A robust optimization approach to reduce the bullwhip effect of supply chains with vendor order placement lead time delays in an uncertain environment

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Abstract
Supply chain management is important for companies and organizations to improve their business and enhance competitiveness in the global marketplace. The bullwhip effect problem of supply chain systems with vendor order placement lead time delays in an uncertain environment is addressed in this paper. Among the numerous causes of bullwhip effect, we focus on uncertainties with respect to demand, production process, supply chain structure, inventory policy implementation and especially vendor order placement lead time delays. Minimizing the negative effect of these uncertainties in inducing bullwhip effect creates a need for developing dynamical inventory policy that increases responsiveness to demand and decreases volatility in inventory replenishment. First, a dynamic model of supply chain with above uncertainties is developed. Then, a novel uncertainty-dependent robust inventory control method using inventory position information is proposed. Additionally, the maximum allowable vendor order placement lead time delay that ensures the stability of supply chains and the suppression of bullwhip effect under the proposed inventory control policy is explored and measured. We find that vendor order placement lead time delays do play important role in supply chain dynamics and contribute to its turbulence and volatility. The effectiveness and flexibility of proposed method is verified through simulation study.

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1. Introduction

As a typical network system which mainly consists of suppliers, manufacturers, distributors and retailers, supply chains (SC) refer to all the activities associated with the transformation of raw materials, products and the distribution of finished products to customers. There are three important flows among SC entities, namely material flows, information flows and financial flows [1]. As a general topic in the process of globalization with intensifying global competition, supply chain management (SCM) has received considerable attention from both SC practitioners and academics. The underlying motivation of the need for efficient SCM lies in reducing multifarious costs (e.g., the rising cost of manufacturing, inventory and transportation), improving service (e.g., improving stock availability and asset utilization), satisfying customer demands (e.g., accelerating the speed of response, improving after-sale service) and finally bringing competitive advantages to companies [2–5]. An effective SCM will integrate internal and external SC resources, optimize resource allocation and production processes, and finally improve supply chain efficiency.
An undesirable phenomenon, also known as "the bullwhip effect" (the observed variation amplification of order quantities moving up a supply chain from end-customers to raw material suppliers), is common in real world and receives much attention [6]. Having its root in J. Forrester's Industrial Dynamics [7], the bullwhip effect (or whiplash effect) has been recognized as one of the most problematic issues to be faced in SCM [8,9]. It not only indicates the vulnerability of supply chain system to external interference, especially the demand fluctuations, but also imposes great human, financial and resource burden on supply chain participants. Numerous researches have been carried out to study its causes and countermeasures. Lee et al. [10] studied the effect of demand processing, game rationalization, order lot sizing and price fluctuations on bullwhip effect from the perspective of information distortion. Through comparative analysis of a variety of traditional contract types, He et al. [5] pointed out that only a properly designed returns policy with sales rebate and penalty contract is able to achieve channel coordination and lead to a Pareto situation for SC members who face stochastic demands. A similar study can be found in [11]. Not only the direct factor of end demand fluctuations, other factors such as batching of orders, pricing policy, lead time delays, etc., have been identified as major causes of bullwhip effect [12,13].

These causes, on the one hand, are related to operational complications (or more precisely the various uncertainties in supply chain process), such as demand changes, price fluctuations, lead time variabilities and information sharing [14,15], and on the other hand, have to do with cognitive limitations on the part of supply chain participants, such as overreaction to backlogs and underlying tendency of underweighting supply lines [16,17]. In this paper, we study the bullwhip effect from the perspective of analytical rather than behavioral approaches and focus explicitly on uncertainties among its various causes.

So far there is no ideal practical method for avoiding the bullwhip effect in SC. Even some advanced SCM method may induce amplification of order information distortion and stock variations [18,19]. Nevertheless, some feasible methods have been proposed in SCM researches, and most of them focus on controlling the uncertainty causes of bullwhip effect. Seferlis and Gianello [20] developed a two-layered hierarchical decentralized inventory control policy which applies an autoregressive integrated moving average (ARIMA) forecasting model for demand prediction. Similar studies about the value of reducing information uncertainty in SCM can be found in [21,22]. Pishvae et al. [4] proposed a robust optimization model and relative robust counterpart for handling the inherent uncertainty of input data in a closed-loop supply chain network design problem. To the stochastic lead time problem, Abghinehchi and Farahani [43] developed a model for optimal suppliers. In addition, study carried out by Springer and Kim [24] showed that dynamical order pipeline provides better customer service and reduces supply chain volatility. Different analytical methods have been applied in these studies: dynamical programming algorithm [25,26], heuristic methodology [27,28], simulations [29,30], for instance.

