Elliptic \((N, N')\)-soliton solutions of the lattice Kadomtsev-Petviashvili equation

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Elliptic soliton solutions, comprising a soliton-type hierarchy of functions based on an elliptic seed solution, are constructed using elliptic Cauchy matrices, for a family of integrable lattice equations of Kadomtsev-Petviashvili (KP) type. This family consists of the lattice KP, modified KP, and Schwarzian KP equations as well as Hirota’s bilinear KP equation, and their successive continuum limits. The reduction to the elliptic soliton solutions of KdV type lattice equations is also discussed.

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I. THE LATTICE KP AND HIROTA EQUATIONS

The study of the discrete versions of soliton systems, i.e., systems given by integrable partial difference equations (\(P\Delta Es\)), has become in recent years a focus of attention within the theory of integrable systems. Among those systems, the discrete analogue of Kadomtsev-Petviashvili (KP) equations, which are defined on a three-dimensional lattice, seem to form a universal class of systems. The first equation of this type, found by Hirota in Ref. 15 and referred to as DAGTE (Discrete analogue of generalized Toda equation), is the bilinear equation

\[
\left(a_1 e^{\delta_1 D_1} + a_2 e^{\delta_2 D_2} + a_3 e^{\delta_3 D_3}\right) \tau \cdot \tau = 0,
\]

in which the exponentials of the Hirota operators \(\exp (\delta_i D_i)\), \((i = 1, 2, 3)\) produce finite-step (with step parameter \(\delta_i\)) forward-and backward shifts in the arguments \(\xi_i\), of a pair of functions \(f(\xi_1, \xi_2, \xi_3)\), \(g(\xi_1, \xi_2, \xi_3)\) in the corresponding lattice direction, i.e., by definition

\[
e^{\delta_i D_i} f = f(\ldots, \xi_i + \delta_i, \ldots), g(\ldots, \xi_i - \delta_i, \ldots).
\]

This equation is integrable for arbitrary parameters \(a_1, a_2, a_3\), in the sense of singularity confinement, see Ref. 32, and also in the sense of multidimensional consistency, see Ref. 2. However, special attention was given to the case that the coefficients \(a_1, a_2, a_3\) satisfy the condition \(a_1 + a_2 + a_3 = 0\), in which case the equation admits rational soliton solutions, and allows reductions to certain two-dimensional equations. For that restricted case Miwa, in Ref. 22, reparametrized the equation in a form that is nowadays often referred to as the Hirota-Miwa equation. Sometimes the full equation (representing the bilinear discrete KP equation) is also (in our view erroneously) referred to as the Hirota-Miwa equation. However, in Ref. 22 only the restricted case was considered and generalized to a four-term bilinear equation which is nowadays referred to as the Miwa equation. In this paper, we will investigate a class of elliptic solutions (1.1) and of related three-dimensional lattice equations, comprising the following family of \(P\Delta Es\):

The bilinear lattice KP equation

\[
\tilde{f} \tilde{f} + \tilde{f} \tilde{f} - \tilde{f} \tilde{f} = 0.
\]

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The lattice KP equation

\[(\tilde{w} - \hat{w})(v - \tilde{w}) = (\tilde{w} - \hat{w})(\tilde{v} - \tilde{w}).\]  

(1.3)

The lattice modified KP (MKP) equation

\[\frac{\tilde{v} - \hat{v}}{\tilde{v}} + \frac{v - \tilde{v}}{v} + \frac{\tilde{v} - v}{\tilde{v}} = 0.\]  

(1.4)

The asymmetric modified KP equation

\[\frac{\tilde{u} - \hat{u}}{\tilde{u}} + \frac{u - \tilde{u}}{u} = 0.\]  

(1.5)

The lattice Schwarzian KP equation

\[\frac{(\tilde{z} - \hat{z})(\tilde{z} - \tilde{z})}{(\tilde{z} - \tilde{z})(\tilde{z} - \tilde{z})} = 1.\]  

(1.6)

The notation adopted in (1.2)–(1.6) is as follows: all dependent variables \(f, w, v, u,\) and \(z,\) are functions defined on the three-dimensional lattice with discrete coordinates \((n, m, h) \in \mathbb{Z}^3,\) e.g., \(f = f_{n,m,h},\) and the operations \(f \mapsto \tilde{f}, f \mapsto \hat{f},\) and \(f \mapsto \tilde{f}\) denote elementary shifts of the lattice, i.e., \(\tilde{f} = f_{n+1,m,h}, \hat{f} = f_{n,m+1,h},\) and \(\tilde{f} = f_{n,m,h+1},\) while for the combined shifts we have: \(\dot{f} = f_{n+1,m+1,h}, \ddot{f} = f_{n,m+1,h+1},\) and \(\ddot{f} = f_{n+1,m,h+1} = f_{n,m,h+1}.\) We can also think of these equations being defined for functions of \(f(\xi_1, \xi_2, \xi_3)\) in terms of continuous variables \(\xi_i\) as in (1.1) each shifting by elementary shifts of step size \(\delta_i,\) along the \(i\)th direction. However, setting \(\dot{\xi}_i = \xi_i^0 + n_i \delta_i,\) and \(\ddot{\xi}_i = \xi_i^0 + m_i \delta_i,\) this situation can without loss of generality be recast in the form of functions on the lattice labelled by integers, where we identify \(f_{n,m,n_3} := f(\xi_1, \xi_2, \xi_3).\) To avoid a plethora of indices we rename these variables \(n_1 = n, n_2 = m, n_3 = h.\) The first equation (1.2) is Hirota’s DAGTE after a change of independent variables and a point transformation. The other KP type lattice equations (1.3)–(1.6) were established in Ref. 28, and appear in a form of the equation containing two additional parameters which can be chosen such that it reduces to either one of these equations up to a point transformation. In particular, the Schwarzian KP (1.6) was first given in this normalised form in Ref. 12, and was subsequently rediscovered in Ref. 7, while its geometric significance was explored in Refs. 19, 17, and 11. All Eqs. (1.3)–(1.6) appear as distinct parameter choices of the general lattice KP type equation. The first equation (1.2) is Hirota’s DAGTE after a change of independent variables and a point transformation. The other KP type lattice equations (1.3)–(1.6) were established in Ref. 28, and appear in a form of the equation containing two additional parameters which can be chosen such that it reduces to either one of these equations up to a point transformation. In particular, the Schwarzian KP (1.6) was first given in this normalised form in Ref. 12, and was subsequently rediscovered in Ref. 7, while its geometric significance was explored in Refs. 19, 17, and 11. All Eqs. (1.3)–(1.6) appear as distinct parameter choices of the general lattice KP type equation. We also mention that matrix analogues to these equations were given in Refs. 23 and 26.

With regard to terminology, we prefer to reserve the name lattice (potential) KP equation for (1.3) rather than for (1.2) or for (1.1) (as is commonly done in the literature), because it is actually the former equation that is more directly related to the actual continuous (potential) KP equation while the latter two equations yield only the bilinear form of the KP equation through a continuum limit. Similarly, Eqs. (1.4) and (1.6) can be shown to be directly related to the continuous (potential) modified and the Schwarzian KP, respectively, equations through continuum limits, cf. Refs. 27, 28, and 35, hence their names. According to the paper, which addresses the classification problem of scalar lattice equations of octahedral type subject to the requirement that they are multidimensionally consistent when embedded in the four-dimensional lattice, the five canonical equations mentioned above exhaust all possible cases of such octahedral type lattice equations.

Some of these equations have been quite widely studied. In particular, Eq. (1.1) has arisen in many contexts over the last two decades, notably in connection with affine Weyl group description of discrete Painlevé equations. An interesting connection was found with the spectrum of Bethe Ansatz states of quantum solvable models, the eigenvalues of which were shown to obey specific versions of Hirota’s bilinear equation, cf. Refs. 21 and 38. Furthermore, related to this observation is a connection found in Ref. 29 between, on the one hand, pole-type solutions (in the sense of Airault et al. and Krichever) giving rise to a class of discrete-time many-body systems, and, on the other hand, soliton solutions
of the lattice KP equation. This has led to the formulation of an exact time-discretizations of the Ruijsenaars model\textsuperscript{31,33} in terms of an integrable correspondence (i.e., a multivalued map), which has exactly the same form as the Bethe-Ansatz equations for the excitations of quantum integrable models solvable by the quantum inverse scattering transform. It seems from all these intriguing connections which are still far from understood at this juncture, that the lattice KP equations exhibit a certain universality in their connection with various types of integrable systems involving at the same time partial difference equations, classical many-body systems and quantum solvable systems.

In this paper, we consider a novel class of solution of these lattice equations, namely, soliton type solutions based on an elliptic seed solution. The construction exploits elliptic $N \times N'$ Cauchy-matrices, and involve also $N' \times N$ coefficient matrices, which in principle allows for a classification of the solutions according to the Schubert decomposition of a corresponding Grassmannian. Thus, our main result, which provides a general expression for what we accordingly call the elliptic $(N, N')$-soliton solutions, could form a basis for a similar analysis as was performed in the case of the continuous KP equation by Kodama in Ref.\textsuperscript{18}, to describe the soliton taxonomy of the KP equation. Furthermore, our approach yields the various Miura type relations between the different equations and allows to study the various intermediate continuum limits, as well as reductions to lower dimensional equations, in a systematic way.

II. CAUCHY MATRIX SCHEME

We will develop now a scheme along the lines of the paper\textsuperscript{24,25} where the elliptic soliton solutions for the ABS (Adler/Bobenko/Suris, cf. Ref. 1) list of two-dimensional lattice equations were studied, based on elliptic Cauchy matrices. That construction provided solutions for all equations of the ABS list up to and including $Q_3$, whereas the case of $Q_4$ was treated separately using a different approach, cf. Ref. 6.

