Mixed Beta Regression: A Bayesian Perspective

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Abstract
This paper builds on recent research that focuses on regression modeling of continuous bounded data, such as proportions measured on a continuous scale. Specifically, it deals with beta regression models with mixed effects from a Bayesian approach. We use a suitable parameterization of the beta law in terms of its mean and a precision parameter, and allow both parameters to be modeled through regression structures that may involve fixed and random effects. Specification of prior distributions is discussed, computational implementation via Gibbs sampling is provided, and illustrative examples are presented.

Key words: Bayesian analysis; Beta distribution; Beta regression; Continuous proportions; Mixed models.

1. Introduction

Mixed-effects models have been widely employed in statistical analyses, mainly in the area of health, and their study has been primarily restricted to response variables with normal or, at least, symmetrical distributions. Such models are not appropriately applicable when the response variable has a support restricted to a doubly bounded interval. Limited-range variables are, however, common in practice; for example, proportions are bounded between zero and one. This paper proposes a Bayesian analysis for mixed-effects regression models that are tailored
for situations where the response variable is measured on a continuous scale and is restricted to the unit interval \((0, 1)\). Situations where the response, say \(y\), is limited to a known interval \((a, b)\) is also accommodated through the transformation \(y^* = (y - a) / (b - a)\). The response variable is assumed to be beta distributed with mean \(\mu\) and possibly a precision parameter \(\phi\) modeled using fixed and random effects. The substantial advantage to consider a beta modeling is due to the flexibility that it provides. In fact, the beta family includes left or right skewed, symmetric, J-shaped, and inverted J-shaped distributions.

Following Ferrari and Cribari-Neto (2004), our proposed model uses a parameterization of the beta law in terms of its mean \(\mu\) and an additional positive parameter \(\phi\) that can be regarded as a precision parameter. The mean of the response variable is conveniently linked with a mixed-effects regression structure by the logit link function. An extended version of such a model is also considered. It assumes that the precision parameter is not constant over the observations, but rather it is related to a mixed-effects function through a log link.

To formulate our proposed models, we adopt a Bayesian approach. We address the issues of model fitting via Gibbs sampling, choice of prior distributions, and model selection based on the deviance information criterion. Simulated and real data analyses are presented for illustration. An appendix presents various pieces of BUGS code used for fitting the mixed beta regression.

2. Bayesian mixed beta regression

Due to the flexibility of the beta distribution in terms of the variety of density shapes that can be accommodated, this distribution is a natural choice for modeling continuous data that are restricted to the interval \((0, 1)\). The probability density function of a variable \(y\) following a beta distribution parameterized in terms of its mean \(\mu\) and a precision parameter \(\phi\) is given by

\[
f(y|\mu, \phi) = \frac{\Gamma(\phi)}{\Gamma(\mu\phi)\Gamma((1 - \mu)\phi)} y^{\mu\phi - 1} (1 - y)^{(1 - \mu)\phi - 1}, \quad 0 < y < 1, \tag{1}\]

where \(\Gamma(\cdot)\) denotes the gamma function, \(\mu = E(y) (0 < \mu < 1)\), and \(\phi > 0\) can be interpreted as a precision parameter, since \(\text{Var}(y) = \mu(1 - \mu)/(1 + \phi)\) and, hence, for each fixed value of the mean \(\mu, \phi\) is inversely proportional to the variance of \(y\). If \(y\) has density function (1), we write \(y \sim \text{beta}(\mu\phi, (1 - \mu)\phi)\).

Now, let \(y_1, \ldots, y_n\) be \(n\) independent random variables such that \(y_i \sim \text{beta}(\mu\phi, (1 - \mu)\phi)\). The definition of a beta regression model requires a transformation of the mean \(\mu_i\) of \(y_i, \ i = 1, \ldots, n\), that maps the interval \((0, 1)\)

\[
\mu_i = \frac{a + \mu\phi_i}{a + b - \mu\phi_i},
\]

where \(a, b\) are the known interval limits.
onto the real line. A convenient and popular link function is the logit link. It is then assumed that
\[ \ln \left\{ \frac{\mu_i}{1 - \mu_i} \right\} = x_i^T \beta, \]
where \( x_i \) is a vector of known covariates for the \( i \)-th subject and \( \beta \) denotes a vector of regression coefficients. The first element of \( x_i \) is usually taken as 1 to allow an intercept.

The precision parameter \( \phi \) may be assumed to be constant over observations (Ferrari and Cribari-Neto, 2004) or it may be modeled in terms of a regression structure (Smithson and Verkuilen, 2006). Since the precision parameter is strictly positive, the log link function is a natural choice. It is then assumed that
\[ \ln(\phi_i) = w_i^T \delta \]
where \( w_i \) is a vector of covariates and \( \delta \) denotes a vector of unknown regression coefficients. Again, it is convenient to take the first element of \( w_i \) as 1 to allow an intercept for the precision description. There is no restriction on whether or not the \( w_i \)'s contain the same predictor variables as \( x_i \)'s.

The beta regression model described above does not involve random effects. Extending previous work on Bayesian generalized linear models (Dey et al., 2000) and Bayesian beta regression (Branscum et al., 2007), we define below two mixed beta regression models, the first of which assumes that the precision parameter is the same for all the observations, and the second involves a mixed-effects model for the precision parameter.

Let \( y_1, \ldots, y_m \) be independent continuous random vectors, where \( y_i = (y_{i1}, \ldots, y_{im})^T \) represents an observed response vector for a sample unit \( i \) and for which each of its components, \( y_{ij} \), takes values on the interval \((0, 1)\). Consider also a regression model with the following structure:
\[ G(\mathbb{E}(y_i|b_i)) = X_i \beta + Z_i b_i, \]
\[ i = 1, \ldots, m, \]
where \( G(\cdot) \) is a vector-function linking the conditional mean response vector \( \mathbb{E}(y_i|b_i) \) with the linear mixed model \( \eta_i = X_i \beta + Z_i b_i \), for which \( X_i \) is the design matrix of dimension \( n_i \times p \) corresponding to the vector \( \beta = (\beta_1, \ldots, \beta_p)^T \) of regression coefficients (the fixed effects) and \( Z_i \) is the design matrix of dimension \( n_i \times q \) associated with the vector \( b_i = (b_{i1}, \ldots, b_{iq})^T \) (the random effects).