In previous studies, most methods related to the reduction of bullwhip effect, however, are static – either they formulate the problem as a static deterministic problem where the supply chain object is modeled based on its average performance and steady-state conditions (supply chain structure, partner relationship, transaction costs, etc.) [31,32], or they are based on standard static inventory control policies where certain stationary demand patterns are relatively pre-established [33,34], such as the static re-order point and the static order-up-to-level policies. Considering the dynamic characteristics of SC in real-world situation, it is clear that these models and methods are insufficient for they neglect, from a dynamic perspective, the endogenous trigger mechanism of bullwhip effect and thus are not able to provide appropriate remedies for this problem. The goals of this paper is, first, to shed light on the endogenous mechanism and dynamic characteristics of bullwhip effect in an uncertain environment from the perspective of inventory dynamics, and second, to find possible countermeasures and develop effective strategies (more specifically, inventory control policies) to improve supply chain management.

Recently, many researchers have addressed the bullwhip effect problem from the perspective of control system [35–37]. Due to the resemblance of supply chain systems to dynamical control systems in engineering, methods in control theory are applied in SC studies, especially in analyzing SC dynamics and in designing SCM strategies for reducing bullwhip effect, such as the classical linear control theory [38,39], model predictive control [40,41] and so forth. Applying robust control theory, Boukas et al. [42] studied the asymptotic stability of an inventory-production system with uncertainty in processing time. They also extended their study by taking into consideration other uncertain system parameters. In their study (also in many other SC researches, e.g., [43,44]), they considered the inventory-production system as a continuous system. But many activities in SC, such as ordering and inventory replenishment, should be described as discrete events. In addition, given that uncertainties exist in nearly all aspects of supply chain systems, researches limited to only one or few sources of uncertainty would affect the accuracy of study and influence their guidance in practical application.

In this paper, we address the bullwhip effect control problem in context of a supply chain system that faces uncertainties with respect to demand, production process, supply chain structure, inventory policy implementation and vendor order placement lead time delays. Based on SC endogenous dynamics, we build a SC state transition model and focus on the effect of uncertain vendor order placement lead time delays on replenishment performance and on supply chain dynamics. In addition, we derive the sufficient linear matrix inequality conditions, corresponding to the proposed inventory control policy in this paper, for the suppression of bullwhip effect and the improvement of SC stability. The rest of this paper is organized as follows: In Section 2, we introduce the SC state transition model and its corresponding inventory control policy. In Section 3, we propose an optimization model for the maximum allowable vendor order placement lead time delay that ensures the stability of supply chain systems and the suppression of volatility in inventory replenishment. We extend our analysis in Section 4 through simulation studies. Finally, conclusions with future research directions are presented in Section 5.
2. Supply chain system model under uncertainty

A typical supply chain system includes several entities: customers, retailers, distributors, manufacturers and suppliers. Along with the product flows there are other two important flows among SC members - the information flows and the financial flows. In this section, we first describe the basic state transition model of SC system as well as its inventory control method which uses inventory position information. Then we extend the model to include uncertainties with respect to demand, production process, supply chain structure, inventory policy implementation and vendor order placement lead time delays.

2.1. Basic state transition model of supply chain systems

Supply chain activities transform natural resources, raw materials and components into finished products. Among SC members, manufacturers play a vital role in these activities, especially in the value-added links of such value chain. They allocate resources, schedule production and provide final products and services with added value to customers [45,46].

Notations used in the formulation of supply chain models in this paper are summarized as follows:

- \( d_{il}(t) \) order from site \( i \) to site \( l \) during time period \( t \)
- \( s_{il}(t) \) quantity of products shipped from site \( l \) to site \( i \) during time period \( t \)
- \( b_i(t) \) backlogged demand faced by site \( i \) during time period \( t \)
- \( x_i(t) \) inventory position of site \( i \) during time period \( t \), consists of raw material inventory \( x_{i1}(t) \) and finished goods inventory \( x_{i2}(t) \)
- \( m_i(t) \) amount of raw material used in production process by site \( i \) during time period \( t \)
- \( \delta x_i(t) \) inventory deviation from the normal inventory position of site \( i \) during time period \( t \)
- \( \delta d_{il}(t) \) order variation from site \( i \) to site \( l \) during time period \( t \)
- \( f_i \) the proportion of raw materials consumed in production in the total inventory of site \( i \)
- \( p_i \) the output of finished goods per unit of raw material input during an observation period, site \( i \)
- \( c_{ij} \) the proportional coefficient used in order fluctuation (from site \( j \)) prediction by site \( i \)