In this section, we first give the main ingredients and derive the basic relations underlying the scheme. Second, we define the relevant $\tau$-function and introduce other quantities in terms of which we will eventually obtain nonlinear closed-form lattice equations. The derivation of the latter is attained through a number of steps, moving through the derivation of some basic linear equations (which constitute the “universal form” of the relevant Lax pairs) to the nonlinear equations for the basic objects. The latter can then be identified (up to some simple transformations) with the KP lattice equations (1.2)–(1.6), the general elliptic $(N, N')$-soliton solution for which are then implied by the construction. The resulting explicit form of the elliptic soliton solutions is presented in “Hirota form” at the end of the section.

A. Basic ingredients

In this paper, we will work throughout exclusively in terms of Weierstrass functions, and we employ primarily the basic addition formulas for these functions. In what follows, $\sigma(x) = \sigma(x|2\omega, 2\omega')$, $\zeta(x) = \zeta(x|2\omega, 2\omega')$, $\varphi(x) = \varphi(x|2\omega, 2\omega')$, denote the standard Weierstrass $\sigma$, $\zeta$, and $\varphi$-functions with simple periods $2\omega, 2\omega' \in \mathbb{C}$, see, e.g., Ref. 4.

At this point we find it useful to introduce the Lamé function. The inclusion of the exponential factor in (2.1) together with the $\Phi$-function, which breaks the symmetry between the argument of the function and the suffix, is not essential for most of the results in this paper, but it is only needed to guarantee that the solutions are proper elliptic functions in terms of one of the variables (in fact, the initial value $\xi$ introduced below). Omitting all exponential factors throughout the main results still hold true, but the solutions may only be quasi-periodic with respect to the periods of the Weierstrass elliptic functions

$$\Psi_\xi(\kappa) := \Phi_\xi(\kappa) e^{-\zeta(\xi)\kappa}, \quad (2.1)$$

where

$$\Phi_\xi(\kappa) = \frac{\sigma(\kappa + \xi)}{\sigma(\xi)\sigma(\kappa)}. \quad (2.2)$$
The basic identities for the $\Psi$ function are the following:

\[
\Psi_\xi(\kappa)\Psi_\xi(\lambda) = e^{\eta_\kappa}\Psi_{\xi+\delta}(\kappa)\Psi_{\xi}(\lambda-\kappa) + e^{\eta_\lambda}\Psi_{\xi}(\kappa-\lambda)\Psi_{\xi+\delta}(\lambda),
\]  
(2.3a)

\[
\Psi_\xi(\kappa)\Psi_{\xi}(\kappa) = e^{\eta_\kappa}\Psi_{\xi+\delta}(\kappa) [\xi(\xi) + \xi(\delta) + \xi(\kappa) - \xi(\xi+\delta+\kappa)],
\]  
(2.3b)

\[
\Psi_\xi(\kappa)\Psi_{\xi}(\lambda) = \Psi_{\xi}(\kappa+\lambda) [\xi(\xi) + \xi(\kappa) + \xi(\lambda) - \xi(\xi+\kappa+\lambda)],
\]  
(2.3c)

in which we have introduced

\[
\eta_\kappa := \eta_\kappa(\xi) = \xi(\xi+\delta) - \xi(\xi) - \xi(\delta) = \frac{1}{2} \frac{\phi'(\xi)}{\phi(\xi)} - \frac{\phi'(\xi)}{\phi(\xi)}.
\]  
(2.4)

Furthermore, we have the symmetry: $\Psi_{\xi}(\kappa-\kappa) = -\Psi_{-\xi}(\kappa)$.

The starting point for our construction is the “bare” non-autonomous $N \times N$ Cauchy matrix

\[
M^0 = \left( M^0_{i,j}(\kappa, \lambda) \right)_{i,j=1, \ldots, N}, \quad M^0_{i,j}(\xi) \equiv \Psi_{\xi}(\kappa_i + \kappa_j),
\]  
(2.5)

depending on an auxiliary variable $\xi$ given by

\[
\xi = \xi_0 + n\delta + m\epsilon + h\lambda,
\]  
(2.6)

linear in the independent discrete variables $n, m, h \in \mathbb{Z}$, and where the complex variables $\delta, \epsilon, \lambda, \xi \in \mathbb{C}$ are the corresponding lattice parameters, where $\xi_0$ is an initial value independent of $n, m, h$. In addition, the solutions will depend on two sets of complex-valued “rapidity parameters,” $\{\kappa_i, i = 1, \ldots, N\}$ and $\{\kappa'_j, j = 1, \ldots, N'\}$, which are chosen such that $\kappa_i + \kappa'_j \neq 0$ (modulo the period lattice of the $\sigma$-function). Furthermore, abbreviating

\[
p_{\kappa} = \Psi_{\xi}(\kappa), \quad q_{\kappa} = \Psi_{\xi}(\kappa), \quad l_{\kappa} = \Psi_{\xi}(\kappa),
\]  
(2.7)

and setting $\kappa = \pm \kappa_i, \kappa' = \pm \kappa'_j$, we can derive from the basic addition formula (2.3a) the following dynamical equations of the elliptic Cauchy matrix:

\[
M^0_{i,j}(\kappa) = \Psi_{\xi}(\kappa_i + \kappa'_j)\Psi_{\xi}(\kappa'_j)
= e^{\eta_{\kappa_i+\kappa'_j}}\Psi_{\xi}(\kappa_i + \kappa'_j)\Psi_{\xi}(\kappa'_j) + e^{\eta_{\kappa'_j}}\Psi_{\xi}(\kappa'_j)\Psi_{\xi}(\kappa_j)
= \tilde{M}^0_{i,j}(\kappa_{-\xi}) = e^{\eta_{\kappa_i+\kappa'_j}}\Psi_{\xi}(\kappa_i + \kappa'_j)\Psi_{\xi}(\kappa_j),
\]

where $\tilde{M}^0_{i,j}(\xi) = M^0_{i,j}(\xi + \delta)$. We introduce now the plane-wave factors (i.e., discrete exponential functions),

\[
\rho(\kappa) = (e^{-\zeta(\delta)\kappa}p_{-\kappa})^m (e^{-\zeta(\epsilon)\kappa}q_{-\kappa})^m (e^{-\zeta(\lambda)\kappa}l_{-\kappa})^h e^{\zeta(\lambda)\kappa}p^0(\kappa),
\]  
(2.8a)

\[
v(\kappa') = (e^{\zeta(\delta)\kappa'}p_{\kappa'})^m (e^{\zeta(\epsilon)\kappa'}q_{\kappa'})^m (e^{\zeta(\lambda)\kappa'}l_{\kappa'})^h e^{\zeta(\lambda)\kappa'}v^0(\kappa'),
\]  
(2.8b)

in which $\rho(\kappa)$ and $v(\kappa')$ denote some initial values independent of $n, m, h$. For the specific values $\kappa = \kappa_i$, we set $p_{\kappa} := \rho(\kappa_0)$ and $q_{\kappa'} := v(\kappa'_j)$. The latter quantities, viewed as functions of $n, m, h$, obey the shift relations

\[
\frac{\bar{\rho}_i}{\rho_i} = e^{\eta_{\kappa_i}}p_{-\kappa_i}, \quad \frac{\bar{\rho}_i}{\rho_i} = e^{\eta_{\kappa_i}}q_{-\kappa_i}, \quad \frac{\bar{\rho}_i}{\rho_i} = e^{\eta_{\kappa_i}}l_{-\kappa_i}
\]  
(2.9a)

and

\[
\frac{\bar{v}_j}{v_j} = e^{\eta_{\kappa'_i}(p_{\kappa'_i})}, \quad \frac{\bar{v}_j}{v_j} = e^{\eta_{\kappa'_i}(q_{\kappa'_j})}, \quad \frac{\bar{v}_j}{v_j} = e^{\eta_{\kappa'_i}(l_{\kappa'_i})},
\]  
(2.9b)

where as before the superscripts “$\sim_\kappa$,” “$\sim_{\kappa'}$,” and “$\sim_\kappa$” denote the lattice shifts related to the shift by one unit in the variables $n, m, h$, respectively, making use also of the relations $\xi = \xi + \delta$, $\xi = \xi + \epsilon$, and $\xi = \xi + \lambda$ which sit inside the coefficients $\eta_{\kappa}, \eta_{\kappa'},$ and $\eta_{\kappa}$. 
Now we can introduce the \(N\)- and \(N'\)-component vectors
\[
\mathbf{r} = (\rho_j \Psi_\xi(k_j))_{j=1,\ldots,N}, \quad \mathbf{s} = (v_j \Psi_\xi(k'_j))_{j=1,\ldots,N'},
\]
which we can define now the “dressed” Cauchy matrix:
\[
\tilde{\mathbf{M}} = (\tilde{M}_{i,j})_{i=1,\ldots,N; j=1,\ldots,N'}, \quad \tilde{M}_{i,j} = \rho_i M_0^{i,j} v_j.
\] (2.11)
As a consequence of the relations given earlier, and employing the definitions of the plane-wave factors, we can now describe the discrete dynamics as follows.

**Lemma 2.1.** The dressed Cauchy matrix \(\tilde{\mathbf{M}}\), as defined in (2.11), obeys the following linear relations under elementary shifts of the independent variables \(n\):
\[
\tilde{\mathbf{M}} = \mathbf{M} - \mathbf{r} \tilde{s}^T,
\] (2.12)
and under shifts of the variable \(m\) the similar relations:
\[
\tilde{\mathbf{M}} = \mathbf{M} - \mathbf{r} \tilde{s}^T,
\] (2.13)
and under shifts of the variable \(h\) the similar relations:
\[
\tilde{\mathbf{M}} = \mathbf{M} - \mathbf{r} \tilde{s}^T,
\] (2.14)
where, as before, \(\tilde{\mathbf{M}}, \hat{\mathbf{M}}, \tilde{\mathbf{M}}\), and \(\overline{\mathbf{M}}\) denote the shifted Cauchy matrices.

In what follows we will employ the relations (2.12)–(2.14) to obtain nonlinear shift relations for specific objects defined in terms of the Cauchy matrix \(\mathbf{M}\).

