The foundation of the grey matrix and the grey input–output analysis

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Received 28 November 2006; accepted 6 December 2006
Available online 26 January 2007

Abstract

The grey systems theory aims at the objects that their information is inadequate and this situation is general in reality. It has been urgent work to study the uncertain problems using the missing information. With the help of the simple introduction of grey systems theory, we further study the covered operation and get some calculation rules about grey number. The definition of grey matrix (GM) and its covered operation are proposed. Particularly, some results of the inverse grey matrix are obtained. Also with the help of the proposed grey matrix theory and the traditional input–output analysis, we propose the grey input–output analysis. The most important results are the computational formulas and their rigorous proofs of the matrix-covered set of the inverse grey Leontief coefficient’s matrix. It provides an effective tool to study an economic system by the input–output analysis under the uncertain situation. The modified case verifies the effectiveness of our methodology.

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Keywords: Grey systems theory; Grey matrix; Input–output analysis; Covered operation; Uncertainty

1. Introduction

In the medium of the 18th century, the second law of thermodynamics took us into an uncertain world and caused the research of stochastic process. Combined with the probability and mathematical statistics, the research of stochastic process had greatly promoted the development of modern science. Consequently, we had been forced to face the recognizable difficulty as the object’s boundary is undistinguishable. In 1965, Zadeh proposed the fuzzy systems theory to resolve this type of uncertain problems with the membership function [1]. For another specific phenomenon with inadequate or missing information, however, we could not give the accurate values but their boundary. Based on the obtained information, this phenomenon can be divided...
into the white box whose information is completely knowable, the black box whose information is absolutely unknowable, and the grey box whose information is partly knowable and partly unknowable, respectively. The Chinese scholar, Deng (see [2]) originated the grey systems theory in 1982. It provided a methodology to resolve this grey phenomenon.

The grey systems theory, fuzzy systems theory, as well as the probability and mathematical statistics are three typical tools to resolve the uncertain problems, but their solutions are different [3,4]. The probability and mathematical statistics does with the stochastic uncertain phenomenons. We can get the probability of every emerged event through large numbers of statistic materials. The fuzzy systems theory is involved in the cognitive uncertain problems and it depends on the experience of decision-makers. The grey systems theory aims at studying the uncertain situation with inadequate information [5–10]. The object to be researched has the feature that the extension is known but the connotation. For example, the sum of people in the world will be controlled between 60 and 80 hundred million in 2010. It means that the true number, which is the connotation, must be in [60,80], but we do not know it. Since the grey systems theory was proposed, it has been greatly developed by many scholars’ efforts and successfully applied in many fields, such as economics, social problems, science and technology, agriculture, ecology, biology, etc. (see [11–18] etc.).

The input–output analysis was proposed by Leontief while he analyzed the US economy [19]. It explained the interconnection among sectors of complex economic systems, which may be national, regional, or enterprise type. Through the development about 60 years, the input–output analysis has been viewed as one of the effective and normative tools by the United Nation. Most of countries and regions in the world set up their input–output tables for every several years to forecast and control the economic development using these coefficients, such as the direct and complete consumption ones, the influence and the interaction ones, etc. (see [20–28] etc.).

But most of results about the input–output analysis are deterministic. In reality, the statistical datum are inaccurate and the forecasting may be unprecise, so it had been urgent work to explain the economic and technological interconnection among sectors of complex economic systems under uncertain situation. Many researchers have discussed it and obtained some confident results using other uncertain methods (see [29–32] etc.). But we consider that the economic system has more grey properties than stochastic and fuzzy. For example, when determining the sum of the input that sector \( i \) throws into sector \( j \), we can only get the range around the true, such as that between 100 and 120 tons. On the other hand, when forecasting the future economic system, we can also give its range, such as the fact that the value of the final consumption product of sector \( i \) will be between 20 and 22 million dollars next year. For all of these discussed situations above, we only know that the true numbers must be in the estimated ranges. Then the grey systems theory can be utilized to analyze the input/output relation among sectors and it is called grey input–output analysis (GIOA).

The established model and table are grey input–output model (GIOM) and grey input–output table (GIOT), respectively. In this paper, we only deal with the value type, i.e., the unit of all inputs and outputs is currency.

