there exists a homomorphism $\overline{\alpha} : Q^X \to Q$ such that $\alpha = \overline{\alpha}\rho\iota$, hence the sequence $0 \to L \xrightarrow{\iota} M \to M/\iota(L) \to 0$ is a Q-cobalanced sequence. Using Proposition 3.7 we deduce that $M/\iota(L) \in 1\text{-cop}(Q)$ and the proof is complete. □

4. Connections with cotilting

The main aim of this section is to prove that for every $n > 0$ there exists a finitistic $(n+1)$-self-cotilting module which is not a finitistic n-self-cotilting module. To obtain this we will use some cotilting theory. Recall that the right R-module Q is an *n-cotilting* module if it has the following properties:

(C1) $\mathrm{id}(Q) \leq n$;
(C2) $\mathrm{Ext}^i(Q^X, Q) = 0$ for all integers $i > 0$ and all sets X;
(C3) There exists an injective cogenerator E and a long exact sequence $0 \to Q_r \to \cdots \to Q_1 \to Q_0 \to E \to 0$, where all Q_k are isomorphic to direct summands of direct products of copies of Q.

Bazzoni proved in [8, Theorem 3.11] that the right R-module Q is n-cotilting if and only if $n\text{-Cogen}(Q) = {}^{\perp}Q_R$, where the class $n\text{-Cogen}(Q)$ is defined as the class $n\text{-cop}(Q)$ deleting the condition "X_i are finite", and ${}^{\perp}Q_R = \{M \in \text{Mod-}R \mid \mathrm{Ext}^i_R(M, Q) = 0 \text{ for all } i > 0\}$. Moreover, in [4] and [8, Lemma 3.12] it is proved that in the definition of n-cotilting modules the condition (C3) can be replaced by ${}^{\perp}Q_R \subseteq \mathrm{Cogen}(Q)$.

If k is a field, and Q is a finitely generated module over a finite dimensional k-algebra R, then $\mathrm{Cogen}(Q) = \mathrm{Add}(Q)$ by [12, Corollary 1.3.3], where $\mathrm{Add}(Q)$ is the class of all direct summands of direct sums of copies of Q, hence we can replace the condition $(C2)$ by

(C2') $\mathrm{Ext}^i(Q, Q) = 0$ for all integers $i > 0$.

Moreover, we can transfer the proof presented in [4] (and in this case there exists a finitely generated injective cogenerator), we observe that we can replace the condition (C3) by

(C3') ${}^{\perp}Q_R \cap \mathrm{FP}_1(R) \subseteq 1\text{-cop}(Q)$,

and we obtain a version of Bazzoni's characterization [8, Theorem 3.11]:

Theorem 4.1. *Let k be a field, Q a finitely generated right module over a finite dimensional k-algebra, R, and $n > 0$ an integer. The following are equivalent:*

a) ${}^{\perp}Q_R \cap \mathrm{FP}_1(R) = n\text{-cop}(Q)$*;*
b) *Q is an n-cotilting module.*

Proof. a)⇒b) Using a proof which is similar to the proof presented for [8, Lemma 3.9], using this time [12, Lemma 5.6.3] and [12, Lemma 4.2.3], we obtain that $\mathrm{Ext}^{n+1}_R(M, Q) = 0$ for all finitely generated R-modules M. If E is a (finitely generated) injective cogenerator of Mod-R, we can construct a resolution $0 \to Q \to E^{X_0} \to \cdots \to E^{X_{n-1}} \to N \to 0$, and using the dimension shifting formula we deduce $\mathrm{Ext}^1_R(M, N) = 0$ for all finitely generated modules M. Therefore, by Baer's criterium, N is injective, hence the injective dimension of Q is at most n. The conditions (C2) and (C3') are obvious.

b)⇒a) Let $L \in {}^{\perp}Q_R \cap \mathrm{FP}_1(R)$. Then $\Delta(L)$ is finitely generated, and there exists a Q-cobalanced exact sequence $0 \to L \to Q^X \to M \to 0$ with X a finite set as a consequence of [12, Lemma 4.2.3] since L is Q-cogenerated. Hence $M \in {}^{\perp}Q_R \cap \mathrm{FP}_1(R) \subseteq 1\text{-cop}(Q)$. We can iterate this reasoning to obtain $L \in n\text{-cop}(Q)$, hence ${}^{\perp}Q_R \cap \mathrm{FP}_1(R) \subseteq n\text{-cop}(Q)$.