**B. The \(\tau\)-function and related basic objects**

Introduce now the \(\tau\)-function:
\[
\tau = \tau_{n,m,h} = \det_{N \times N} (1 + \mathbf{MC}) = \det_{N' \times N'} (1 + \mathbf{CM}),
\] (2.15)
where, since \(\mathbf{M}\) is in general not a square matrix, the \(N' \times N\) constant matrix \(\mathbf{C}\) is introduced to compensate for the discrepancy. The matrix \(\mathbf{1}\) is either the \(N' \times N'\) unit matrix, respectively, in both determinants. The latter identity is a consequence of the general Weinstein-Aronszajn formula,
\[
\det_{N \times N} \left( 1 + \sum_{i=1}^{N'} m_i c_i^T \right) = \det_{N' \times N'} (1 + c_i^T m_i),
\] (2.16)
where \(m_i = (M_{i,j})_{j=1,\ldots,N}\) and \(c_i^T = (C_{i,j})_{j=1,\ldots,N'}\), \(i = 1, \ldots, N'\) are the \(N'\)-component column- and row-vectors from the matrices \(\mathbf{M}\) and \(\mathbf{C}\), respectively, and where \(c_i^T m_i\) denote their dot products.

From the definition of the \(\tau\)-function and the relations for the dressed Cauchy matrix \(\tilde{\mathbf{M}}\) we obtain
\[
\tilde{\tau} = \det_{N \times N} (1 + \tilde{\mathbf{M}} \mathbf{C}) = \det_{N \times N} \left[ 1 + (\mathbf{M} - \mathbf{r} \tilde{s}^T) \mathbf{C} \right]
\]
\[
= \det_{N \times N} \left[ (1 + \mathbf{MC}) \left[ 1 - (1 + \mathbf{MC})^{-1} r \tilde{s}^T C \right] \right]
\]
\[
= \tau \det_{N \times N} \left[ 1 - (1 + \mathbf{MC})^{-1} r \tilde{s}^T C \right].
\]

For convenience we will introduce now the following symbol, which we will use throughout the paper,
\[
\chi_{\alpha,\beta} := \chi_{\alpha,\beta}(\xi) = \xi(\alpha) + \zeta(\beta) + \zeta(\xi) - \xi(\xi + \alpha + \beta) = \Phi_{\alpha}(\xi) \Phi_{\beta}(\xi) \Phi_{\alpha + \beta}(\xi),
\] (2.17)
(where the last equality follows from the basic identity for the function \(\Phi\)), and adopt the convention that functions containing the diagonal matrices \(\mathbf{K} = \text{diag}(\kappa_1, \ldots, \kappa_N)\) and \(\mathbf{K}' = \text{diag}(\kappa'_1, \ldots, \kappa'_N)\)
are simply the diagonal matrices whose diagonal entries are given by the evaluation of the function in question at the entries $\kappa_j$ and $\kappa'_j$, respectively. Any scalar term appearing in such functions is understood to be multiplied by the relevant unit matrix (which we suppress for notational convenience), and when we take inverses of those diagonal matrices we simply write them as reciprocals where their meaning cannot lead to any confusion. With these conventions we can easily establish the relation

$$\tilde{s} = e^{\eta_i K} (p_{K'})^{-1} \frac{\Psi_{\tilde{\xi}}(K')}{\Psi_{\tilde{\xi}}(K)} s = \left[\chi_{\delta,K'}\right]^{-1} s,$$

(2.18)

where $p_{K'}$ is the diagonal matrix with entries $p_{\kappa_j}$, and $\Psi_{\tilde{\xi}}(K')$ is the diagonal matrix with entries $\Psi_{\tilde{\xi}}(\kappa'_j)$. Following our convention, the notation $\chi_{\delta,K'}$ denotes the diagonal matrix with entries $\chi_{\delta,j}$, ($j = 1, \ldots, N'$).

Using (2.18), we find from our computation of $\tilde{\tau}$ the result

$$\frac{\tilde{\tau}}{\tau} = 1 - s^T \left[ x_{\delta,K'} \right]^{-1} C \left[ 1 + MC \right]^{-1} r =: W_\delta,$$

(2.19)

introducing the new object of interest $W_\delta$. More generally for arbitrary parameters $\alpha$, $\beta$ let us introduce

$$V_\alpha = 1 - s^T \left[ 1 + C M \right]^{-1} C \left[ x_{\alpha,K} \right]^{-1} r,$$

(2.20a)

$$W_\beta = 1 - s^T \left[ x_{\beta,K} \right]^{-1} C \left[ 1 + MC \right]^{-1} r,$$

(2.20b)

where $V_\alpha$ is a natural companion object which appears when we do the computation for the $\tau$-function in reverse. Thus, proceeding as follows:

$$\tau = \det_{N \times N} (1 + MC) = \det_{N \times N} \left\{ 1 + (\tilde{M} + r\tilde{s}^T)C \right\}$$

$$= \det_{N \times N} \left\{ 1 + (\tilde{M}C) \left[ 1 + (1 + MC)^{-1} r\tilde{s}^T C \right] \right\}$$

$$= \tilde{\tau} \det_{N \times N} \left\{ \left[ 1 + (1 + MC)^{-1} r\tilde{s}^T C \right] \right\}$$

and using also the fact that

$$\tilde{r} = e^{\eta_i K} p_{-K} \frac{\Psi_{\tilde{\xi}}(K)}{\Psi_{\tilde{\xi}}(K)} r = - \left[ \tilde{\chi}_{-\delta,K} \right] r,$$

(2.21)

we obtain

$$\frac{\tau}{\tilde{\tau}} = 1 - \tilde{s}^T \left[ 1 + MC \right]^{-1} C \left[ \tilde{\chi}_{-\delta,K} \right]^{-1} \tilde{r} =: \tilde{V}_{-\delta}.$$

(2.22)

Thus, we conclude that

$$\frac{\tilde{\tau}}{\tau} = W_\delta = \frac{1}{V_{-\delta}}.$$

(2.23)

We will now proceed with the derivation of more general dynamical relations for $V_\alpha$ and $W_\beta$.

### C. Basic linear relations

In order to derive dynamical relations for the objects $V_\alpha$ and $W_\beta$ themselves it is convenient to introduce the following $N$-component column vectors $u_\alpha$ and $N'$-component row vectors $u_\beta$, \n
$$u_\alpha = (1 + MC)^{-1} (\chi_{\alpha,K})^{-1} r,$$

(2.24a)

$$u_\beta = s^T (\chi_{\beta,K'})^{-1} (1 + CM)^{-1},$$

(2.24b)
where the “adjoint” function \( \cdot \beta^\prime \) is indicated by a left superscript \( \prime \), in order to stress the fact that this object is not obtained simply by matrix transposition (indicated by a right superscript \( T \)). Using these definitions, we can write the functions \( V \) and \( W \) in the forms

\[
V_\alpha = 1 - s^T C u_\alpha, \quad (2.25a)
\]

\[
W_\beta = 1 - \cdot \beta^\prime C r. \quad (2.25b)
\]

Performing the following calculation:

\[
\tilde{u}_\alpha = (1 + \tilde{M} C)^{-1}(\tilde{\chi}_\alpha, K)^{-1} r = (1 + \tilde{M} C)^{-1}(\tilde{\chi}_\alpha, K)^{-1} e_{\alpha K}^p \frac{\psi_{\xi}(K)^{p - K}}{\psi_{\xi}(K)} - r 
\]

\[
\Rightarrow \quad (1 + \tilde{M} C)\tilde{u}_\alpha = \frac{\Phi_{\xi + \alpha}(K)}{\Phi_{\xi}(K)} \frac{\psi_{\delta}(K)}{\psi_{\xi}(K)} e_{\alpha K}^p \frac{\psi_{\xi}(K)^{p - K}}{\psi_{\xi}(K)} - r = -\frac{\Phi_{\xi + \alpha}(K)}{\Phi_{\xi}(K)} \frac{\psi_{\xi}(K)^{p - K}}{\psi_{\xi}(K)} - r
\]

\[
\Rightarrow \quad [(1 + MC) - r s^T C] \tilde{u}_\alpha = -\frac{\zeta(K) + \zeta(\xi + \alpha) - \zeta(\delta) - \zeta(K + \xi + \alpha)}{\zeta(K) + \zeta(\xi) + \zeta(\alpha) - \zeta(K + \xi + \alpha)} r = \left(-1 + \frac{\chi_{\alpha, \delta}}{\chi_{\alpha, K}}\right) (2.26)
\]

where we have used the definition of the quantities \( \chi \), the relation \((2.21)\) and a basic addition formula for the function \( \Phi \) implied in \((2.3b)\), we can now multiply both sides of \((2.26)\) by \((1 + MC)^{-1}\) and use the definitions \((2.25a)\) as again \((2.24a)\) to obtain the relation

\[
\tilde{u}_\alpha = -u_\alpha \tilde{V}_\alpha + \chi_{\alpha, \delta} u_\alpha , \quad (2.27)
\]

where we have introduced the vector

\[
u_0 := (1 + MC)^{-1} r. \quad (2.28)
\]

Obviously, a similar relation like \((2.27)\) holds for the shifts in the other lattice directions. A similar set of relations can be derived for the adjoint vectors \((2.24b)\) which involves the adjoint vector to \((2.28)\), namely,

\[
\cdot \beta^\prime u_0 \equiv s^T (1 + CM)^{-1} , \quad (2.29)
\]

and obviously these relations all have their counterparts involving the other lattice shift related to shifts in the discrete independent variables \( m \) and \( h \) instead of \( n \).

Summarising the results of these derivations, we have the following statement.