For the logit link function, the \( j \)-th component of (2) is
\[ \ln \left\{ \frac{\mu_{ij}}{1 - \mu_{ij}} \right\} = \eta_{ij} = x_{ij}^\top \beta + z_{ij}^\top b_i, \]
where \( \mu_{ij} = \mathbb{E}(y_{ij}|b_i) \), \( x_{ij} = (x_{ij1}, \ldots, x_{ijp})^T \), and \( z_{ij} = (z_{ij1}, \ldots, z_{ijq})^T \), which is equivalent to
\[ \mu_{ij} = \frac{\exp(\eta_{ij})}{1 + \exp(\eta_{ij})} = \frac{\exp(x_{ij}^\top \beta + z_{ij}^\top b_i)}{1 + \exp(x_{ij}^\top \beta + z_{ij}^\top b_i)}. \]
In this work, we first assume that for \( i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n_i \)
\[
y_{ij} | b_i, \beta, \phi \sim \text{beta}(\mu_{ij} \phi, (1 - \mu_{ij}) \phi),
\]
i.e., conditionally on \( b_i, \beta, \phi \), the \( y_{ij} \)'s are independent and have probability density function given by (1), with \( \mu \) replaced by \( \mu_{ij} \), which is specified by (3). Note that in this formulation, \( \phi \) represents a common precision parameter.

In mixed models, the random effects \( b_1, \ldots, b_m \) are typically assumed to be independent and normally distributed, namely
\[
b_i | \Sigma_b \sim N_q(0, \Sigma_b), i = 1, \ldots, m,
\]
where \( \Sigma_b \) is a positive-definite matrix. The normality assumption, however, can be inappropriate in practical applications where the measurements present outliers. In these cases, it is more adequate to consider multivariate distributions with heavier-than-normal tails for the random effects. Consequently, a multivariate \( t \)-distribution with \( \nu_b > 0 \) degrees of freedom, location vector \( \mu_b = 0 \in \mathbb{R}^q \) and positive-definite dispersion matrix \( \Sigma_b \) is a better candidate to model the random effects \( b_i \)'s, i.e., \( b_i | \nu_b, \Sigma_b \sim t_q(\nu_b, 0, \Sigma_b), i = 1, \ldots, m \). It should be noticed here that for large values of \( \nu_b \) the multivariate \( t \)-distribution is approximately a multivariate normal distribution.

In the mixed beta regression model proposed above, the precision parameter \( \phi \) is constant over the observations. For a more general formulation of this model, we consider a different precision parameter, say \( \phi_{ij} \), for each response \( y_{ij} \). We then assume a mixed linear model for the logarithm of \( \phi_{ij} \), namely
\[
\ln(\phi_{ij}) = \tau_{ij} = w_{ij}^\top \delta + h_{ij}^\top d_i,
\]
where \( w_{ij}^\top = (w_{ij1}, \ldots, w_{ijp^*}) \) is the design vector corresponding to the \( p^* \times 1 \) vector \( \delta \) of fixed effects and \( h_{ij}^\top = (h_{ij1}, \ldots, h_{ijq^*}) \) is the design vector corresponding to the \( q^* \times 1 \) vector \( d_i \) of random effects. Note that, the design matrices \( W_i = (w_{i1}, \ldots, w_{in_i})^\top \) and \( H_i = (h_{i1}, \ldots, h_{in_i})^\top \) may, but are not required to, contain the same predictor variables as the matrices \( X_i = (x_{i1}, \ldots, x_{in_i})^\top \) and \( Z_i = (z_{i1}, \ldots, z_{in_i})^\top \), respectively. Here, it may be assumed that \( d_i | \Sigma_d \sim N_q(0, \Sigma_d), i = 1, \ldots, m \), where \( \Sigma_d \) is a positive-definite matrix. Alternatively, we may assume that \( d_i | \nu_d, \Sigma_d \sim t_q(\nu_d, 0, \Sigma_d), i = 1, \ldots, m \).

In order to complete the Bayesian specification of the beta mixed models described above, elicitation of prior distributions for all unknown parameters is required. Multivariate normal prior distributions are typically considered for the fixed effects, i.e., \( \beta \sim N_p(\mu_\beta, \Sigma_\beta) \). Vague priors are usually specified by taking large values for the prior variances. However, the impact of the scale choice under
the normal model cannot be neglected. An alternative strategy is to consider a multivariate \( t \)-distribution, i.e., \( \beta \sim t_{p}(\nu_{\beta}, \mu_{\beta}, \Sigma_{\beta}) \) and to specify an appropriated value for \( \nu_{\beta} \), the degrees of freedom parameter. If the vector of random effects is assumed to follow a multivariate \( t \)-distributed, i.e., \( b_{i} | \nu_{b}, \mu_{b}, \Sigma_{b} \sim t_{q}(\nu_{b}, 0, \Sigma_{b}) \), then the prior distribution for the degrees of freedom can be discrete as in Albert and Chib (1993) and Besag et al. (1995), or continuous as in Geweke (1993). We have chosen the latter alternative. More specifically, we consider an exponential prior distribution with mean \( 1/a \) for the degrees of freedom, say \( \varepsilon(a) \). The prior distribution for the scale matrix of random effects \( \Sigma_{b} \) is chosen, mainly for computational simplicity, to be an inverted Wishart distribution as in Fong et al. (2010), i.e., \( \Sigma_{b} \sim IW_{q}(\psi, c) \). An alternative prior distribution for \( \Sigma_{b} \) is a constrained Wishart distribution (Everson and Morris, 2000).

We now turn to the specification of prior distributions for the precision parameter. As mentioned above, in this paper we study the following beta mixed regression models.