The paper [33] discussed the GIOA and obtained the global optimal solution by the grey mathematical programming based on the genetic algorithm. It gave the formula of the inverse grey matrix but the manipulation process, so we could not obtain a sort of coefficients, which are essential to analyze the technological relation among sectors. The paper [34] only discussed grey factors that are difficult to be quantified, such as the weather condition. On the other hand, the numbers in this paper are also deterministic. The paper [35] utilized the upper and lower bounds to get the range of the inverse Leontief coefficient’s matrix, but they did not verify it and the computation process was not satisfied with grey meaning. In this paper, we propose the grey matrix theory and give the manipulation formulas of the inverse grey matrix. Then some coefficients of the input–output analysis can be obtained under the uncertain situation and we can use them to forecast and analyze the economic system.

This paper is organized as follows. In Section 2, we introduce some basic knowledge of grey systems theory, such as the grey hazy set, grey number and its covered operation. The further study about grey number is enumerated. In Section 3, we propose the grey matrix theory and stress the difference between the matrix-covered set and the interval matrix. In Section 4, the grey input–output analysis is proposed and some important results are obtained. We analyze a modified case by our grey input–output analysis in Section 5. At last, the conclusion is given.
2. Preliminaries

In this section, we will introduce some basic knowledge of grey systems theory. There are four parts below and the Parts I, II and IV can be seen in [36–43], etc. For systematic comprehension, they can be found in book [44]. The Part III is our work and we give some remarks in this section.

2.1. Grey hazy set and grey number

Just as the fact that the Cantor Set and Fuzzy Set are the foundation of probability and mathematical statistics and fuzzy systems theory, respectively, the base of grey systems theory is Grey Hazy Set.

**Definition 2.1.** Supposing that \( \lambda \) is a proposition and \( \lambda(\theta) \) is a set of information about it. If \( \lambda(\theta) \) is (1) propositional; (2) complete; (3) objective; (4) true; (5) whitened or non-whitened, then we call it the proposition–information field about \( \lambda \).

**Definition 2.2.** If a set satisfies the 4IN properties: (a) intention; (b) interim; (c) intangible; and (d) inkling, then we call it Hazy Set (HS). The point of HS is Hazy Point (HP).

**Remark 2.1.** The meaning of the 4IN properties is as the following:

1. Intention shows that we collect the information to obtain the truth of the proposition.
2. Interim shows that the obtained information is temporary and we can get more information using other methods.
3. Intangible shows that we could not justify whether the obtained information about the proposition is true/objective or not.
4. Inkling shows that the truth of the proposition is difficult to be whitened and we cannot get it based on the obtained information.

**Definition 2.3.** The Hazy Set that includes the proposition–information field is called kernelled Hazy Set.

**Definition 2.4.** These subsets, i.e., the Embryo Set, the Growing Set, the Mature Set and the Evidence Set, which are evolved from Kernelled Hazy Set, are called Grey Hazy Set. That is to say,

\[
\text{Grey Hazy Set} = \{\text{Embryo Set, Growing Set, Mature Set, Evidence Set}\}.
\]

In fact, the Embryo Set is a black box, and the Evidence Set is a white one, and the Growing and Mature Sets are grey ones. When the information about the object/proposition is supplemented, the subsets may be gradually evolved from the Embryo Set, the Growing Set, the Mature Set and the Evidence Set, respectively. It is apparent that the Grey Hazy Set is dynamic and its evolutive order is fixed, i.e., Embryo Set \( \Rightarrow \) Growing Set \( \Rightarrow \) Mature Set \( \Rightarrow \) Evidence Set.

**Note.** The Kernelled Hazy Set assures that the obtained information is correct and objective, and the truth is always within these subsets.

**Example 2.1.** we guess the stature of someone and suppose that it is 178 cm. Because the stature of a man is between 0 cm to 300 cm, we have that \([0,300]\) is the Embryo Set. If we know that he is an adult, then his stature is between 140 cm and 300 cm, and \([140,300]\) is the Growing Set; when knowing other information, for example, his stature is normal, we get that \([170,180]\) is the Mature Set; At last, we measure him and get his true stature, then \([178]\) is the Evidence Set.