**Lemma 2.2.** The \( N \)- and \( N' \)-component vectors given in \((2.24)\), together with the ones given in \((2.28)\) and \((2.29)\) obey the following sets of linear shift relations

\[
\tilde{u}_\alpha = -u_\alpha \tilde{V}_\alpha + \chi_{\alpha, \delta} u_\alpha , \quad (2.30a)
\]

\[
\cdot \beta^\prime u_\beta = W_\beta \tilde{u}_\beta - \tilde{\chi}_{\beta, - \delta} \tilde{u}_\beta , \quad (2.30b)
\]

for the shifts in the variable \( n \),

\[
\tilde{u}_\alpha = -u_\alpha \tilde{V}_\alpha + \chi_{\alpha, \epsilon} u_\alpha , \quad (2.31a)
\]

\[
\cdot \beta^\prime u_\beta = W_\beta \tilde{u}_\beta - \tilde{\chi}_{\beta, - \epsilon} \tilde{u}_\beta , \quad (2.31b)
\]

for the shifts in the variable \( m \), and

\[
\tilde{u}_\alpha = -u_\alpha \tilde{V}_\alpha + \chi_{\alpha, \lambda} u_\alpha , \quad (2.32a)
\]

\[
\cdot \beta^\prime u_\beta = W_\beta \tilde{u}_\beta - \tilde{\chi}_{\beta, - \lambda} \tilde{u}_\beta , \quad (2.32b)
\]

involving the shifts in the variable \( h \).
These linear relations form the “universal form” for the derivation of Lax pairs of nonlinear equations for the coefficients $V_\alpha$ and $W_{\beta}$. We will now proceed by deriving closed-form nonlinear equations for these and related objects.

### D. Basic nonlinear relations

In order to derive closed-form nonlinear equations for objects such as $V_\alpha$ and $W_{\beta}$ we first need to derive some basic nonlinear relations relating different quantities which we will introduce as we go along. Such relations would play the role of Miura type relations between various quantities obeying different nonlinear equations. Thus, one type of relation can be obtained by multiplying (2.30a) from the left by $s^T C$, leading to

$$s^T C \hat{u}_\alpha = -s^T C u_0 \hat{V}_\alpha + \chi_{\alpha,\delta} s^T C u_\alpha \Rightarrow \hat{s}^T \chi_{\delta, K} C \hat{u}_\alpha = -w_0 \hat{V}_\alpha + \chi_{\alpha, \delta} (1 - V_\alpha), \quad (2.33)$$

where we have introduced the variable

$$w_0 := s^T C u_0 = s^T C (1 + MC)^{-1} r. \quad (2.34a)$$

Similar relations can obviously be obtained for the other lattice directions, namely,

$$\hat{s}^T \chi_{\epsilon, K} C \hat{u}_\alpha = -w_0 \hat{V}_\alpha + \chi_{\alpha, \epsilon} (1 - V_\alpha), \quad (2.34b)$$

$$\hat{s}^T \chi_{\lambda, K} C \hat{u}_\alpha = -w_0 \hat{V}_\alpha + \chi_{\alpha, \lambda} (1 - V_\alpha). \quad (2.34c)$$

Subtracting the \( \hat{\cdot} \)-shift of (2.33) from the \( \hat{\cdot} \)-shift of (2.34b), and taking into account the explicit expressions for $\chi_{\delta, K}$ and $\chi_{\epsilon, K}$, in the objects on the left-hand sides, we obtain

$$\hat{V}_\alpha (\hat{w} - \hat{w}) = \hat{\chi}_{\alpha, \delta} \hat{V}_\alpha - \hat{\chi}_{\alpha, \epsilon} \hat{V}_\alpha, \quad (2.35)$$

where we have introduced the important quantity. (Note that the extra constant (with respect to the discrete variables $n$, $m$, and $h$) term $-\xi(\xi_0)$ is needed to render $w$ an elliptic function in terms of the initial value $\xi_0$.)

$$w := \xi(\xi) - \xi(\xi_0) - w_0 - n\xi(\delta) - m\xi(\epsilon) - h\xi(\gamma). \quad (2.36)$$

Similarly, multiplying (2.30b) from the right by $C \hat{T}$, we obtain

$$W_{\beta}(\hat{w} - \hat{w}) = \hat{\chi}_{\beta, -\delta} \hat{W}_{\beta} - \hat{\chi}_{\beta, -\epsilon} \hat{W}_{\beta}. \quad (2.37)$$

We now can write $\hat{w} - \hat{w}$ in the following form:

$$\hat{w} - \hat{w} = \frac{1}{\hat{V}_\alpha} \left( \frac{\Phi_\alpha(\xi) \Phi_\alpha(\delta)}{\Phi_\alpha(\xi + \delta)} \hat{V}_\alpha - \frac{\Phi_\alpha(\xi) \Phi_\alpha(\epsilon)}{\Phi_\alpha(\xi + \epsilon)} \hat{V}_\alpha \right)$$

$$= \frac{1}{\hat{W}_{\beta}} \left( \frac{\Phi_\beta(\xi) \Phi_\beta(-\delta)}{\Phi_\beta(\xi - \delta)} \hat{W}_{\beta} - \frac{\Phi_\beta(\xi) \Phi_\beta(-\epsilon)}{\Phi_\beta(\xi - \epsilon)} \hat{W}_{\beta} \right). \quad (2.38)$$

In order to simplify the factors containing $\xi$ and its shift, we introduce some yet new notation: we introduce the scaled functions

$$\hat{V}_\alpha = \Phi_\alpha(\hat{\xi}) \hat{V}_\alpha, \quad \hat{W}_{\beta} = \Phi_\beta(\hat{\xi}) \hat{W}_{\beta},$$

as well as the re-scaled parameters

$$\hat{\varphi}_a := \Phi_\alpha(\delta), \quad \hat{\varphi}_a := \Phi_\alpha(\epsilon), \quad \hat{\varphi}_a := \Phi_\alpha(\lambda), \quad (2.40)$$

in order to cast the resulting equations in autonomous form in terms of which we now have

$$\hat{\dot{w}} - \hat{\dot{w}} = \frac{1}{\hat{V}_\alpha} \left( \hat{\varphi}_a \hat{V}_\alpha - \hat{\varphi}_a \hat{V}_\alpha \right) = \frac{1}{\hat{W}_{\beta}} \left( \hat{\varphi}_{\beta, \delta} \hat{W}_{\beta} - \hat{\varphi}_{\beta, \epsilon} \hat{W}_{\beta} \right). \quad (2.41a)$$
Obviously, this “Miura type” relation has its counterpart involving the other lattice directions, namely,

\[
\hat{w} - \hat{\omega} = \frac{1}{\hat{V}_a} \left( \hat{q}_a \hat{V}_a - \hat{q}_a \hat{V}_a \right) = \frac{1}{{\hat{W}_\beta}} \left( \hat{l}_- \hat{W}_\beta - \hat{q}_- \hat{W}_\beta \right),
\]

\[
\hat{w} - \hat{\omega} = \frac{1}{\hat{V}_a} \left( \hat{p}_a \hat{V}_a - \hat{p}_a \hat{V}_a \right) = \frac{1}{{\hat{W}_\beta}} \left( \hat{p}_- \hat{W}_\beta - \hat{l}_- \hat{W}_\beta \right).
\]

(2.41b)

Using (2.41a)–(2.41c), we thus have as a consequence

\[
\frac{\hat{p}_a \hat{V}_a - \hat{q}_a \hat{V}_a}{\hat{V}_a} + \frac{\hat{q}_a \hat{V}_a - \hat{l}_a \hat{V}_a}{\hat{V}_a} + \frac{\hat{l}_a \hat{V}_a - \hat{p}_a \hat{V}_a}{\hat{V}_a} = 0
\]

(2.42)
or equivalently

\[
\frac{\hat{p}_- \hat{W}_\beta - \hat{q}_- \hat{W}_\beta}{\hat{W}_\beta} + \frac{\hat{q}_- \hat{W}_\beta - \hat{l}_- \hat{W}_\beta}{\hat{W}_\beta} + \frac{\hat{l}_- \hat{W}_\beta - \hat{p}_- \hat{W}_\beta}{\hat{W}_\beta} = 0
\]

(2.43)

and both these two equations can be identified with the lattice modified KP equation.

Taking \( \alpha = - \delta \), and using the fact that \( \hat{p}_{-3} = 0 \), (2.42) simplifies to

\[
\frac{\hat{q}_{-3} \hat{V}_{-3} - \hat{l}_{-3} \hat{V}_{-3}}{\hat{V}_{-3}} + \frac{\hat{l}_{-3} \hat{V}_{-3} - \hat{q}_{-3} \hat{V}_{-3}}{\hat{V}_{-3}} = 0.
\]

(2.44)

Similarly, taking \( \beta = \delta \), (2.43) becomes

\[
\frac{\hat{q}_{-3} \hat{W}_{-3} - \hat{l}_{-3} \hat{W}_{-3}}{\hat{W}_{-3}} + \frac{\hat{l}_{-3} \hat{W}_{-3} - \hat{q}_{-3} \hat{W}_{-3}}{\hat{W}_{-3}} = 0
\]

(2.45)

Since the choice of parameters leading to (2.44) and (2.45) breaks the covariance of the lattice system (as the choice favours one of the lattice directions) we refer to these two equations as the asymmetric modified KP equations.

Furthermore, for \( \alpha = - \delta \), we find from (2.41) that

\[
\hat{w} - \hat{\omega} = - \hat{q}_{-3} \hat{V}_{-3}/\hat{V}_{-3},
\]

(2.46a)

and combining these two relations, by eliminating the \( \hat{V}_s \)’s, yields

\[
(\hat{w} - \hat{\omega})(\hat{w} - \hat{\omega}) = \hat{w}(\hat{w} - \hat{\omega}),
\]

(2.47)

which is the lattice KP equation.

Finally, using (2.23), we can write (2.46a) and (2.46b) in the forms

\[
\hat{w} - \hat{\omega} = - \hat{q}_{-3} \frac{\tau}{\tau},
\]

(2.48a)

while from (2.41b), setting \( \alpha = - \varepsilon \), we also have

\[
\hat{w} - \hat{\omega} = - \hat{l}_{-3} \frac{\tau}{\tau},
\]

(2.48b)

The combination of (2.48a), (2.48b), and (2.49) gives

\[
\hat{l}_{-3} \tau \tau - \hat{q}_{-3} \frac{\tau}{\tau} - \hat{q}_{-3} \frac{\tau}{\tau} = 0,
\]

(2.50)
which leads to Hirota’s DAGTE in the following form:

\[
\sigma(\lambda - \delta)\sigma(\varepsilon)\Theta + \sigma(\delta - \varepsilon)\sigma(\lambda)\Theta + \sigma(\varepsilon - \lambda)\sigma(\delta)\Theta = 0.
\]  
(2.51)

We note that in the case of rational soliton solutions, the summation of coefficients would add up to be zero, as in the “Hirota-Miwa equation” (in our understanding of the terminology). However, in the case of elliptic soliton solutions, this condition is no longer valid.