**Model 1:** It considers the mixed regression model (3) for the location parameters \( \mu_{ij} \) and a common precision parameter \( \phi \) for each observation \( y_{ij} \). In the Bayesian context, a natural choice for the prior distribution of the precision parameter would be an inverse gamma distribution. If a slightly informative prior is required, it can be assumed that \( \phi \sim IG(\epsilon, \epsilon) \), with a small fixed positive value for \( \epsilon \). Gelman (2006) suggests that the prior distribution \( \phi = U^{2} \) with \( U \sim U(0, a) \) with large \( a \) (\( a = 50 \) for example) is less informative than an inverse gamma prior. Here, we propose a more flexible prior distribution for \( \phi \) that includes Gelman’s prior distribution as a special case. More specifically, we propose the following prior specification for \( \phi: \phi = (aB)^{2} \), where \( B \sim \text{beta}(1 + \epsilon, 1 + \epsilon) \), for given positive values for \( a \) and \( \epsilon \).

**Model 2:** It considers the mixed regression model (3) for the location parameter \( \mu_{ij} \) and a different precision parameter \( \phi_{ij} \) for each \( y_{ij} \), where \( \phi_{ij} \) is modeled as in (4). Here, the specification of prior distributions for \( \delta \) and the parameters of the distribution of the \( d_{i} \)s is similar to that used for \( \beta \) and the parameters of the distribution of the \( b_{i} \)s.

### 3. Model fitting using Markov chain Monte Carlo sampling

Let \( y^{\top} = (y_{1}^{\top}, \ldots, y_{m}^{\top}) \) and \( \eta^{\top} = (\eta_{1}^{\top}, \ldots, \eta_{m}^{\top}) \), where \( \eta_{i}^{\top} = (\eta_{i1}, \ldots, \eta_{im}) \). Note that, by assumption, conditionally on \( \beta, \Sigma_{b}, \) and \( \nu_{b} \), the \( \eta_{i} \)’s are independent
and have density function $f(\eta_i | \beta, \Sigma_b, \nu_b) \propto f(b_i | \beta, \Sigma_b, \nu_b)$, $i = 1, \ldots, m$. We now present the following results for the joint posterior distribution under models 1 and 2 described in the previous section.

Under model 1, and the assumption that the parameters $\Sigma_b$, $\nu_b$, $\phi$, and $\beta$ are independent, the joint posterior density is

$$f(\beta, \Sigma_b, \nu_b, \phi, \eta | y) \propto \prod_{i=1}^{m} \prod_{j=1}^{n_i} f(y_{ij} | \eta_{ij}, \phi) \left[ \prod_{i=1}^{m} f(\eta_i | \beta, \Sigma_b, \nu_b) \right] f(\Sigma_b) f(\nu_b) f(\phi) f(\beta).$$

Gibbs sampling can be used to generate a Monte Carlo sample from the joint posterior density, $f(\beta, \Sigma_b, \nu_b, \phi, \eta | y)$. The Gibbs sampler in this context involves iteratively sampling from the full conditional distributions: $f(\Sigma_b | \nu_b, \beta, \phi, \eta, y)$, $f(\nu_b | \Sigma_b, \beta, \phi, \eta, y)$, $f(\beta | \Sigma_b, \nu_b, \phi, \eta, y)$, and $f(\eta_i | \eta_k, \beta, \Sigma_b, \nu_b, \phi, y_i)$, $i, k = 1, \ldots, m$, $i \neq k$, which can be implemented in the WinBUGS software. Posterior inferences on $\beta$, $\Sigma_b$, and $\phi$ and, more importantly, on the mean responses $(\mu_{ij}; i = 1, \ldots, m, j = 1, \ldots, n_i)$ are readily obtained in WinBUGS. Hypothesis testing regarding regression coefficients and mean responses are also straightforward.

We now turn to model 2. Let $\tau^T = (\tau_1^T, \ldots, \tau_m^T)$, where $\tau_i^T = (\tau_{i1}, \ldots, \tau_{in_i})$. By assumption, conditionally on $\delta$, $\Sigma_d$, and $\nu_d$, the $\tau_i$s are independent and have density function $f(\tau_i | \delta, \Sigma_d, \nu_d) \propto f(d_i | \delta, \Sigma_d, \nu_d)$, $i = 1, \ldots, m$. Assuming prior independence of $\Sigma_b$, $\nu_b$, $\Sigma_d$, $\nu_d$, $\delta$, and $\beta$, we obtain the posterior density given by

$$f(\beta, \Sigma_b, \nu_b, \delta, \Sigma_d, \nu_d, \eta, \tau | y) \propto \left[ \prod_{i=1}^{m} \prod_{j=1}^{n_i} f(y_{ij} | \eta_{ij}, \tau_{ij}) \right] \left[ \prod_{i=1}^{m} f(\eta_i | \beta, \Sigma_b, \nu_b) \right] \times \left[ \prod_{j=1}^{m} f(\tau_i | \delta, \Sigma_d, \nu_d) \right] \times f(\Sigma_b) f(\nu_b) f(\delta) f(\Sigma_d) f(\nu_d) f(\beta).$$

Similar to model 1, the Gibbs sampling can be used to generate a Monte Carlo sample from $f(\beta, \Sigma_b, \nu_b, \delta, \Sigma_d, \nu_d, \eta, \tau | y)$. In this case, the Gibbs sampler involves iteratively sampling from the following full conditional distributions:

- $f(\Sigma_b | \nu_b, \beta, \delta, \Sigma_d, \nu_d, \eta, \tau, y)$, $f(\nu_b | \Sigma_b, \beta, \delta, \Sigma_d, \nu_d, \eta, \tau, y)$, $f(\beta | \Sigma_b, \nu_b, \delta, \Sigma_d, \nu_d, \eta, \tau, y)$, $f(\delta | \beta, \Sigma_b, \nu_b, \Sigma_d, \nu_d, \eta, \tau, y)$, $f(\Sigma_d | \beta, \Sigma_b, \nu_b, \delta, \nu_d, \eta, \tau, y)$, $f(\nu_d | \beta, \Sigma_b, \nu_b, \delta, \Sigma_d, \eta, \tau, y)$, $f(\eta_i | \eta_k, \tau_k, \beta, \Sigma_b, \nu_b, \delta, \Sigma_d, \nu_d, \eta_i)$, $f(\tau_i | \tau_k, \eta_k, \beta, \Sigma_b, \nu_b, \delta, \Sigma_d, \nu_d, \eta_i)$, $i, k = 1, \ldots, m$, $i \neq k$, which can be also implemented in the WinBUGS software. Thus, posterior inferences on $\beta$, $\Sigma_b$, and $\phi_{ij}$, for $i = 1, \ldots, m$, $j = 1, \ldots, n_i$, and on the mean responses $\mu_{ij}$, for $i = 1, \ldots, m$, $j = 1, \ldots, n_i$ are easily obtained in WinBUGS. Again, hypothesis testing regarding regression coefficients and mean responses are also straightforward.
4. Illustration via simulations

To illustrate the proposed methodology, we consider the following mixed beta regression model with simulated data (model 1):

\[ y_{ij} | b_i, \phi, \beta \sim \text{beta}(\mu_{ij}\phi, (1 - \mu_{ij})\phi), \]

where \( \beta = (\beta_1, \beta_2, \beta_3)^\top \), \( b_i = (b_{i1}, b_{i2})^\top \),

\[ \ln \left\{ \frac{\mu_{ij}}{1 - \mu_{ij}} \right\} = \eta_{ij} = (\beta_1 + b_{i1}) + (\beta_2 + b_{i2})x_{ij2} + \beta_3x_{ij3}, \]

\( i = 1, \ldots, 100 \), \( j = 1, \ldots, 5 \), and \( b_i | \nu_b, \Sigma_b \sim t_2(\nu_b, 0, \Sigma_b) \). For our simulation study, the values of the covariates were generated from a uniform distribution in the unit interval, and we set \( \nu_b = 1000 \), \( \phi = 49 \), \( \beta = (-2, 1, 2)^\top \), and

\[ \Sigma_b = \begin{pmatrix} 1.09 & -0.36 \\ -0.36 & 0.13 \end{pmatrix}. \]

As was proposed in Section 2, we adopted prior the specifications \( \nu_b \sim \varepsilon(a) \), \( \Sigma_b \sim IW_2(\Psi, c) \), and \( \beta = (\beta_1, \beta_2, \beta_3)^\top \sim t_3(\nu_\beta, \mu_\beta, \Sigma_\beta) \) with

\( a = 0.001 \), \( \Psi = \begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix} \), \( c = 5 \), \( \nu_\beta = 20 \), \( \mu_\beta = (0, 0, 0)^\top \), \( \Sigma_\beta = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \).

In order to evaluate the influence of the prior specification for the precision parameter on the posterior inference, we considered different prior distributions for \( \phi \), namely: (i) \( \phi \sim IG(\epsilon, \epsilon) \), with \( \epsilon = 0.001 \) (model 1a); (ii) \( \phi = U^2 \), with \( U \sim U(0, 50) \) (model 1b); (iii) \( \phi = (50B)^2 \) where \( B \sim \text{beta}(1 + \epsilon, 1 + \epsilon) \), with \( \epsilon = 0.1 \) (model 1c) and 0.5 (model 1d); (iv) \( \ln(\phi) \sim t(\nu_\beta, \mu_\beta, \sigma_\beta^2) \) with \( \nu_\beta = 20 \), \( \mu_\beta = 0 \), \( \sigma_\beta^2 = 5 \) (model 1e). Note that (iii) corresponds to our proposal (see Section 2). A sensitivity analysis for prior specification of the precision parameter can be carried out from the figures in Table 1, which reports the deviance information criterion (DIC) for the fitted models with different prior distributions for \( \phi \). We observe that the different proposed priors lead to similar DICs. However, model 1d shows a slightly better fit than the other proposals.

In Table 2, we report the parameter estimates for model 1d. These results show that the estimated parameters from the Bayesian methodology proposed here are
Table 1: DIC for the fitted models with different prior specifications for the precision parameter under model 1; simulated data

<table>
<thead>
<tr>
<th>Model</th>
<th>Prior for $\phi$</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>model 1a</td>
<td>$\phi \sim IG(0.01, 0.01)$</td>
<td>−1245.02</td>
</tr>
<tr>
<td>model 1b</td>
<td>$\phi = U^2, U \sim U(0, 50)$</td>
<td>−1245.14</td>
</tr>
<tr>
<td>model 1c</td>
<td>$\phi = (50B)^2, B \sim \text{beta}(1.1, 1.1)$</td>
<td>−1245.15</td>
</tr>
<tr>
<td>model 1d</td>
<td>$\phi = (50B)^2, B \sim \text{beta}(1.5, 1.5)$</td>
<td>−1245.41</td>
</tr>
<tr>
<td>model 1e</td>
<td>$\ln(\phi) \sim t(10, 0, 5)$</td>
<td>−1244.85</td>
</tr>
</tbody>
</table>

similar to the true values of the model parameters. In our simulation study, we consider 100,000 Monte Carlo iterations and the results are presented considering the last 90,000 iterations. In addition, the necessary diagnostic tests (such as convergence, autocorrelation, history) were performed, from which desirable behaviors were observed in the chains (see Table 3, Figures 1 and 2, and comments below). We also conducted a sensitivity analysis with respect to the prior specifications of the regression coefficients and the dispersion matrix of the random effects coefficients. In each case, the posterior inferences were not appreciably altered in comparison with the results presented in Table 2.