To describe the micro-dynamics of SC systems, we first consider a local supply chain shown in Fig. 1. During the time period \( t \), site \( i \) (e.g., a manufacturer) sends its order \( d_{il}(t) \) (e.g., raw material order or components order) to supplier site \( l \), and receives related materials or goods \( s_{il}(t) \) from supplier \( l \). Similarly, \( d_{ij}(t) \) and \( d_{ik}(t) \) represent orders received by site \( i \) from downstream sites \( j \) and \( k \), while \( s_{ij}(t) \) and \( s_{ik}(t) \) denote goods in shipment, respectively. Taking into consideration the possibility of inventory shortage, let \( b_i(t) \) denote the backlogged demand faced by site \( i \), we obtain:

\[
\begin{align*}
    x_i(t+1) &= x_i(t) + s_{il}(t) - (s_{il}(t) + s_{ik}(t)), \\
    s_{il}(t) + s_{ik}(t) &= \min \{ x_i(t), b_i(t) + d_{ij}(t) + d_{ik}(t) \}, \\
    b_i(t+1) &= b_i(t) + d_{ij}(t) + d_{ik}(t) - s_{ij}(t) - s_{ik}(t),
\end{align*}
\]

where the inventory position \( x_i(t) \) mainly consists of raw materials inventory and finished goods inventory. To describe the relationship between inventory level and production process, we introduce into our model the productivity parameter \( p_i \) - a measure of output from a production process, per unit of input, during an observation period. In addition, it is assumed that the production time of manufacturers, namely the time needed to move materials from raw materials, process them, and convert them into finished goods, is one period. When there is no inventory shortage, the material balance equations of site \( i \) at \( t \)th observation period satisfy:
\[ x_{i1}(t + 1) = x_{i1}(t) + d_{li}(t) - m_i(t), \]
\[ x_{i2}(t + 1) = x_{i2}(t) + p_i m_i(t) - (d_{ij}(t) + d_{ik}(t)), \]
\[ x_i(t + 1) = x_{i1}(t + 1) + x_{i2}(t + 1) = x_i(t) + (p_i - 1)m_i(t) + d_{il}(t) - (d_{ij}(t) + d_{ik}(t)). \]

According to parameter definitions in above notations, we have \( m_{i}(t) = f_{x_{i}}(t) \). To SC members, overstocking raw materials or finished goods will increase inventory costs, while understocking raw materials will disrupt production process and inventory shortage of finished goods might result in the loss of sales and customers. Generally, production rate is proportional to manufacturer’s inventory level or, in other words, the greater or faster the production, the higher the inventory level. Taking these into account, parameter \( f_{i} \) is defined as time-invariant constant.

When variations of end demands in SC exceed the range of initial stable state, bullwhip effect – oscillating demand magnified upstream – is likely to emerge from the broken balance. When this happens, inventory deviation dynamics of site \( i \) can be derived from Eq. (6):
\[ \delta x_{i}(t + 1) = \delta x_{i}(t) + (p_i - 1)f_{i}\delta x_{i}(t) + \delta d_{ij}(t) - (\delta d_{ij}(t) + \delta d_{ik}(t)). \]

From the view point of suppressing bullwhip effect, the control mechanism in our model can be expressed as follows: to each SC member (e.g., site \( i \)), select appropriate order compensation \((\delta d_{ij})\) for the “standard” order deviations so as to minimize inventory fluctuations \((\delta x_{i})\), also the measure of bullwhip effect in this paper. We assume that placing orders is synchronous and thus the variations of downstream orders are unknown to upstream members. This is also what SC members usually face in real-world situations. On the other hand, predicting downstream order variations is an important and indispensable part in order decision-making of upstream members. In this paper, we introduce the following forecasting method into our model to imitate the prediction mechanism in order decision-making of SC members. Take facility \( i \) as an example:
\[ \delta d_{ij}(t) = -c_{ij}\delta x_{i}(t), \]
where \( c_{ij} \) is the proportional coefficient. Negative sign in Eq. (8) indicates the opposite variation trend of inventory position and order changes. Careful analysis reveals two additional important implications of parameter \( c_{ij} \): First, it indicates the supply–demand relationship between different SC members; Second, the value of \( c_{ij} \) reflects the importance of facility \( i \) to facility \( j \), namely, the larger its value, the more important facility \( i \) is among suppliers of facility \( j \). To simplify notation, \( \delta x_{i}(t) \) and \( \delta d_{ij}(t) \) are denoted by \( x_{i}(t) \) and \( u_{i}(t) \) (the control parameter) in the rest of this paper, respectively. Then, Eq. (7) becomes:
\[ x_i(t + 1) = x_i(t) + (p_i - 1)f_i x_i(t) + u_i(t) + (c_i x_i(t) + c_{ii} x_i(t)). \]