Just as the fuzzy number of fuzzy systems theory, the “number” of grey systems theory is grey number (GN) and it is based on the Grey Hazy Set. Before introducing the GN, we will give the following definitions.
Definition 2.5. Supposing that $\odot$ is an operator, and $x_i (i \in I$ and $I$ is the set of natural numbers) is a number (real, fuzzy or grey), and $\delta$ is a real number. If $x_i \odot x_i = \delta$ holds for all $i \in I$, then we call $\odot$ a common operation.

For example, if $x_i (i \in I)$ are real numbers, we can get a real number $0$ and have that $x_i - x_i = 0$ holds for all $i \in I$. Then the operator “−” is a common operation.

Definition 2.6. A number is called an absolutely conceptional number (ACN), if it only exists in our mind and its specific value need not be paid attention in operation, such as the $1$ and the $e$; It is a relatively conceptional number (RCN), if it belongs to the conception in mind. The specific value of RCN may or may not be paid attention in operation, such as the grey number that we will discuss later.

Definition 2.7. The ACN and the RCN are integrated by conceptional number (CN).

Definition 2.8. A RCN is called the Potential Number (PN) if it satisfies the property of the common operation, and we denote it as $d^\ast/C^\ast$. It means that $d^\ast/C^\ast/C^\ast = 0$ holds even if we do not know the specific value of $d^\ast/C^\ast$.

Note that the two ACNs $\infty$ and $\varepsilon$ are not PN, because $\infty - \infty$ and $\varepsilon - \varepsilon$ have no meaning.

Theorem 2.1. The CN and the PN are uncertain numbers.

Definition 2.9. If we have no reason to deny that the real number $a$ is true, then we call $a$ an acquiescing number (AN).

Based on the definitions above, we can give the definition of grey number.

Definition 2.10. Let $\lambda$ be a proposition and $\lambda(\theta)$ be its proposition–information field. $D$ is the number field and $\odot$ is the uncertain number about $\lambda$. Supposing that $\odot$ and $d^\ast$ are the AN and only potential true number of $\odot$, respectively, and $\tilde{D}$ is a set of real numbers. If $\odot$ satisfies the following condition:

$$\forall \tilde{\odot} \text{ of } \odot \Rightarrow \tilde{\odot} \in D(\lambda),$$

where $D(\lambda) = \{ \tilde{\odot} | \tilde{\odot} \text{ approaches } \lambda(\theta), \text{ and } \exists d^\ast \text{ approaches } \lambda(\theta); \ d^\ast, \tilde{\odot} \in \tilde{D} \subset D; \ d^\ast \text{ is the true number about } \lambda; \text{ When } d^\ast \text{ occurs, } d^\ast = d^\ast, \text{ and } \tilde{\odot} \text{ and } \odot \text{ vanish at the same time}\}$, then we call that

1. $\odot$ is a grey number (GN) about the proposition $\lambda$;
2. $\tilde{\odot}$ is the whitened number of $\odot$;
3. $\tilde{D}$ is the number-covered set of $\odot$;
4. $\lambda(\theta)$ and $d^\ast$ (or $d^\ast$) are the information background of $\odot$.

Remark 2.2. The definition above points out that the grey number is based on the proposition $\lambda$ and its number-covered set is a Grey Hazy Set, which is based on the proposition–information field. The number-covered set is dynamic and evolutive (see Example 2.1).

Definition 2.11. Let $\tilde{D}$ be the number-covered set of $\odot$.

1. If $\tilde{D}$ is a dispersed set, then $\odot$ is called dispersed grey number (DGN).
2. If $\tilde{D}$ is a continuous set, then $\odot$ is called continuous grey number (CGN).

Note. For a GN’s number-covered set $\tilde{D}$,

1. a real number $a$ is a special GN, and its number-covered set $\tilde{D} = \{a\}$ is Evidence Set.
2. if $\tilde{D}$ is a disperse type, then the permutation of numbers in $\tilde{D}$ is that the left element is always smaller than the right one;
3. if $\tilde{D}$ is a continuous type, i.e., $\tilde{D} = [a, b]$, we always have $a < b$. 
Definition 2.12. Supposing that \( D \) is the number-covered set of \( \otimes \). Let \( A(D) = x^+ - x^- \), where \( x^+ \) and \( x^- \) are the maximum and the minimum numbers of \( D \), respectively. Then we call that \( A(D) \) is the chaos of \( \otimes \).