Table 2: True mean and estimated posterior medians and means, 95% credibility intervals (CI) for model 1d; simulated data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True</th>
<th>Mean</th>
<th>MC Error</th>
<th>Median</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>−2.000</td>
<td>−2.065</td>
<td>0.002</td>
<td>−2.065</td>
<td>(−2.281, −1.856)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.000</td>
<td>1.037</td>
<td>0.001</td>
<td>1.035</td>
<td>(0.904, 1.171)</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>2.000</td>
<td>1.949</td>
<td>0.001</td>
<td>1.949</td>
<td>(1.824, 2.065)</td>
</tr>
<tr>
<td>$\phi$</td>
<td>49.000</td>
<td>44.160</td>
<td>0.071</td>
<td>44.090</td>
<td>(38.340, 50.610)</td>
</tr>
<tr>
<td>$\nu_b$</td>
<td>1000.000</td>
<td>975.600</td>
<td>33.540</td>
<td>672.100</td>
<td>(19.570, 3429.500)</td>
</tr>
<tr>
<td>$\Sigma_{b11}$</td>
<td>1.090</td>
<td>0.981</td>
<td>0.004</td>
<td>0.964</td>
<td>(0.709, 1.330)</td>
</tr>
<tr>
<td>$\Sigma_{b12}$</td>
<td>−0.360</td>
<td>−0.257</td>
<td>0.002</td>
<td>−0.252</td>
<td>(−0.408, −0.131)</td>
</tr>
<tr>
<td>$\Sigma_{b22}$</td>
<td>0.130</td>
<td>0.080</td>
<td>0.000</td>
<td>0.077</td>
<td>(0.035, 0.146)</td>
</tr>
</tbody>
</table>
The multivariate version of Gelman and Rubin’s convergence diagnostic proposed by Brooks and Gelman (1998) indicates that the chain is convergent since the multivariate proportional scale reduction factor (mprf) equals 1.01. Analyzing the convergence of the chain for each parameter, we may conclude that convergence is achieved for each chain. The latter conclusion is corroborated by three different convergence tests, namely Gelman and Rubin’s convergence diagnostic (Gelman and Rubin, 1992), Geweke’s diagnostic (Geweke, 1992), and Heidelberg and Welch’s diagnostic (Heidelberger and Welch, 1981) and Heidelberger and Welch (1983), which were obtained using the libraries lattice and coda (Plummer et al., 2006) of the R software (freely available from http://www.r-project.org/). To obtain Gelman and Rubin’s convergence diagnostic, we started two chains in different initial points and performed 100,000 Monte Carlo iterations, considering the last 90,000 iterations. The diagnostic results are presented in Table 3. In addition, Figures 1 and 2 suggest that the chain for each parameter is not correlated and stationary, respectively. These results are essential to achieve an adequate estimation of the parameters.

We now use the same set of simulated data to fit model 2, with five different regression structures for the precision parameter. Note that the true (unknown)
model is a mixed beta regression model with constant precision, and hence only model 2a corresponds to the true model (see Table 4). Prior distributions for the parameters $\nu_b$, $\Sigma_b$ and $\beta$ are the same as those proposed for model 1. Also, for model specifications that include random effects for the precision parameter, we assume that the precision random effects $d_i$ have the same distribution as the location random effects $b_i$, namely $t(\nu_b, 0, \Sigma_b)$. Table 4 reports the DIC for the five fitted models using simulated data. We observe that the submodels for the precision parameter that do not include random effects achieve the best fits to our data, models 2a, 2c and 2d being similarly good. However, model 2a, the model under which the data were simulated and which is equivalent to model 1e, provides a better fit than the other proposals. Therefore, the best fitted model agrees with the true model.

Table 4: **DIC for the fitted models with different specifications of the precision parameter (model 2); simulated data**

<table>
<thead>
<tr>
<th>Model</th>
<th>Location Mixed Model</th>
<th>Precision Mixed Model</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>model 2a</td>
<td>$(\beta_1 + b_{i1}) + (\beta_2 + b_{i2})x_{ij2} + x_{ij3}\beta_3$</td>
<td>$\delta_1$</td>
<td>$-1244.85$</td>
</tr>
<tr>
<td>model 2b</td>
<td>$(\beta_1 + b_{i1}) + (\beta_2 + b_{i2})x_{ij2} + x_{ij3}\beta_3$</td>
<td>$\delta_1 + d_{i1}$</td>
<td>$-1221.18$</td>
</tr>
<tr>
<td>model 2c</td>
<td>$(\beta_1 + b_{i1}) + (\beta_2 + b_{i2})x_{ij2} + x_{ij3}\beta_3$</td>
<td>$\delta_1 + \delta_3x_{ij3}$</td>
<td>$-1242.94$</td>
</tr>
<tr>
<td>model 2d</td>
<td>$(\beta_1 + b_{i1}) + (\beta_2 + b_{i2})x_{ij2} + x_{ij3}\beta_3$</td>
<td>$\delta_1 + \delta_2x_{ij2} + \delta_3x_{ij3}$</td>
<td>$-1241.53$</td>
</tr>
<tr>
<td>model 2e</td>
<td>$(\beta_1 + b_{i1}) + (\beta_2 + b_{i2})x_{ij2} + x_{ij3}\beta_3$</td>
<td>$(\delta_1 + d_{i1}) + (\delta_2 + d_{i2})x_{ij2} + \delta_3x_{ij3}$</td>
<td>$-1223.74$</td>
</tr>
</tbody>
</table>

Table 5 reports the parameter estimates under model 2a. It can be seen that the estimates obtained through the Bayesian methodology proposed here are similar to the corresponding true values of the parameters. To fit model 2a, we considered 100,000 Monte Carlo iterations and the estimates were obtained using the last 90,000 iterations. We obtained $\text{mprf} = 1.00 < 1.2$, indicating that the chain is convergent. Figures 3 and 4 suggest that the chain for each parameter is not correlated and stationary, respectively. Hence, our estimates are reliable.
Table 5: True mean and estimated posterior medians and means, 95% credibility intervals (CI) for model 2a; simulated data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>True Mean</th>
<th>MC Error</th>
<th>Median</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>-2.000</td>
<td>-2.065</td>
<td>0.004</td>
<td>-2.062</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.000</td>
<td>1.034</td>
<td>0.002</td>
<td>1.033</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>2.000</td>
<td>1.947</td>
<td>0.002</td>
<td>1.947</td>
</tr>
<tr>
<td>$\delta_1$</td>
<td>3.892</td>
<td>3.783</td>
<td>0.002</td>
<td>3.784</td>
</tr>
<tr>
<td>$\nu_b$</td>
<td>1000.000</td>
<td>971.200</td>
<td>33.640</td>
<td>675.700</td>
</tr>
<tr>
<td>$\Sigma_{b11}$</td>
<td>1.090</td>
<td>0.976</td>
<td>0.005</td>
<td>0.970</td>
</tr>
<tr>
<td>$\Sigma_{b12}$</td>
<td>-0.360</td>
<td>-0.257</td>
<td>0.002</td>
<td>-0.255</td>
</tr>
<tr>
<td>$\Sigma_{b22}$</td>
<td>0.130</td>
<td>0.080</td>
<td>0.001</td>
<td>0.076</td>
</tr>
</tbody>
</table>