On the one hand, Eq. (9) implies that, to suppress inventory variation of facility \( i \), inventory level of both its own and its direct downstream customers should be considered in the designing of control parameter \( u_{i}(t) \). On the other hand, reducing bullwhip effect should not be limited to certain facilities, but should cover all members in supply-demand network system. Thus, from Eq. (9), we derive the basic supply chain system state transition model below:
\[
\begin{align*}
X(t + 1) &= AX(t) + BU(t) + CX(t), \\
A &= I + \text{diag}\{(p_i - 1)f_i\}, \\
C &= [c_{ii}],
\end{align*}
\]
where \( A \) represents the productivity-related coefficients matrix, \( C \) is the proportional coefficients matrix, \( B \) and \( i \) are identity matrices with appropriate dimension. \( X(t) \) denotes the system state vector at time step \( t \). System inventory control variable is represented by vector \( U(t) \). Here, from the perspective of system optimization, the method of suppressing bullwhip effect can now be described as follows: given inventory status of analyzed SC system \( X(t) \), select appropriate order compensation for each SC member (components of system vector \( U(t) \)) so as to improve the stability of SC system.

### 2.2 Supply chain model with vendor order placement lead time delays in an uncertain environment

Based on state transition model (10), the related supply chain model with vendor order placement lead time delays in an uncertain environment can be defined as follows:
\[
\begin{align*}
X(t + 1) &= (A + \Delta A)X(t) + (B + \Delta B)U(t) + (C + \Delta C)X(t - \tau(t)), \\
A &= I + \text{diag}\{(p_i - 1)f_i\}, \\
C &= [c_{ii}], \\
0 &\leq \tau(t) \leq T.
\end{align*}
\]

Here uncertainties in supply chain system are represented by three newly introduced parameters: \( \Delta A \) indicates the uncertain change in production process, which usually results from the change of production technology or productivity; \( \Delta B \) denotes uncertainty in inventory policy implementation, such as technical and cultural factors that affect the implementation of designed inventory policy in practice; \( \Delta C \) denotes SC structure changes, for example, the change of buyer-supplier partnership and market expansion. In Eq. (11), \( \tau(t) \) is the time-delay parameter, which indicates the vendor order placement lead
time that it takes for an order to be received. In reality, the delayed orders, especially those resulting from the physical goods supply lead time, are usually predictable, controllable and explicitly considered in order policy designing. The vendor order placement lead time, on the other hand, is uncertain, mainly determined by decision-making process and appreciably affected by subjective factors of policy makers. While only accounting for a small part of the total supply lead time in supply chain, the vendor order placement lead time has greater unpredictable impact on the volatility of inventory system. Thus, this paper focuses on the vendor order placement lead time delay rather than the more general “lead time”. Considering the bullwhip effect control mechanism described at the end of Section 2.1, we propose an uncertainty-dependent robust inventory control method using information of inventory positions, which is given by:

\[ U(t) = KX(t), \]  

where \( K \) is the state feedback gain matrix. It indicates that inventory control parameter \( U(k) \) is determined based on inventory position \( X(k) \). Then Eq. (11) can be rewritten as:

\[ X(t + 1) = ((A + \Delta A) + (B + \Delta B)K)X(t) + (C + \Delta C)X(t - \tau(t)). \]  

According to above descriptions, we obtain the supply chain state transition model with vendor order placement lead time delays in an uncertain environment equation (11), and its corresponding model under the proposed robust inventory control policy equation (13). Based on these models, we will explore the effect of uncertain vendor order placement lead time delays on supply chain dynamics and measure the maximum allowable vendor order placement lead time delay that ensures the suppression of bullwhip effect in an uncertain environment.

3. Maximum allowable vendor order placement lead time delay model

As seen in literatures, inventory variations in the presence of vendor order placement lead time delays would become undesirably oscillatory. The various uncertainties in SC make the role of vendor order placement lead time delays in system dynamics as well as in bullwhip effect phenomenon more complex and indefinable, and increase the difficulty of SCM. Although uncertainties in supply chain systems can not be eliminated for various technical, economic or social reasons, most vendor order placement lead time delays arise from, e.g., poor management, improper production scheduling and planning, and thus can be avoided. Here a natural question arises: given the range of uncertainties in SC systems and relevant inventory policy, what is the maximum allowable vendor order placement lead time delay that ensures the stability of SC systems and the suppression of bullwhip effect?

Before further analysis, we introduce the following concept of exponential stabilization for supply chains in an uncertain environment, and relevant stability properties.