E. Schwarzian KP variables

To obtain elliptic soliton solutions for the Schwarzian KP equation (1.6), we consider the following quantity:

\[
S_{\alpha,\beta} := s^T [X_{\beta,\gamma}^{-1} C [1 + MC]^{-1} [X_{\alpha,\gamma}]^{-1} r
\]

\[
= 'u_{\beta} C [X_{\alpha,\gamma}]^{-1} r = s^T [X_{\beta,\gamma}]^{-1} C u_{\alpha}.
\]

(2.52)

and we set out to derive some basic relations for this object. To achieve that we start from the relation (2.30a) for \(u_{\alpha}\) and multiply it from the left by the row-vector \(s^T (X_{\beta,\gamma})^{-1} C\) leading to the following computation:

\[
s^T (X_{\beta,\gamma})^{-1} C (\chi_{\alpha,\beta} u_{\alpha} - \tilde{V}_{\alpha} u_{\alpha}) = s^T \Phi_{\alpha}(K') \Phi_{\beta}(K') e^{-q_{\beta} K'} u_{\alpha}
\]

\[
= \left( 1 + \zeta(\delta) - \zeta(\beta) + \zeta(\xi + \beta) - \zeta(\xi + K + \beta) \right) u_{\alpha} = (1 - \tilde{V}_{\alpha}) - \tilde{\chi}_{\alpha,\beta} \tilde{S}_{\alpha,\beta}.
\]

(2.53a)

from which we obtain the following relation:

\[
\tilde{V}_{\alpha} W_{\beta} = 1 - \tilde{\chi}_{\alpha,\beta} - \chi_{\alpha,\beta}.
\]

Similarly, we have for the other lattice shifts

\[
\tilde{V}_{\alpha} W_{\beta} = 1 - \tilde{\chi}_{\alpha,\beta} - \chi_{\alpha,\beta},
\]

(2.53b)

\[
W_{\beta} V_{\alpha} = 1 - \tilde{\chi}_{\alpha,\beta} - \chi_{\alpha,\beta}.
\]

(2.53c)

Alternatively, multiplying the relation (2.30b) for \(u_{\beta}\) by the column vector \(C [X_{\alpha,\gamma}]^{-1} r\) from the right, and performing a similar computation, we also arrive at Eq. (2.53a).

In order to get rid of the quantities \(\chi\) in (2.53a) and its counterparts, we next multiply the relation by a factor \(\Phi_{\alpha}(\xi)\Phi_{\beta}(\xi)\) and perform the following computation:

\[
(\Phi_{\alpha}(\xi)\tilde{V}_{\alpha} \Phi_{\beta}(\xi) W_{\beta}) = - \Phi_{\beta}(\xi) \Phi_{\alpha}(\xi) \Phi_{\alpha}(\delta) \tilde{S}_{\alpha,\beta} - \Phi_{\beta}(\xi) \Phi_{\alpha}(\xi) \Phi_{\alpha}(\delta) \tilde{S}_{\alpha,\beta}
\]

\[
+ \Phi_{\alpha}(\xi) \Phi_{\alpha}(\xi) \Phi_{\alpha}(\delta) = \Phi_{\alpha}(\xi) \Phi_{\beta}(\xi) (\tilde{S}_{\alpha,\beta} - \tilde{S}_{\alpha,\beta}).
\]

(2.54)

Introducing the new variable

\[
Z_{\alpha,\beta} = \Phi_{\alpha}(\xi) (1 - \chi_{\alpha,\beta} S_{\alpha,\beta}),
\]

(2.55)

we can write (2.54) in the form

\[
\tilde{V}_{\alpha} W_{\beta} = p_{\alpha} Z_{\alpha,\beta} - p_{\beta} \tilde{Z}_{\alpha,\beta}.
\]

(2.56)

Similarly, we have

\[
\tilde{V}_{\alpha} W_{\beta} = q_{\alpha} Z_{\alpha,\beta} - q_{\beta} \tilde{Z}_{\alpha,\beta}.
\]

(2.57)
\[ \nabla_\alpha \mathcal{W}_\beta = l_\alpha Z_{\alpha,\beta} - l_{-\beta} \overline{Z}_{\alpha,\beta}. \]  

Using the identity
\[ \frac{(\tilde{V}_\alpha \tilde{W}_\beta)}{(V_\alpha W_\beta)} = \frac{(\tilde{V}_\alpha \tilde{W}_\beta)}{(V_\alpha W_\beta)} \frac{(\tilde{V}_\alpha \tilde{W}_\beta)}{(V_\alpha W_\beta)}, \]
we can derive the equation
\[ \frac{(p_\alpha \tilde{Z}_{\alpha,\beta} - p_{-\beta} \tilde{Z}_{\alpha,\beta})}{(q_\alpha \tilde{Z}_{\alpha,\beta} - q_{-\beta} \tilde{Z}_{\alpha,\beta})} = \frac{(l_\alpha \tilde{Z}_{\alpha,\beta} - l_{-\beta} \tilde{Z}_{\alpha,\beta})}{(l_\alpha \tilde{Z}_{\alpha,\beta} - l_{-\beta} \tilde{Z}_{\alpha,\beta})} \]
which is the “Schwarzian lattice KP equation,” first given the explicit form in Ref. 12.

Remark. We note that the elliptic soliton type solution, by the choice of the scaling factor in objects \( \mathcal{V}_\alpha \) and \( \mathcal{W}_\beta \) in (2.39) and in the parameters \( p_\alpha, q_\alpha, \) and \( l_\alpha, \) etc., are not truly elliptic functions (in terms of the background variable \( \xi \)) as they are only doubly quasi-periodic. This can be repaired by choosing the scaling factor to be \( \Psi_\xi(\alpha) \) instead of \( \Phi_\xi(\alpha) \). However, in that case the resulting equations become non-autonomous with coefficients depending on the variable \( \xi \) through exponential factors of the type \( \exp(-\eta_\alpha \alpha) \), etc. Of course, this only amounts to non-autonomous point symmetry, or choice of gauge, and is nothing but a matter of convention. Thus, for instance, the relation (2.38) could be recast in the form
\[ \hat{w} - \tilde{w} = \frac{1}{\tilde{V}_\alpha} (p_\alpha e^{-\eta_\alpha} \tilde{V}_\alpha - q_\alpha e^{-\eta_\alpha} \tilde{V}_\alpha) = \frac{1}{W_\beta} (p_{-\beta} e^{\eta_\beta} \tilde{W}_\beta - q_{-\beta} e^{\eta_\beta} \tilde{W}_\beta), \]
and similarly for the other relations involving the third lattice shift, and where now \( \mathcal{V}_\alpha = \Psi_\xi(\alpha) \mathcal{V}_\alpha \), \( \mathcal{W}_\beta = \Psi_\xi(\beta) \mathcal{W}_\beta \). Similarly, the relation (2.53a) would take the form
\[ \tilde{V}_\alpha \mathcal{W}_\beta = p_\alpha e^{-\eta_\alpha} \tilde{V}_\alpha - p_{-\beta} e^{\eta_\beta} \tilde{V}_\alpha, \]
if we would define the variable \( Z_{\alpha,\beta} \) by
\[ Z_{\alpha,\beta} = \Psi_\xi(\alpha + \beta)(1 - \chi_{\alpha,\beta} S_{\alpha,\beta}). \]
The effect on the corresponding lattice equations can be easily deduced, leading to (mildly) non-autonomous versions of the lattice equations mentioned earlier, and we refrain here from giving the explicit formulas.

F. Hirota form of the elliptic N-soliton solution

In this section, we will derive some explicit formulas for the soliton solutions in terms of the \( \tau \)-function, which allow us to study their properties. Using the fact that the matrix \( M \) is actually a Cauchy matrix, the \( \tau \)-function can be explicitly computed by using the expansion
\[ \tau = \det (1 + MC) = 1 + \sum_{i=1}^{N} \left| B_{i,i} \right| + \sum_{i<j} B_{i,j} B_{j,i} + \cdots + \det(B), \]
where \( B = MC \).

Lemma 2.3. Cauchy-Binet formula: For an arbitrary \( N \times M \) matrix \( A \) and \( M \times N \) matrix \( B \) we have the following formula for the \( N \times N \) determinant of the product:\textsuperscript{14}
\[ \det_{N \times N} (AB) = \begin{cases} 0 & \text{if } M < N \\ \det(A) \det(B) & \text{if } M = N \\ \sum_{1 \leq l_1 \leq \cdots \leq l_N \leq M} \det(A_{(l_1, \ldots, l_N)}) \det(B_{(l_1, \ldots, l_N)}) & \text{if } M > N \end{cases} \]
in which $A_{(1,\ldots,N)|(l_1,\ldots,l_N)}$ denotes the matrix obtained by selecting the $l_1,\ldots,l_N$ columns from the matrix $A$ and $B_{(l_1,\ldots,l_N)|(1,\ldots,N)}$ is the matrix obtained by selecting the $l_1,\ldots,l_N$ rows from the matrix $B$.

Using the Cauchy-Binet formula, we may express the $\tau-$function in the form

$$
\tau = 1 + \sum_{i=1}^{N} \left( \sum_{l=1}^{N} M_{i,l} C_{li} \right) + \sum_{i<j} \left( \sum_{l_1<l_2} \begin{vmatrix} M_{i,l_1} & M_{i,l_2} \\ M_{j,l_1} & M_{j,l_2} \end{vmatrix} \begin{vmatrix} C_{il_1} & C_{il_2} \end{vmatrix} \begin{vmatrix} C_{jl_1} & C_{jl_2} \end{vmatrix} \right) + \ldots + \sum_{l_1<l_2<\ldots<l_N} \det M_{(1,\ldots,N)|(l_1,l_2,\ldots,l_N)} \det C_{(l_1,l_2,\ldots,l_N)|(1,\ldots,N)}, \tag{2.61}
$$

where $C_{ij}$ are the entries of the matrix $C$ and $M_{(1,\ldots,N)|(l_1,l_2,\ldots,l_N)}$ denotes the matrix obtained by selecting the $(l_1, l_2, \ldots, l_N)$ columns from the matrix $M$ and $C_{(l_1,l_2,\ldots,l_N)|(1,\ldots,N)}$ is the matrix obtained by selecting the $(l_1, l_2, \ldots, l_N)$ rows from $C$.