Figure 1: Autocorrelation under model 1d; simulated data
Figure 2: History under model 1d; simulated data

Figure 3: Autocorrelation under model 2a; simulated data
5. A real data application

We now consider the dataset reported by Prater (1956). The response variable is the proportion of crude oil converted into gasoline after distillation and fractionation. The dataset contains 32 observations on the response and on other variables. By sorting the data, it is clear that there are only 10 crudes involved. A potentially useful covariate is the end point \((EP)\), i.e., the temperature (in degrees Fahrenheit) at which all gasoline has vaporized. Ferrari and Cribari-Neto (2004) fitted a beta regression model with constant precision to these data, in which the batches of crude oil are treated as a fixed factor with ten levels and with a fixed slope for the end point. Instead, Venables (2000) suggested that the batches should be viewed as a random factor. Graphical inspection of the data suggests that a location submodel with random intercepts and a common slope may be suitable for the data.

At the outset, we consider a mixed beta regression model with a constant precision parameter (model 1). The location submodel involves random intercepts and a common slope. Table 6 gives the DIC for the model fitting with different prior specifications for the precision parameter \(\phi\). As before, for the parameters \(\nu_b\), \(\Sigma_b\), and \(\beta\) we considered the prior distributions \(\nu_b \sim \varepsilon(a)\), \(\Sigma_b \sim IW_2(\Psi, c)\), and \(\beta \sim t_2(\nu_\beta, \mu_\beta, \Sigma_\beta)\) with

\[
a = 0.001, \quad \Psi = \begin{pmatrix} 1000 & 0 \\ 0 & 1000 \end{pmatrix}, \quad c = 4, \quad \nu_\beta = 10, \quad \mu_\beta = (0, 0)^T, \quad \Sigma_\beta = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.
\]
It can be noticed that the different proposed priors provide similar DIC values, the smallest DIC, however, being obtained by a beta prior with $\epsilon = 0.5$.

Table 6: DIC for the fitted models with different prior specifications of the precision parameter under model 1; Prater’s data

<table>
<thead>
<tr>
<th>Model</th>
<th>Location submodel</th>
<th>Prior for $\phi$</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>model 1.1</td>
<td>$(\beta_1 + b_{1i}) + \beta_2 EP_{ij}$</td>
<td>$\phi \sim IG(0.01, 0.01)$</td>
<td>$-142.474$</td>
</tr>
<tr>
<td>model 1.2</td>
<td>$(\beta_1 + b_{1i}) + \beta_2 EP_{ij}$</td>
<td>$\phi = u^2, u \sim U(0, 50)$</td>
<td>$-142.555$</td>
</tr>
<tr>
<td>model 1.3</td>
<td>$(\beta_1 + b_{1i}) + \beta_2 EP_{ij}$</td>
<td>$\phi = (50B)^2, B \sim beta(1.1, 1.1)$</td>
<td>$-142.567$</td>
</tr>
<tr>
<td>model 1.4</td>
<td>$(\beta_1 + b_{1i}) + \beta_2 EP_{ij}$</td>
<td>$\phi = (50B)^2, B \sim beta(1.5, 1.5)$</td>
<td>$-142.595$</td>
</tr>
</tbody>
</table>

We now assume that $\phi$ is not constant through the observations. Again, the location submodel assumes random intercepts and a common slope. As in the simulation study, we consider that both random effects, the $b_i$s and $d_i$s, are identically distributed with distribution $t(\nu, 0, \Sigma)$ (so that $\nu_d = \nu_b = \nu$ and $\Sigma_b = \Sigma_d = \Sigma$ in our previous notation). Table 7 gives the DIC for the model fitting under different precision submodels (models 2.1–2.6). Note that model 2.1 is the same as model 1, i.e., it implies constant precision but with a different prior for $\phi$, namely $\ln(\phi) \sim t(10, 0, 5)$. Tables 8 and 9 give the posterior estimates of the parameters associated with models 1.4 and 2.5, which provide the best fits for constant and nonconstant precision, respectively. It can be noticed that for both models, the estimates for the degrees of freedom ($\nu$) is approximately 1,000, which suggests that the random effects $b_i$s and $d_i$s could be modeled using a normal distribution. Between the constant precision model (modelo 1.4) and the variable precision model (model 2.5), the DIC values suggest that the later is the best. It means that not only the location submodel but also the precision submodel are affected by a random additive effect and the end point ($EP$). Also, there is no evidence of association between the random effects since zero belongs to the credibility interval for $\Sigma_{12}$. It can also be noticed that the covariate $EP$ affects both the mean and the precision of the proportion of crude oil converted into gasoline positively.