**Definition 3.1.** The uncertain supply chain system (Eq. (11)) is exponential stabilizable if there exists a linear state feedback control law \( U(t) = K(t)X(t), K \in \mathbb{R}^{n \times n} \) such that the trivial solution \( Z(t) = 0 \) of system equation (13), after applying the following state parameter transformation, is exponential stable:

\[ Z(t) = e^{it}X(t), \quad \lambda > 0. \]  

The stability theory underlying above definition of exponential stabilizable is provided by Razumikhin theorems [47,48].

**Theorem 3.1.** The equilibrium at the origin for system \( \dot{x} = f(x,t) \) is globally asymptotically stable if there exists a positive-definite, radially unbounded function \( V: \mathbb{R}^n \to \mathbb{R} \), and a continuous function \( x: \mathbb{R}^{n \times n} \to \mathbb{R}^n \), with \( x(x) > 0 \) for \( x > 0 \), such that

\[ V = L_i V \leq -x(\theta) \quad \text{whenever} \quad \pi(V(x)) \geq V(x_\theta), \quad \text{for all} \quad \theta \in [-r,0], \]  

where the continuous non-decreasing function \( \pi: \mathbb{R}^n \to \mathbb{R}^n \) satisfies \( \pi(s) > s \) for all \( s > 0 \).

To describe the uncertainties in supply chain systems, we assume that they satisfy the following form:

\[ \Delta A(t) = H_aF_a(t)E_a, \quad \Delta B(t) = H_bF_b(t)E_b, \quad \Delta C(t) = H_cF_c(t)E_c, \]  

where \( F_a(t), F_b(t), F_c(t) \) are unknown real time-varying matrices with Lebesgue measurable elements satisfying:

\[ F_i(t) \leq I, \quad i = a,b,c. \]  

\[ H_a, E_i(i = a,b,c) \]  

are known real constant matrices which indicate the effect of uncertain parameters \( F_a(t), F_b(t), F_c(t) \) on system. Through adjusting parameter values, Eq. (15) can describe a variety of uncertainty. Based on the above assumptions, the main issues to be addressed in this section are to determine the maximum allowable vendor order placement lead time delay that ensures SC system satisfy stability **Definition 3.1**, namely the suppression of bullwhip effect, and to explore the effect of proposed inventory control policy in supply chain performance.

3.1. Model description

Given parameters of supply chain dynamics \((A,B,C)\) and their uncertainties \((F_i^c(t), H_i, E_i(i = a,b,c))\), the maximum allowable vendor order placement lead time delay \( \tau_m \) of uncertain supply chain system (13), in the sense of **Definition 3.1** (with decay rate \( \lambda \)), is determined by the following linear optimization model:

\[ \text{minimize} \quad \tau_m \]  

**subject to**

\[ \Delta A(t) = H_aF_a(t)E_a, \quad \Delta B(t) = H_bF_b(t)E_b, \quad \Delta C(t) = H_cF_c(t)E_c, \]
Proof of Theorem 3.2.

Let \( \varepsilon_i \) and the constraints in model (16) are just the robust stability conditions for the system. Because of this, we derive and present the maximum allowable vendor order placement lead time delay model (16) in the form of LMI optimization problem. From the perspective of system stability in sense of Definition 3.1, the maximum allowable vendor order placement lead time delay model leads to the following conclusions:

Theorem 3.2. Suppose that uncertain vendor order placement lead time delays change within the allowable range that is determined by model (16). Then supply chain system (11) is exponential stabilizable with decay rate \( \lambda \), in sense of Definition 3.1, and the constraints in model (16) are just the robust stability conditions for the system.

Before the proof of Theorem 3.2, we introduce some matrix inequalities [50] which are essential for the model analysis:

**Proposition 3.1.** Let \( A, B, C \) and \( D \) be real matrices with appropriate dimensions, and matrix \( D \) satisfies: \( D^T D \leq I \). For any scalar \( \varepsilon (\varepsilon > 0) \), then:

\[
BDC + C^T D^T B^T \leq \varepsilon BB^T + \varepsilon^{-1} C^T C.
\]

**Proposition 3.2.** Let \( A, B, C \) and \( D \) be real matrices with appropriate dimensions, and matrix \( D \) satisfies: \( D^T D \leq I \). If \( \varepsilon I - CPC^T \geq 0 \) for any symmetric positive definite matrix \( P \) and scalar \( \varepsilon (\varepsilon > 0) \), then:

\[
(A + BDC)P(A + BDC)^T \leq APA^T + APC^T (\varepsilon I - CPC^T)^{-1} CPA^T + \varepsilon BB^T.
\]