At this stage we can use the Frobenius determinant formula,\(^{13}\) for elliptic Cauchy matrices to compute the minors in the expansion (2.61). The Frobenius formula gives us the evaluation

$$
\det \left( \rho_i \Phi_{\kappa_i + \kappa_i'}(\xi) v_j \right) = \left( \prod_{i} \rho_i v_i \right) \frac{\sigma(\xi + \sum (\kappa_i + \kappa_{i'}))}{\sigma(\xi)} \times \frac{\prod_{i<j} \sigma(\xi + \kappa_i + \kappa_{i'}) \sigma\left(\kappa_i + \kappa_{i'}\right)}{\prod_{i,j} \sigma(\kappa_i - \kappa_{i'})}. \tag{2.62}
$$

Introducing the notations

$$
e^{A_{ij}} \equiv \sigma(\kappa_i - \kappa_j), \quad e^{\theta_i} = \rho_i e^{-\xi \kappa_i},$$

$$e^{E_{ij}} \equiv \sigma(\kappa_i' - \kappa_{i'}), \quad e^{\nu_i} = \nu_i e^{-\xi \kappa_{i'}},$$

the Hirota formula for the $\tau-$function thus takes the form

$$
\tau = 1 + \sum_{i=1}^{N} e^{\theta_i} \sum_{l=1}^{N} e^{\theta_{1\ldots l}} C_{l}^{(1\times 1)} \left| \frac{\sigma(\xi + \kappa_i + \kappa_{i'})}{\sigma(\xi) \sigma(\kappa_i + \kappa_{i'})} \right| + \sum_{i<j} e^{\theta_i + \theta_j + A_{ij}} \sum_{l_1<l_2} e^{\theta_1 + \theta_2 + E_{12}} C_{l_1l_2}^{(2\times 2)} \left| \frac{\sigma(\xi + \kappa_i + \kappa_j + \kappa_{i'} + \kappa_{i'})}{\sigma(\xi) \prod_{j=1}^{2} \sigma(\kappa_i + \kappa_{i'})} \right| + \ldots + \sum_{l_1<l_2<\ldots<l_N} \left| C_{l_1l_2\ldots l_N}^{(N\times N)} \right| \left| \frac{\sigma(\xi) \prod_{j=1}^{N} \sigma(\kappa_i + \kappa_{i'}) \sigma(\kappa_j + \kappa_{i'}) \sigma(\kappa_{i'} + \kappa_{i''})}{\sigma(\xi) \prod_{j=1}^{N} \sigma(\kappa_i + \kappa_{i'})} \right| + \ldots, \tag{2.63}
$$

where $C_{l_1l_2\ldots l_N}^{(N\times N)}$ is the determinant of the $N \times N$ matrix of the selected entries of $C$.

The dependence of $\tau$ in (2.63) on the discrete dynamical variables $n, m, h$ enters only through the $\theta_i$ and $\eta_i$, which in turn depend on $n, m, h$ via the plane wave factors (2.8a) and (2.8b).

We collect now the solutions of the various lattice KP equations expressed in terms of the $\tau-$function (2.63). This requires one further extension, namely, the inclusion of plane wave factors

\[^{13}\text{Frobenius determinant formula (Ref. 12).}\]
associated with the parameters $\alpha$ and $\beta$ considered as lattice parameters in their own right (i.e., treating these parameters on the same footing as the parameters $\delta$, $\epsilon$, and $\lambda$). The multidimensional consistency allows us to make such an extension without this affecting in any way the relations and the form of the equations thus far obtained. On the level of the plane wave-factors, this amounts to the inclusion of factors

$$\rho^0(\kappa) = \left(e^{-\xi(\alpha)_{\kappa}} a_{\kappa}\right)^{N_a} \left(\Psi_{\beta}(\kappa_{\kappa}) b_{\kappa}\right)^{N_b} \rho^{00}(\kappa), \tag{2.64a}$$

$$v^0(\kappa') = \left(e^{\xi(\alpha)_{\kappa'}} a_{\kappa'}\right)^{-N_a} \left(\Psi_{\beta}(\kappa'_{\kappa}) b_{\kappa'}\right)^{-N_b} v^{00}(\kappa), \tag{2.64b}$$

where $a_{\kappa} := \Psi_{-\alpha}(\kappa)$, $b_{\kappa} := \Psi_{\beta}(\kappa)$ similar to (2.7) and where $N_a$, $N_b \in \mathbb{Z}$ denote associated discrete lattice variables, in the initial conditions $\rho^0(\kappa)$ and $v^0(\kappa)$ of (2.8), as well as the inclusion of terms $\xi_0 = \xi_{00} - N_a\alpha + N_b\beta$ in the initial condition $\xi_0$ of the auxiliary variable $\xi$ of (2.6).

Implementing these extensions in the formula (2.63) we can now express all solutions of the lattice KP systems in terms of the $\tau$-function and its shifts $T_{\alpha}\tau$ and $T_{\beta}\tau$ in the discrete variables $N_a$, $N_b$ (thus $T_{-\alpha}\tau := \tau|_{N_a \rightarrow N_a+1}$ and $T_{\beta}\tau := \tau|_{N_b \rightarrow N_b+1}$). These expressions are

$$V_{\alpha} = \Phi_{\alpha}(\xi) \left(\frac{T_{-\alpha}\tau}{\tau}\right), \quad V_{\beta} = \Phi_{\beta}(\xi) \left(\frac{T_{\beta}\tau}{\tau}\right), \quad Z_{\alpha,\beta} = \Phi_{\alpha+\beta}(\xi) \left(\frac{T_{-\alpha}T_{\beta}\tau}{\tau}\right), \tag{2.65}$$

whereas the lattice KP variable $w$ is related to the $\tau$-function through Eqs. (2.48).

### III. REDUCTION TO THE LATTICE KdV EQUATIONS

Dimensional reductions of the lattice KP systems can be obtained by imposing certain symmetry conditions. This process could be achieved by imposing the reversal symmetry condition $T_{-\delta} \circ T_{\delta} u = u$, and similarly for all lattice directions. Here, as in Subsection 2.1F, the $T_\delta$ denotes the shift operator in the variable associated with the lattice parameter $\delta$, i.e., the operator acting by $T_\delta := u|_{n \rightarrow n+1}$. Note that so far we have not implied anywhere that the shift operator $T_{-\delta}$ coincides with the inverse of the shift operator $T_{\delta}$, and in fact this is not true for general solutions of the KP lattice equations: the imposition of this condition, in fact, provokes a dimensional reduction of the equation, as we shall now demonstrate.

We now consider the plane-wave factors shifted in the “$\sim$” direction

$$\tilde{\rho} = \Phi_{\delta}(\kappa) \rho, \tag{3.1}$$

$$\tilde{v} = \Phi^{-1}_{\delta}(\kappa') v. \tag{3.2}$$

We find that

$$T_{-\delta} \circ T_{\delta} (\rho \nu) = \frac{\varphi(\kappa) - \varphi(\delta)}{\varphi(\kappa') - \varphi(\delta)} \rho \nu, \tag{3.3}$$

from which we conclude that if we set $\kappa = \kappa'$ we have $T_{-\delta} \circ T_{\delta} (\rho \nu) = \rho \nu$, i.e., in that case the action of $T_{-\delta}$ can be identified with the action of the inverse of the shift operator $T_\delta$. To do this on the soliton solutions, we have to identify the rapidity $\kappa_j$ with the rapidity parameters $\kappa'_j$ and hence we should set $N = N'$, implying that all the Cauchy matrices are square in this case and generically the constant matrix $C$ can be absorbed in the initial values of the vectors $r$ or $s$.

Let us consider what this reduction implies for the lattice KP (2.47) equation written in the following form:

$$\hat{\bar{w}} - \hat{\bar{w}} = \hat{\bar{w}} - \hat{\bar{w}} = \hat{\bar{w}} - \hat{\bar{w}}. \tag{3.4}$$

(It can easily be checked that the two equality signs in (3.4) do not yield two separate equations, but one is a trivial consequence of the other.) Setting $\lambda = -\delta$, we have $T_\delta \circ T_{\delta} w = \bar{\bar{w}} = w \Rightarrow \bar{w} = \hat{w}$, where $\hat{w} = w(n-1, m)$, yielding

$$(\hat{\bar{w}} - \hat{\bar{w}})(w - \hat{\bar{w}}) = T_\delta \left[ (\hat{\bar{w}} - \hat{\bar{w}})(w - \hat{\bar{w}}) \right]. \tag{3.5}$$
Similarly, setting $\lambda = -\varepsilon$, we have $T_{\delta} \circ T_{\varepsilon} w = \tilde{w} = \hat{w} = w(n, m - 1)$, yielding
\[(\hat{w} - \tilde{w})(w - \hat{w}) = T_{\delta} \left[(\hat{w} - \tilde{w})(w - \hat{w})\right]. \tag{3.6}\]
This implies that the product on the right-hand sides of these relations is conserved in all directions, and then constant
\[(\hat{w} - \tilde{w})(w - \hat{w}) = \text{constant}, \tag{3.7}\]
which is actually the lattice potential KdV equation (or the H1 equation in the terminology of Ref. 1).

From (2.41b), if we set $\lambda = -\delta$ we find that
\[\hat{\phi} - \tilde{\phi} = 1 \hat{\Gamma}_{\alpha} + \hat{\phi} \tilde{\phi} = \hat{\phi} \tilde{\phi} \tag{3.8}\]
Combining this equation with (2.41c) and using the lattice KdV, we obtain
\[\hat{\phi} \tilde{\phi} = \hat{\phi} \tilde{\phi} \tag{3.9}\]
which is the “lattice modified KdV” equation.