Some technical details relating to the fit of the models are now in order. We considered 200,000 Monte Carlo iterations and our results were obtained considering the last 190,000 iterations. Additionally, we performed the diagnostic tests reported for the simulated data, all of which suggested suitable behavior of the chains. For model 1.4, the multivariate version of Gelman and Rubin’s conver-
Table 7: DIC for the fitted models with different specifications for the precision parameter under model 2; Prater’s data

<table>
<thead>
<tr>
<th>Model</th>
<th>Location submodel</th>
<th>Precision submodel</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>model 2.1</td>
<td>$(\beta_1 + b_{i1}) + \beta_2 EP_{ij}$</td>
<td>$\delta_1$</td>
<td>-140.584</td>
</tr>
<tr>
<td>model 2.2</td>
<td>$(\beta_1 + b_{i1}) + \beta_2 EP_{ij}$</td>
<td>$\delta_1 + d_{i1}$</td>
<td>-140.506</td>
</tr>
<tr>
<td>model 2.3</td>
<td>$(\beta_1 + b_{i1}) + \beta_2 EP_{ij}$</td>
<td>$\delta_2 EP_{ij}$</td>
<td>-145.296</td>
</tr>
<tr>
<td>model 2.4</td>
<td>$(\beta_1 + b_{i1}) + \beta_2 EP_{ij}$</td>
<td>$\delta_1 + \delta_2 EP_{ij}$</td>
<td>-144.883</td>
</tr>
<tr>
<td>model 2.5</td>
<td>$(\beta_1 + b_{i1}) + \beta_2 EP_{ij}$</td>
<td>$d_{i1} + \delta_2 EP_{ij}$</td>
<td>-146.771</td>
</tr>
<tr>
<td>model 2.6</td>
<td>$(\beta_1 + b_{i1}) + \beta_2 EP_{ij}$</td>
<td>$(\delta_1 + d_{i1}) + \delta_2 EP_{ij}$</td>
<td>-145.858</td>
</tr>
</tbody>
</table>

Table 8: Estimated posterior medians and means, 95% credibility intervals (CI) for the mixed beta regression model 1.4; Prater’s data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>MC Error</th>
<th>Median</th>
<th>Posterior Inference</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>-5.121</td>
<td>0.009</td>
<td>-5.122</td>
<td>(-5.600, -4.612)</td>
<td></td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$10.710 \times 10^{-3}$</td>
<td>0.015 $\times 10^{-3}$</td>
<td>$10.740 \times 10^{-3}$</td>
<td>(9.606 $\times 10^{-3}$, 11.750 $\times 10^{-3}$)</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>1043.000</td>
<td>26.440</td>
<td>716.000</td>
<td>(21.790, 3726.000)</td>
<td></td>
</tr>
<tr>
<td>$\phi$</td>
<td>300.900</td>
<td>1.890</td>
<td>290.200</td>
<td>(144.300, 517.700)</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{11}$</td>
<td>960.900</td>
<td>55.260</td>
<td>443.200</td>
<td>(5.449, 4889.000)</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{12}$</td>
<td>-67.020</td>
<td>106.700</td>
<td>-85.760</td>
<td>(-4021.000, 3981.000)</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{22}$</td>
<td>4087.000</td>
<td>85.220</td>
<td>3449.000</td>
<td>(478.100, 11700.000)</td>
<td></td>
</tr>
</tbody>
</table>

gence diagnostic ([Brooks and Gelman](1998)) indicates that the chain is convergent (mprf = 1.09 < 1.2). Also, Figures 3 and 4 suggest that the chain for each parameter is not correlated and stationary, respectively, while Figure 5 demonstrates that the posterior density function for each parameter does not present multimodality; it should be noted that multimodality can be accompanied by convergence problems. We can then assume that the estimates reported in Table 7 are reliable. For model 2.5, similar diagnostic evidence was obtained. Here, mprf = 1.06 and Figures 6–8 suggest that the results in Table 8 can be trusted.
Table 9: Estimated posterior medians and means, 95% credibility intervals (CI) for the mixed beta regression model 2.5; Prater’s data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>MC Error</th>
<th>Median</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>-4.775</td>
<td>0.010</td>
<td>-4.766</td>
<td>(-5.341, -4.199)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$9.819 \times 10^{-3}$</td>
<td>0.020 $\times 10^{-3}$</td>
<td>$9.819 \times 10^{-3}$</td>
<td>(8.531 $\times 10^{-3}$, 11.020 $\times 10^{-3}$)</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>$17.110 \times 10^{-3}$</td>
<td>0.030 $\times 10^{-3}$</td>
<td>17.180 $\times 10^{-3}$</td>
<td>(14.630 $\times 10^{-3}$, 19.240 $\times 10^{-3}$)</td>
</tr>
<tr>
<td>$\nu$</td>
<td>1002.000</td>
<td>23.570</td>
<td>685.600</td>
<td>(19.650, 3776.000)</td>
</tr>
<tr>
<td>$\Sigma_{11}$</td>
<td>972.800</td>
<td>66.950</td>
<td>468.200</td>
<td>(6.542, 4627.000)</td>
</tr>
<tr>
<td>$\Sigma_{12}$</td>
<td>-146.200</td>
<td>132.100</td>
<td>-109.000</td>
<td>(-4243.000, 3769.000)</td>
</tr>
<tr>
<td>$\Sigma_{22}$</td>
<td>4079.000</td>
<td>87.220</td>
<td>3417.000</td>
<td>(488.700, 10970.000)</td>
</tr>
</tbody>
</table>

Figure 5: Autocorrelation; Prater’s data; constant $\phi$
Figure 6: History; Prater’s data; constant $\phi$; model 1.4

Figure 7: Density; Prater’s data; constant $\phi$; model 1.4
Figure 8: Autocorrelation; Prater’s data; non constant $\phi$; model 2.5

Figure 9: History – Prater’s data; non constant $\phi$; model 2.5
6. Discussion

Beta regression modeling has gained increasing popularity after the work of Ferrari and Cribari-Neto (2004), who described a beta regression model parameterized in terms of the mean response and a common precision parameter, and developed frequentist inference and basic diagnostic tools for the proposed model. A complementary approach proposed by Smithson and Verkuilen (2006) considers that the precision parameter is not fixed but, instead, is modeled in a regression manner. A Bayesian beta regression model was studied by Branscum et al. (2007). In this paper, we extended these ideas for a mixed beta regression model under a Bayesian perspective.