**Proof of Theorem 3.2.** To simplify notation, we set: \( \tilde{A} = A + \Delta A, \tilde{B} = B + \Delta B, \tilde{C} = C + \Delta C \). Before proceeding further, we introduce the following differential equation into model (11):
X(t − τ(t)) = X(t) − \sum_{i=1}^{t}(\bar{A} + BK - I)X(t - τ(i)) + \bar{C}X(t - τ(t) - τ(t - τ(i))))(τ(i) - τ(i - 1)). \tag{21}

To analyze supply chain performance in reducing bullwhip effect, we define the following Lyapunov function for system:

\[ V(Z(t)) = Z^T(t)PZ(t), \tag{22} \]

where \( P \in \mathbb{R}^{n \times n} \) is a symmetric positive definite matrix, \( Z(t) \) is the new system state parameter derived from state transformation equation (14). The differential equation of \( V(Z(t)) \) along trajectory of uncertain supply chain system (13) is:

\[ \Delta V(Z(t)) = V(Z(t + 1)) - V(Z(t)) = e^{2(t+1)}X^T(t + 1)PX(t + 1) - e^{-2}X^T(t)PX(t). \tag{23} \]

Considering differential equation (21) and Proposition 3.1, we have:

\[ \Delta V(Z(t)) \leq e^{2(t+1)}X^T(t)(\bar{A} + BK + C)^T\bar{P}(\bar{A} + BK + C)X(t) + M^T (d + C^TPC)M + \varepsilon^{-1}X^T(t)(\bar{A} + BK + C)^T\bar{P}CC^T\bar{P}(\bar{A} + BK + C)X(t) - e^{-2}X^T(t)PX(t), \tag{24} \]

where \( M = \sum_{i=1}^{t}(\bar{A} + BK - I)X(t - τ(i)) + \bar{C}X(t - τ(i) - τ(t - τ(i))))(τ(i) - τ(i - 1)), \varepsilon \in \mathbb{R}^+ \). Inspired by the Razumikhin stability theorem (see Theorem 3.1), we assume that there is a scalar \( \delta > 1 \) such that:

\[ V(X(\varphi)) \leq \delta V(X(t)), \quad t - 2τ_m \leq \varphi \leq t. \tag{25} \]

Then inequality (24) becomes:

\[ \Delta V(Z(t)) \leq e^{2(t+1)}X^T(t)(\bar{A} + BK + C)^T\bar{P}(\bar{A} + BK + C)X(t) + \tau_m^2\delta^2X^T(t)(\bar{A} + BK - I + C)^T(d + C^TPC)(\bar{A} + BK - I + C)X(t) + \varepsilon^{-1}X^T(t)(\bar{A} + BK + C)^T\bar{P}CC^T\bar{P}(\bar{A} + BK + C)X(t) - e^{-2}X^T(t)PX(t). \tag{26} \]

Repeatedly apply Proposition 3.2 and consider Eqs. (17) and (18), inequality (26) finally yields:

\[ \Delta V(Z(t)) \leq e^{2(t+1)}X^T(t)T_1P_1E_i^T(\varepsilon_3I - E_iE_i^T)^{-1}E_iE_i^TP_1T_1^T + T_1P_1T_1^T + \tau_m^2\delta^2\varepsilon T_2T_2^T + \varepsilon_3H_1H_1^T + \tau_m^2\delta^2CT_2P(CT_2)^T + \varepsilon_3H_1H_1^T + \tau_m^2\delta^2EE_3H_1H_1^T + \tau_m^2\delta^2E_2H_2H_2^T + e^{-2}P(X(t), \tag{27} \]

where:

\[ \begin{align*}
T_1 & = A + K + C, \\
T_2 & = A + K + C - I, \\
P_1 & = P + \varepsilon^{-1}PQP, \\
Q & = CC^T + CE_i^T(\varepsilon_3I - E_iE_i^T)^{-1}E_iC^T + E_2H_2H_2^T, \\
\varepsilon_3I - E_iE_i^T & > 0, \\
\varepsilon_3I - E_iP_i^TP_i & > 0, \\
\varepsilon_4I - E_iE_i^T & > 0, \\
\varepsilon_4I - E_iP_i^TP_i & > 0, \\
\varepsilon_4I - E_iE_i^T & > 0, \\
\varepsilon_4I - E_iP_i^TP_i & > 0, \tag{28} \end{align*} \]

and \( P_1 \) is a symmetric positive definite matrix. We set:

\[W(\delta) = T_1P_1E_i^T(\varepsilon_3I - E_iP_iE_i^T)^{-1}E_iP_iE_i^TP_1T_1^T + T_1P_1T_1^T + \tau_m^2\delta^2\varepsilon T_2T_2^T + \tau_m^2\delta^2CT_2P(CT_2)^T + \varepsilon_3H_1H_1^T + \tau_m^2\delta^2EE_3H_1H_1^T + \tau_m^2\delta^2E_2H_2H_2^T + e^{-2}P. \tag{29} \]

It is clear that \( W(\delta) \) is monotone increasing with respect to parameter \( \delta \). Given a certain \( \tau_m \), if \( W(\delta) < 0 \) then there exists a sufficiently small \( \delta > 1 \) such that for any \( 0 \leq \tau(t) \leq \tau_m, W(\delta) < 0 \). Based on the Razumikhin theorem (Theorem 3.1), the analyzed system is globally uniformly asymptotically stable. According to Definition 3.1, this implies the exponential stabilization of supply chain system with vendor order placement lead time delays in an uncertain environment (model (11)).