From (2.50), if we set $\lambda = -\delta$ we get
\[\hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau \tag{3.10}\]
and if we set $\lambda = -\varepsilon$, we get
\[\hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau \tag{3.11}\]
Here, (3.10) and (3.11) are the Hirota bilinear forms for the lattice KdV equations.

From (2.58), if we set $\lambda = -\delta$ we get
\[\hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau \tag{3.12}\]
and if we set $\lambda = -\varepsilon$ we get
\[\hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau \tag{3.13}\]
Using the identity
\[\hat{\phi} \tilde{\phi} \tau = \hat{\phi} \tilde{\phi} \tau \tag{3.14}\]
we obtain the equation for $Z_{\beta,\alpha}$,
\[\frac{p_{\alpha} Z_{\beta,\alpha} - q_{-\beta} Z_{\beta,\alpha}}{q_{\alpha} Z_{\beta,\alpha} - q_{-\beta} Z_{\beta,\alpha}} = \frac{q_{-\beta} Z_{\beta,\alpha} - q_{\alpha} Z_{\beta,\alpha}}{p_{-\beta} Z_{\beta,\alpha} - p_{\alpha} Z_{\beta,\alpha}}, \tag{3.15}\]
which is the “lattice Schwarzian KdV” equation.

The dimensional reductions described here to lattice equations of the KdV class are obtained on the basis of the explicit form for the solutions described in Secs. II A–II F. We would like to mention at this point that in the recent paper,5 it was shown how to embed all equations in the ABS list in the Schwarzian KP equation (1.6), through the canonical definition of a “Schwarzian variable” associated with each member of the list.

IV. CONTINUUM LIMITS

In Sec. II, we have derived the fully discrete KP equations together with their elliptic $(N, N')$-soliton solutions (associated with an $N \times N'$ elliptic Cauchy matrix). From these fully discrete equations, through appropriate continuum limits, we can obtain various semi-discrete KP type equations, i.e., differential-difference equations both with either two discrete and one continuous.
independent variable or with two continuous and one discrete independent variable. Those limits of the lattice KP systems were already studied in a different context in Refs. 28, as well as in Refs. 8, 34, and 35 where the derivation of whole hierarchies of equations arising from the lattice KP equations were systematically studied. Most of these limits were designed having in mind rational soliton solutions (or more generally, solutions obtainable through the direct linearization scheme). Here we will consider continuum limit on the level of the elliptic soliton solution, i.e., establish how the limits work through on the level of those solutions. These limits will be motivated by how they affect the plane-wave factors, but since those quantities appear in the solutions of the whole class of equations, the same limits will apply to all the equations in the class. Hence, we will illustrate the details only for the lattice KP (2.47) equation, as this suffices to establish how these limits subsequently can be applied to the successive equations, and we will present only the final results for the other lattice KP equations. In contrast with the semi-discrete limits, the full continuous limit is much more subtle and it is not clear at this stage whether these limits persist also on the level of the elliptic solutions.

To treat the lattice KP equation, we find that it is more convenient to express the lattice KP equation (2.47) in terms of the variable \( w_0 \) of (2.34a), namely, in the following form:

\[
\begin{align*}
\frac{\left(\hat{w}_0 - \hat{w}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda)\right)}{(\hat{w}_0 - \hat{w}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda))} = \frac{\left(\hat{\nu}_0 - \hat{\nu}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda)\right)}{(\hat{w}_0 - \hat{w}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda))},
\end{align*}
\]  

(4.1)

where \( w_0 = s^T C (1 + MC)^{-1} r \).

**First continuum limit (The skew continuum limit).** We start to perform the continuum limit by making change of discrete variables \((n, m) \rightarrow (N = n + m, m)\) and then taking the limit

\[
\epsilon - \delta = \eta \rightarrow 0, \ m \rightarrow \infty, \ n \rightarrow -\infty, \ \text{s.t.} \ m\eta = \tau \ \text{and} \ N \ \text{fixed}.
\]

The plane wave functions (2.8a) and (2.8b) become

\[
\rho(\kappa) = \left(e^{-\zeta(\delta)\kappa} P_{-\kappa}\right)^N \left(e^{-\zeta(\lambda)\kappa} L_{-\kappa}\right)^h e^{\zeta(\xi)\kappa + \zeta(\delta - \kappa)\tau} \rho_{0, 0, 0}(\kappa),
\]  

(4.2)

\[
v(\kappa') = \left(e^{\zeta(\delta)\kappa'} P_{\kappa'}\right)^{-N} \left(e^{\zeta(\lambda)\kappa'} L_{\kappa'}\right)^{-h} e^{\zeta(\xi)\kappa' + \zeta(\delta + \kappa')\tau} \rho_{0, 0, 0}(\kappa'),
\]  

(4.3)

and \( w_0(n, m, h) \rightarrow w_0(N, \tau, h) \). Then \( w_0(N, \tau, h) \) satisfies the differential-difference equation

\[
\begin{align*}
\frac{\partial w_0}{\partial \tau} + \varphi(\xi + \delta + \lambda) - \varphi(\delta) &= \frac{\hat{\nu}_0 - \hat{\nu}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda)}{\hat{w}_0 - \hat{w}_0 + \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda) - \zeta(\xi + \delta + \lambda)}.
\end{align*}
\]  

(4.4)

**Second continuum limit.** Next, we perform the limit on the discrete variable \( h \),

\[
\lambda \rightarrow 0, \ \ h \rightarrow \infty, \ \ \text{s.t.} \ \gamma = \lambda h \ \text{fixed}.
\]

The plane wave functions (4.2) and (4.3) become

\[
\rho(\kappa) = \left(e^{-\zeta(\delta)\kappa} P_{-\kappa}\right)^N e^{\zeta(\xi)\kappa + (\xi + \delta - \kappa)\tau} \rho_{0, 0, 0}(\kappa),
\]  

(4.5a)

\[
v(\kappa') = \left(e^{\zeta(\delta)\kappa'} P_{\kappa'}\right)^{-N} e^{\zeta(\xi)\kappa' - (\xi + \delta + \kappa')\tau} \rho_{0, 0, 0}(\kappa'),
\]  

(4.5b)

and \( w_0(N, \tau, h) \rightarrow w_0(N, \tau, \gamma) \). Then \( w_0(N, \tau, \gamma) \) now satisfies the partial differential-difference equation

\[
\begin{align*}
- \left(\frac{\partial w_0}{\partial \gamma} + \varphi'(\xi)\right) = \left(\frac{\partial w_0}{\partial \tau} + \varphi(\xi) - \varphi(\delta)\right) (\hat{w}_0 - 2\hat{w}_0 + \hat{w}_0 + 2\zeta(\xi) - \zeta(\xi + \delta) - \zeta(\xi + \delta)).
\end{align*}
\]  

(4.6)
The full continuum limit. This limit can be obtained by investigating the limiting behaviour of the plane wave function. From (4.5a), we have

\[
(e^{-\zeta(\delta \kappa) \rho} \delta_{\omega})^N e^{\zeta(\delta \kappa) \rho + \zeta(\delta \kappa) \sigma - \zeta(\delta \kappa) \gamma} \rightarrow \exp \left( N \ln |\sigma(\kappa - \delta)| - \ln |\sigma(\kappa)| + \zeta(\delta - \kappa) \tau - \zeta(\kappa) \gamma + \zeta(\delta \kappa) \right) \\
\rightarrow \exp \left( N \left( -\delta \zeta(\kappa) - \frac{\delta^2}{2} \phi(\kappa) + \frac{\delta^3}{6} \phi'(\kappa) + \ldots \right) - \zeta(\kappa) \gamma \right) \\
+ \tau \left( -\zeta(\kappa) - \delta \phi(\kappa) - \frac{\delta^2}{2} \phi'(\kappa) + \ldots \right) + \zeta(\delta \kappa) \right) \\
\rightarrow \exp \left( \zeta(\delta \kappa) + \zeta(\kappa) \tau + \zeta(\delta \kappa) \gamma + \phi(\kappa)' \tau + \ldots \right),
\]

(4.7)

where we define \( b = N \delta \) and

\[
x = -b - \tau - \gamma, \quad y = \frac{\delta b}{2} - \tau \delta, \quad \text{and} \quad t = \frac{\delta^2 b}{6} - \frac{\delta^2 \tau}{2}.
\]

(4.8)

Applying the Taylor expansions, we have

\[
\tilde{w}_0 = w_0 + \delta \frac{\partial w_0}{\partial b} + \frac{\delta^2}{2} \frac{\partial^2 w_0}{\partial b^2} + \frac{\delta^3}{6} \frac{\partial^3 w_0}{\partial b^3} + \frac{\delta^4}{24} \frac{\partial^4 w_0}{\partial b^4} + \ldots,
\]

(4.9)

where \( b = b_0 + N \delta \) and using the chain rule formulas, we have

\[
\frac{\partial w_0}{\partial \gamma} = -\frac{\partial w_0}{\partial x},
\]

(4.10a)

\[
\frac{\partial w_0}{\partial \tau} = -\frac{\partial w_0}{\partial x} - \delta \frac{\partial w_0}{\partial y} - \frac{\delta^2}{2} \frac{\partial w_0}{\partial t},
\]

(4.10b)

\[
\frac{\partial^2 w_0}{\partial y \partial \tau} = \frac{\partial^2 w_0}{\partial x^2} + \frac{\delta}{\partial x} \frac{\partial^2 w_0}{\partial y} + \frac{\delta^2}{2} \frac{\partial^2 w_0}{\partial x \partial t}.
\]

(4.10c)

Using (4.9) and (4.10), we recover the “potential KP” equation in order \( O(\delta^2) \),

\[
[w_0]_1 = 6[w_0]_1[w_0]_3 + \frac{3}{2} [w_0]_{3y} + \frac{1}{2} [w_0]_{3xx},
\]

(4.11)

where \( w_0(N, \tau, x, y) \rightarrow w_0(x, y, t) \).