The present paper considered Bayesian inference for mixed beta regression based on two different approaches. First, the precision parameter was assumed to be fixed, i.e., the same for all observations. A linear regression structure was proposed for the mean parameter through a logit link function. Our results are readily extended to other link choices. Specification of different priors for the common precision parameter was studied. We considered a prior distribution for \( \phi \) of the type \( \phi = U^2 \), with \( U \sim U(0, a) \), where it is common to consider as initial value \( a = 50 \) (Gelman, 2006). We also proposed alternative priors, namely, prior distributions of the type \( \phi = (aB)^2 \), with \( B \sim \text{beta}(1 + \epsilon, 1 + \epsilon) \) and \( \epsilon = 0.001, 0.01, 0.1, 0.5, \ldots \), which delivered good results in terms of model fit and performance of diagnostics tests. Second, the precision parameter was modeled through its own linear regression structure using a log link. Again, other
choices of the precision link function can be accommodated. For both the mean and the precision submodels, a mixed-effects model with a multivariate t distribution for the fixed and the random effects was considered. Our empirical applications yielded good results in terms of model fit and diagnostics tests. It is worth mentioning that in this context, it is necessary to perform a careful model selection for the precision modeling including more or fewer fixed and random effects since it is not clear in advance which model is more plausible.

A classic version of this problem was raised by Zimprich (2010), where mixed beta regression models were estimated using the SAS procedure NLMIXED (SAS (2008)), employing adaptive Gaussian quadrature. This approach achieves good results, but the implementation of the mixed beta regression model for random effects that are non-normally distributed is very challenging. In this sense, our approach is more flexible because one can easily implement it when the distribution of the random effects is normal, Student-t, skew normal or other distributions, by using simple and accessible software such as WinBUGS. Another advantage of this approach is the easy implementation for the imputation of missing data (Carrigan et al., 2007), a common situation in practice and for which a classic approach is much more complicated.

7. Appendix: BUGS codes for the mixed beta regression

This appendix presents the various pieces of BUGS code used for fitting the mixed beta regression in the simulated data example.

**Inverse gamma prior for $\phi$**

```plaintext
model {
  for (i in 1 : m) {
    for (j in 1 : n) {
      Y[i, j] ~ dbeta(a1[i,j], a2[i,j])
      a1[i,j] <- mu[i, j]*phi
      a2[i,j] <- (1-mu[i, j])*phi
      logit(mu[i, ,]) <- inprod(x[i, ,], beta[ ])+inprod(z[i, ,], b[i,1,])
    }
    b[i,1:q] ~ dmt(cerovec[], psi[ ,], gl1)
  }
  gl1~dexp(a0)
  beta[1:p] ~ dmt(alpha[ ], V1[ ,], gl2)
  V1[1:p,1:p] <- inverse(V[ , ])
  psi[1:q,1:q] ~ dwish(R0[ ,], c0)
  psiinv[1:q,1:q]<-inverse(psi[1:q,1:q])
  phiinv ~ dgamma(a00,a00)
  phi<-1/phiinv
}
```

20
Uniform prior for $\phi$

```r
model {
  for (i in 1 : m) {
    for (j in 1 : n) {
      Y[i, j] ~ dbeta(a1[i,j], a2[i,j])
      a1[i,j] <- mu[i,j] * phi
      a2[i,j] <- (1-mu[i,j]) * phi
      logit(mu[i,j]) <- inprod(x[i, j, ], beta[ ])+inprod(z[i, j, ], b[i,1,])
    }
    b[i,1,1:q] ~ dmt(cerovec [,psi[,]],gl1)
    gl1~dexp(a0)
    beta[1:p] ~ dmt(alpha[ ], V[1, ],gl2)
    V[1:p,1:p] <- inverse(V[ , ])
    psi[1:q,1:q] ~ dwish(R0[, ], c0)
    psiinv[1:q,1:q]<-inverse(psi[1:q,1:q])
    philr ~ dunif(a00,b01)
    phi<- philr*phi
  }
}
```

Beta prior for $\phi$

```r
model {
  for (i in 1 : m) {
    for (j in 1 : n) {
      Y[i, j] ~ dbeta(a1[i,j], a2[i,j])
      a1[i,j] <- mu[i,j] * phi
      a2[i,j] <- (1-mu[i,j]) * phi
      log(phi[i,j])<-inprod(x[i, j, ], delta[ ])+inprod(z[i, j, ], gama[i,1,])
      logit(mu[i,j]) <- inprod(x[i, j, ], beta[ ])+inprod(z[i, j, ], b[i,1,])
    }
    b[i,1,1:q] ~ dmt(cerovec [,psi[,]],gl1)
    gl1~dexp(a0)
    beta[1:p] ~ dmt(alpha[ ], V[1, ],gl2)
    V[1:p,1:p] <- inverse(V[ , ])
    psi[1:q,1:q] ~ dwish(R0[, ], c0)
    psiinv[1:q,1:q]<-inverse(psi[1:q,1:q])
    phiinicial ~ dbeta(a00,b0)
    phi<-- phiinicial*phi
  }
}
```

Submodel for $\phi$

```r
model {
  for (i in 1 : m) {
    for (j in 1 : n) {
      Y[i, j] ~ dbeta(a1[i,j], a2[i,j])
      a1[i,j] <- mu[i,j] * phi[i,j]
      a2[i,j] <- (1-mu[i,j]) * phi[i,j]
      log(phi[i,j])<-inprod(x[i, j, ], delta[ ])+inprod(z[i, j, ], gama[i,1,])
      logit(mu[i,j]) <- inprod(x[i, j, ], beta[ ])+inprod(z[i, j, ], b[i,1,])
    }
  }
}
```
b[i,1:q] ~ dmt(cerovec[], psi[], gl1)
gama[i,1:q] ~ dmt(cerovec[], psi[], gl1)
}
gl1~dexp(a0)

beta[1:p] ~ dmt(alpha[], V1[], gl2)
delta[1:p] ~ dmt(alpha[], V1[], gl2)
V1[1:p,1:p] <- inverse(V[]

psi[1:q,1:q] ~ dwish(R0[], c0)
psiinv[1:q,1:q]<-inverse(psi[1:q,1:q])
meanphi<-mean(phi[])

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References