Consequently, applying the Schur complement theorem [51] and making the following parameter replacements, we can obtain the constraints in the maximum allowable vendor order placement lead time delay model (16). From the above analysis, it is clear that these constraints ensure supply chain stability in the sense of Definition 3.1. This completes the proof of Theorem 3.2.

\[ X = P + \varepsilon^{-1}PQP, \quad Y = \sqrt{\varepsilon}K, \quad N = KK, \quad R = KP, \]
\[ J = QP, \quad L = \varepsilon Q + PQ, \quad \varepsilon_1 = \sqrt{\varepsilon}. \]

The above analysis implies that there are two important features of the proposed maximum allowable vendor order placement lead time delay model. On the one hand, given relevant system parameters, it can determine the maximum vendor order placement lead time delay that guarantees the suppression of bullwhip effect resulted from uncertainties. This provides a framework and an alternative method for supply chain managers to find the weak links in supply chains, especially those with vendor order placement lead time delays, and thus help develop more targeted and effective management policy to reduce the bullwhip effect. On the other hand, if the range and general form of uncertainties in SC are informed, we can further obtain the theoretical optimal parameters of the proposed inventory control policy, namely the value of feedback gain matrix: \[ K = N \varepsilon^{-1}. \] This provides some theoretical support for actual supply chain planning and management.

### 4. Numerical experiments

To assess the performance of proposed uncertainty-dependent robust inventory control method and to gain further insights into the dynamical characteristics of supply chain systems with vendor order placement lead time delays in an uncertain environment, especially the bullwhip phenomenon, we carry out simulations in this section.

#### 4.1. Experimental design

To simplify analysis, we consider a simple supply chain system depicted in Fig. 2, where digits on arrows are proportional coefficients, corresponding to the system structure coefficients matrix \( C \), and digits in brackets represent parameters of system structure variation \( E_c \). According to model in Section 2, we get system state vector \( X(t) = (x_1(t), x_2(t), \ldots, x_{11}(t))^T \), inventory control vector \( U(t) = (u_1(t), u_2(t), \ldots, u_{11}(t))^T \), and feedback control strength matrix \( B = I \). We assume that uncertainties in simulations satisfy: \( F_i(t) = \sin(t) \), \( H_i = I \) (\( i = a, b, c \)), and set the system decay rate \( \lambda = 0.7 \) with vendor order placement lead time delays \( \tau(t) = \tau_m |\sin(t)| \). Other parameters are summarized in Table 1, from which we can calculate production coefficients matrix \( A \) and other system parameters used in model (11). Note that \( X(1) \) in Table 1 denotes the initial inventory variation.

#### 4.2. Simulation results

Based on the model presented in Section 3, we obtain the maximum allowable vendor order placement lead time delay of simulation system: \( \tau_m \approx 2.83 \). Given this value, our objective then is to extend analysis to the uncertainty-dependent robust stability of system under the proposed inventory control policy. In the following paragraphs we only consider inventory state in upstream nodes \( (x_1(t) \cdots x_{11}(t)) \).

In Figs. 3 and 4, we report system evolution of the first two scenarios whose maximum vendor order placement lead time delays are: \( \tau_1 = 1, \tau_2 = 2 \), respectively. As shown in these figures, initial inventory variations in downstream nodes will spread and be magnified upstream supply chain. Given that the maximum value of uncertain vendor order placement lead

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Fig. 2. The structure of supply chain network used in simulations.
time delay is less than its maximum allowable value \( \tau \leq \tau_m \), it is clear from these figures that the magnified inventory variations will be under control and inventory will finally return to a stable state. This implies that the proposed uncertainty-dependent robust inventory control method is effective in suppressing bullwhip effect. In the meantime, we should not ignore the different system evolution characteristics that emerge in these scenarios.