Conversely, let $M \in n\text{-cop}(Q)$. Then there exists a long exact sequence

$$0 \to M \to Q^{X_0} \to \cdots \to Q^{X_{n-1}} \to N \to 0$$

and it follows that $\mathrm{Ext}^i_R(M, Q) \cong \mathrm{Ext}^{n+i}_R(N, Q) = 0$ for all $i > 0$. □

Theorem 4.2. *Let k be a field, R a finitely dimensional k-algebra and Q a finitely generated right R-module.*

a) *If Q is n-cotilting then it is a finitistic n-self-cotilting module.*

b) *If Q is a finitistic n-self-cotilting module such that $R \in n\text{-cop}(Q)$ then it is an n-cotilting module.*
c) *If Q is an $(n+1)$-cotilting module which is not n-cotilting, then Q is not a finitistic n-self-cotilting module.*

Proof. a) Let Q be a finitely generated right R-module. Then every finitely generated module which is Q-cogenerated is finitely Q-cogenerated as a consequence of [12, Corollary 1.3.3]. Using this together with [8, Proposition 3.6], we obtain $n\text{-cop}(Q) = (n+1)\text{-cop}(Q) \subseteq {}^{\perp}Q_R$. Then Q is a finitistic n-self-cotilting module as a consequence of Proposition 3.11.

b) Since $R \in n\text{-cop}(Q)$, using Corollary 3.9 we deduce that $n\text{-cop}(Q)$ is closed under extensions. Hence $n\text{-cop}(Q) = (n+1)\text{-cop}(Q) \subseteq \{M \in \text{Mod-}R \mid \text{Ext}^1_R(M,Q) = 0\}$ by Proposition 3.11 b).

The rest of proof is dual to the proof of (4) $\Rightarrow$ (1) in [18, Theorem 4.3]. We give some details for reader's convenience.

If $M \in n\text{-cop}(Q)$ and $\ldots \overset{\alpha_{m+1}}{\to} R^{k_m} \to \ldots \overset{\alpha_1}{\to} R^{k_0} \overset{\alpha_0}{\to} M \to 0$ is a free resolution of M then, inductively, $\text{Ker}(\alpha_m) \in n\text{-cop}(Q)$ by Proposition 3.7 a), hence $\text{Ext}^1_R(\text{Ker}(\alpha_i), Q) = 0$ for all $i \geq 0$. Then $\text{Ext}^i_R(M,Q) = 0$ for all $i \geq 1$, and it follows that $n\text{-cop}(Q) \subseteq {}^{\perp}Q_R \cap \text{FP}_1(R)$.

If $M \in {}^{\perp}Q_R \cap \text{FP}_1(R)$, we consider a free resolution $R^{k_n} \overset{\alpha_{n-1}}{\to} \ldots \overset{\alpha_1}{\to} R^{k_0} \overset{\alpha_0}{\to} M \to 0$. Using a dual version of [18, Proposition 3.7] we obtain $\text{Ker}(\alpha_{n-1}) \in n\text{-cop}(Q)$. Suppose that $\text{Ker}(\alpha_{n-i}) \in n\text{-cop}(Q)$ for some $i \in \{1, \ldots, n-1\}$. Then we have an exact sequence $0 \to \text{Ker}(\alpha_{n-i}) \to R^{k_{n-i}} \to \text{Ker}(\alpha_{n-i-1}) \to 0$, which is Q-cobalanced since $\text{Ext}_1(\text{Ker}(\alpha_{n-i-1}), Q) = 0$. Therefore, by Proposition 3.7, $\text{Ker}(\alpha_{n-i-1}) \in n\text{-cop}(Q)$. Then $\text{Ker}(\alpha_0) \in n\text{-cop}(Q)$, and, using the same argument as in the inductive step, we obtain $M \in n\text{-cop}(Q)$. Then ${}^{\perp}Q_R \cap \text{FP}_1(R) \subseteq n\text{-cop}(Q)$.

c) Using the previous theorem we obtain $n\text{-cop}(Q) \neq {}^{\perp}Q_R \cap \text{FP}_1(R) = (n+1)\text{-cop}(Q)$, hence Q is not a finitistic n-self cotilting module. □

Finally, we present connections between some standard notions and finitistic 1-self-cotilting notion for the case R is a finite dimensional k-algebra. I am indepted to Lidia Angeleri Hügel for her help to state these connections.

Proposition 4.3. *Let R be a finite dimensional k-algebra and Q be a finitely generated right R-module. We consider the following statements:*

a) *Q is a finitistic 1-self-cotilting module;*
b) *Q is a costar module;*
c) *Q is an f-cotilting module;*
d) *the dual module $\text{Hom}_k(Q,k)$ is a $\star$-module.*
e) *Q is a finitely cotilting module;*
f) *Q is an cotilting module;*

Then d) $\Leftrightarrow$ e) $\Rightarrow$ a) $\Leftrightarrow$ b) $\Leftrightarrow$ c). *All these are equivalent if Q is a faithful module.*

Proof. a), b) and c) are equivalent as a consequence of Corollary 2.12, [11, Theorem 2.7] and [20, 4.10]. The rest of proof is presented in [20, 3.7 and 3.9]. □

By [20, 3.9], we obtain the following

Example 4.4. *Every semisimple module of injective dimension > 1 over a finite dimensional algebra is a finitistic 1-self-cotilting module which is not a (1-) cotilting module.*

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