We would like to finish the section by listing similar limits as we have performed on the lattice KP case, to other lattice equations that we considered in this paper, in particular, the case of the lattice modified and lattice Schwarzian KP equations.

The discrete modified KP equation:

\[
\frac{\partial}{\partial x} \tilde{V}_a - \frac{\partial}{\partial y} \tilde{V}_a - \frac{\partial}{\partial t} \tilde{V}_a + \frac{\partial}{\partial y} \tilde{V}_a - \frac{\partial}{\partial t} \tilde{V}_a = 0,
\]

(4.12)

where \( V_a = \Phi_v(\xi)[1 - s^T C(1 + MC)^{-1} (x - a, \kappa)^{-1} r] \). Applying in the same way as above, we obtain the differential-difference equation

\[
\tilde{V}_a \nabla_a [\sigma(\delta) \sigma(\lambda) \sigma(\alpha + \delta)(\xi(\delta) - \xi(\alpha + \delta))] \\
+ \nabla_a \frac{\partial}{\partial \tau} [\sigma(\delta) \sigma(\lambda) \sigma(\alpha + \delta) \nabla_a - \sigma^2(\delta) \sigma(\alpha + \lambda) \nabla_a] \\
+ \nabla_a \frac{\partial}{\partial \tau} \nabla_a [\sigma(\delta) \sigma(\lambda) \sigma(\alpha + \delta)(\xi(\delta) - \xi(\alpha + \delta))] \\
+ \nabla_a \frac{\partial}{\partial \tau} [\sigma^2(\delta) \sigma(\alpha + \lambda) \nabla_a - \sigma(\delta) \sigma(\lambda) \sigma(\alpha + \delta)] = 0,
\]

(4.13)

where \( \nabla_a(n, m, h) \rightarrow \nabla_a(N, \tau, h) \).
Taking the subsequent limit on the discrete variable $h$, we obtain the partial differential-difference equation
\[
(\tilde{V}_\alpha V_\alpha V_\alpha - V_\alpha^3)\left[\sigma(\delta)\sigma(\alpha)(\xi(\alpha) - \xi(\delta))\right] + \tilde{V}_\alpha V_\alpha \frac{\partial V_\alpha}{\partial \varphi} \sigma(\delta)\sigma(\alpha)\xi(\delta)
\]
\[
+ \left(\tilde{V}_\alpha V_\alpha \frac{\partial^2 V_\alpha}{\partial \varphi \partial \tau} - \frac{\partial V_\alpha}{\partial \varphi} \frac{\partial V_\alpha}{\partial \tau}\right)\sigma(\delta)\sigma(\alpha - \delta) + \left(V_\alpha^2 - \tilde{V}_\alpha V_\alpha\right) \frac{\partial V_\alpha}{\partial \tau} \sigma(\alpha)\sigma(\delta) = 0,
\]
\[(4.14)\]
where $V_\alpha(N, \tau, h) \rightarrow V_\alpha(N, \tau, \gamma)$.

The lattice Schwarzian KP equation:
\[
\begin{align*}
\left(1 - \tilde{\chi}_{\alpha,\beta} \tilde{S}_{\beta,\alpha} - \tilde{\chi}_{\beta,\alpha} \tilde{S}_{\beta,\alpha}\right)
&\left(1 - \tilde{\chi}_{\alpha,\beta} \tilde{S}_{\beta,\alpha} - \tilde{\chi}_{\beta,\alpha} \tilde{S}_{\beta,\alpha}\right)
\left(1 - \tilde{\chi}_{\alpha,\epsilon} \tilde{S}_{\epsilon,\alpha} - \tilde{\chi}_{\epsilon,\alpha} \tilde{S}_{\epsilon,\alpha}\right)
\left(1 - \tilde{\chi}_{\alpha,\delta} \tilde{S}_{\delta,\alpha} - \tilde{\chi}_{\delta,\alpha} \tilde{S}_{\delta,\alpha}\right)
\end{align*}
\]
\[
= 1,
\]
\[(4.15)\]
where $S_{\alpha,\beta} = s^T \left[X_{\beta,\kappa}\right]^{-1} C \left[1 + MC\right]^{-1} [X_{\alpha,\kappa}]^{-1} r$. The skew limit in this case yields the following differential-difference equation:
\[
\begin{align*}
&\left(\varphi(\tilde{\xi} + \alpha) - \varphi(\tilde{\xi})\right)\tilde{S}_{\beta,\alpha} + \left(\varphi(\tilde{\xi} + \beta) - \varphi(\tilde{\xi})\right)\tilde{S}_{\beta,\alpha} + \tilde{\chi}_{\alpha,\beta} \frac{\partial \tilde{S}_{\beta,\alpha}}{\partial \tau} + \tilde{\chi}_{\beta,\alpha} \frac{\partial \tilde{S}_{\beta,\alpha}}{\partial \tau}
\end{align*}
\]
\[
= 0,
\]
\[(4.16)\]
where $S_{\alpha,\beta}(n, m, h) \rightarrow S_{\alpha,\beta}(N, \tau, h)$.

Finally, from the subsequent limit on the discrete variable $h$, we obtain the partial differential-difference equation
\[
\begin{align*}
&\left[\varphi(\tilde{\xi} + \alpha) - \varphi(\tilde{\xi})\right]S_{\beta,\alpha} + \left[\chi_{\alpha} + \chi_{\beta}\right] \frac{\partial S_{\beta,\alpha}}{\partial \tau} - \frac{\partial^2 S_{\beta,\alpha}}{\partial \varphi \partial \tau}
\end{align*}
\]
\[
= 0,
\]
\[(4.17)\]
where $\chi_{\alpha} = \zeta(\alpha) + \zeta(\xi) - \zeta(\xi + \alpha)$ and $S_{\alpha,\beta}(N, \tau, h) \rightarrow S_{\alpha,\beta}(N, \tau, \gamma)$.

V. SUMMARY

In this paper, we established the explicit form of a class of elliptic soliton solutions for all the lattice KP equations, based on a construction using elliptic Cauchy matrices denoted by $M$. Furthermore, the construction exhibits the various relations between these lattice equations, as well as allows for the construction of the corresponding Lax pairs (which follow straightforwardly from the relations in Lemma 2.2, but we have refrained from working out the explicit Lax pairs in the present paper). The explicit form for the corresponding $\tau$-function depends crucially on the coefficient matrix $C$, which opens the way to classify the various different lattice soliton behaviours according
to the Schubert decompositions of the corresponding Grassmannians, following similar work in the continuous case by Chakravarty and Kodama. Several reductions were considered: (i) dimensional reduction to KdV lattice systems, and (ii) continuum limits to the semidiscrete and fully continuous KP equations. For all these equations the corresponding elliptic soliton solutions are derived in parallel. Furthermore, the result in (2.63) can be simplified to the cases of trigonometric/hyperbolic by taking: \( \sigma(x) \rightarrow \sin(x) \) or \( \sigma(x) \rightarrow \sinh(x) \) and for the rational case we have \( \sigma(x) \rightarrow x \).

There are well-established connections between soliton solution of integrable partial differential equations (PDE) and the integrable many body systems. This can be made most explicit in the rational and trigonometric/hyperbolic cases. The elliptic case of this correspondence is more difficult to establish, but we expect that the elliptic solitons which we have studied here can be connected to elliptic case of the discrete-time Ruijsenaars model constructed in Ref. 29. In Refs. 36 and 37, an explicit connection between the rational discrete-time RS system and the KP lattice was established using the solution structure of the former model.

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APPENDIX: QUADRATIC EIGENFUNCTION EXPANSIONS

In this section, we derive the discrete analogue of the quadratic eigenfunction expansions associated with the soliton-type solutions studied in the main text of the paper. These can be derived in a very general fashion, without specifying any particular (discrete or continuous) independent variables.

Let \( \partial \) denote the derivative with respect to any independent variable on which solely the \( \rho_i \) and all the \( \nu_j \) depend, but not any of the other ingredients in the elliptic soliton solutions. Hence \( \partial \) only acts on the \( \rho_i \) and \( \nu_j \). Assuming that the latter can be written in the following form:

\[
\begin{align*}
\rho_i &= R_i e, \\
\nu_j &= S_j e,
\end{align*}
\]

we can use the following relation for the Cauchy matrix:

\[
\partial M = (\partial R) R^{-1} M + M S^{-1} (\partial S)
\]

to derive the following expression for the action of \( \partial \) on the main variable \( S_{\alpha, \beta} \):

\[
\partial S_{\alpha, \beta} = \tilde{u}_\beta \left[ S^{-1} \partial S C + C \partial R R^{-1} \right] u_\alpha.
\]

This expression, in fact, is the analogue of the square eigenfunction expansion (in the sense of the seminal paper by Deift et al. cf. Ref. 10, who used square eigenfunction expansions in the context of the inverse scattering for two-dimensional continuous soliton equations) of the elliptic soliton solution of a continuous KP equation, e.g., by choosing \( \partial \) to represent the partial derivative with respect to one of the continuous independent variables of the KP equation.

In the discrete case, one can derive a similar equation to (A1). Let “\( \tilde{~} \)” denote here the shift with respect to any discrete variable on which solely the \( \rho_i \) and \( \nu_j \) depend. Using the ingredients of the previous case, one can derive the formula

\[
\tilde{S}_{\alpha, \beta} - S_{\alpha, \beta} = \tilde{u}_\beta \left[ C \tilde{R} R^{-1} - \tilde{S} S^{-1} C \right] u_\alpha.
\]

which constitutes the discrete analogue to (A1). Note that the shift “\( \tilde{~} \)” may also represent a composite shift, or a combined shift with respect to multiple variables, the derivation of (A2) only uses the fact that the shift distributes over products of functions of the discrete variable by \( f g = \tilde{f} \tilde{g} \) and that the discrete variable enters the solutions via the \( \rho_i \) and the \( \nu_j \). There is, however, an important difference between the formulas (A1) and (A2), namely, that the left-hand side of the latter involves
also shifts of the eigenfunctions. We expect that these formulas may prove useful in the derivation of conservation laws and recursion operators for the discrete equations considered in this paper.