In Fig. 3 \((\tau_1 = 1)\), the maximum amplitude of inventory fluctuations caused by end demand magnification appears in node 1, followed by node 2, both near the forth simulation time step. Inventory state of node 1 also determines the control time: it takes near 8 simulation time steps to level off from initial disturbance. Compare with Fig. 3, system fluctuations in Fig. 4

<table>
<thead>
<tr>
<th>Site 1</th>
<th>Site 1</th>
<th>Site 2</th>
<th>Site 3</th>
<th>Site 4</th>
<th>Site 5</th>
<th>Site 6</th>
<th>Site 7</th>
<th>Site 8</th>
<th>Site 9</th>
<th>Site 10</th>
<th>Site 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_i )</td>
<td>0.7</td>
<td>0.5</td>
<td>0.3</td>
<td>0.4</td>
<td>0.35</td>
<td>0.3</td>
<td>0.6</td>
<td>0.7</td>
<td>0.65</td>
<td>0.75</td>
<td>0.7</td>
</tr>
<tr>
<td>( f_i )</td>
<td>0.441</td>
<td>0.467</td>
<td>0.615</td>
<td>0.571</td>
<td>0.519</td>
<td>0.5</td>
<td>0.375</td>
<td>0.471</td>
<td>0.455</td>
<td>0.4</td>
<td>0.412</td>
</tr>
<tr>
<td>( E_s(i) )</td>
<td>0.12</td>
<td>0.1</td>
<td>0.2</td>
<td>0.15</td>
<td>0.15</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
<td>0.15</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>( E_d(i) )</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
</tr>
<tr>
<td>( X(1) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1
Summary of system parameters used in simulations.
\( \tau_2 = 2 \) last a little longer, about 9 simulation time steps. However, amplitudes of inventory variations decrease greatly: to node 1, it drops from \( x_1(4) = -38 \) in scenario 1 to \( x_1(3) = 21 \) in scenario 2. Similar trends can be found in other SC nodes. If we care more about the amplitude of inventory deviations than the slight prolonged time of instability, we can say that Fig. 4 shows better inventory control performance. This seemingly contradicts previous analysis, namely longer time delays will worsen SC performance. But closer examination reveals its possibility: when time-delay rises, the process fluctuations may be counterbalanced and smoothed during the accumulation process, even though the designed control compensation could not reflect the up-to-date fluctuations in time. This explains the emergence of less inventory volatility in Fig. 4.

Fig. 5 shows result of the control experiment, where supply chain system evolves with the possibility that vendor order placement lead time delay exceeds its maximum allowable value: \( \tau_3 = 5 \). To compare its evolution features with those in former scenarios (Figs. 3 and 4), control parameter value used in Fig. 5 is the same as that in Fig. 4. Obviously, inventory fluctuation in Fig. 5 is uncontrollable: when the largest possible time delay rises to \( \tau_3 = 5 \), not only do the amplitudes of inventory deviations enlarge, but also the unstable state continues with system evolution.

Summing up the above comparison and analysis, it is clear that vendor order placement lead time delay determines the evolution characteristics of supply chain system. An appropriate and effective inventory control strategy will reduce unwanted inventory fluctuations resulted from vendor order placement lead time delays or other system uncertainties, such as those in production process, supply chain structure, inventory policy implementation, etc.

5. Conclusions

In this paper, we have studied the bullwhip effect problem in uncertain supply chain systems with vendor order placement lead time delays. Taking into consideration the multiple sources of uncertainty in supply chain systems, we focused on uncertainties with respect to demand, production process, supply chain structure, inventory policy implementation and vendor order placement lead time delays. We first built an optimization model which enables analyzing the maximum allowable vendor order placement lead time delay that ensures the controllability of system in an uncertain environment. From the perspective of control system, we then explored system robustness under the proposed uncertainty-dependent state feedback inventory control method. Simulation study indicates that vendor order placement lead time delays do play an important role in reducing supply chain stability and inducing the bullwhip effect phenomenon. The larger the uncertain range of vendor order placement lead time delay is, the larger the inventory deviation amplitude is and the longer the bullwhip phenomenon exists. On the other hand, upstream suppliers may avoid overreacting to inaccurate downstream orders by extending their reaction time within a certain extent. In other words, slight time delays may counterbalance and smooth inventory fluctuations during system evolution, and thus may play a positive role in reducing bullwhip effect. Simulation results also verify the effectiveness and applicability of the proposed uncertainty-dependent state feedback inventory control method in improving SC robustness with respect to uncertainties in demand, production process, supply chain structure, inventory policy implementation and vendor order placement lead time delays. In practice, our work provides a new method for supply chain managers, allowing them to find the weak links in supply chains, design better inventory control strategy and improve SC performance. It is important to note that our model is open-ended. There are many avenues for future research. Extending the proposed model to more detailed study, such as intensive study of the relationship between different uncertainty sources in SC operation process, is currently under investigation.
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References