While the new information about \( \lambda \) is added to the proposition–information field \( \lambda(\theta) \), the number-covered set \( D \) may gradually evolves from the Embryo Set, the Growing Set, the Mature Set, and then the Evidence Set, and the chaos of \( \otimes \) becomes smaller (see Example 2.1).

Definition 2.13. Supposing that \( D_1 \) and \( D_2 \) are the number-covered sets of \( \otimes \), which are based on the different (or same) proposition–information field. If \( D_1 \subseteq D_2 \) holds, then we call that \( D_1 \) is the relatively superior number-covered set of \( \otimes \) and \( D_2 \) is its relatively inferior one.

Definition 2.13 manifests the dynamic and evolutive property of Grey Hazy Set.

Example 2.2. Let the proposition \( \lambda \) be the achievement of examination, and \( \lambda(\theta) \) denote the obtained information that can correctly reflect the examination, i.e., the proposition–information field. \( D = [0, 100] \) is the number field of \( \lambda \). The achievement of someone is uncertain and it is a GN \( \otimes \) based on the information \( \lambda(\theta) \). He get some estimated numbers \( \otimes \), such as 80, 85 and 90. Then the number-covered set \( D = [80, 90] \) is obtained. After the achievement is known, the true number \( d^* \) is 83, and all of the estimated numbers disappear.

2.2. The covered operation of the GN

In this part, we introduce the basic operation between GNs. Because the expressive form of GN is a set of numbers, i.e., the number-covered set, and the practical participators are the elements of it, we call the operation between GNs as the covered operation.

Definition 2.14. Let \( \otimes_i \) and \( \otimes_j \) be GNs. \( \otimes_i \) and \( \otimes_j \), \( D_i \) and \( D_j \) as well as \( d_i^r \) and \( d_j^r \) be the whitened numbers, the number-covered sets and the only potential true numbers of \( \otimes_i \) and \( \otimes_j \), respectively. \( \otimes_{ij} \) is the result between \( \otimes_i \) and \( \otimes_j \) through the operator \( \otimes \), and \( D_{ij} \) is the number-covered set of \( \otimes_{ij} \). If all of the following conditions:

\[
\otimes_i \otimes \otimes_j = \otimes_{ij}; \quad D_i \otimes D_j = D_{ij}; \quad \forall \otimes_i \text{ of } \otimes_i \text{ and } \forall \otimes_j \text{ of } \otimes_j \Rightarrow \otimes_i \otimes \otimes_j \in D_{ij}
\]

hold, where

\[
D_{ij} = \{ \otimes_{ij} | \forall \otimes_i \in D_i \text{ and } \otimes_j \in D_j \Rightarrow \otimes_{ij} = \otimes_i \otimes \otimes_j; \exists d_i^r \in D_i \text{ and } d_j^r \in D_j \Rightarrow d_{ij}^r = d_i^r \otimes d_j^r 
\]

\[
\in D_i \otimes D_j; \quad d_{ij}^* = d_i^* \otimes d_j^*; \quad \text{If } d_{ij}^* \text{ occurs, then } d_{ij}^r = d_{ij}^* \text{ and all of } \otimes_{ij} \text{ (and } d_{ij}^* \text{) vanish at the same time,}
\]

then we call “\( \otimes \)” the covered operation between the GNs \( \otimes_i \) and \( \otimes_j \), and denote it as \( \otimes_i \otimes \otimes_j \Rightarrow \{ D_i \otimes D_j \} \).

Remark 2.3. Apparently, \( \otimes_{ij} \) is a GN, and \( d_{ij}^* \) and \( D_{ij} \) are its only potential true number and number-covered set, respectively.

Theorem 2.2. For any GN \( \otimes \), we have

\[
\otimes \otimes = \otimes; \quad \text{and if } 0 \text{ is not the only potential true number (or the true number) of } \otimes, \text{ then } \otimes \div \otimes = 1.
\]

Proof. Supposing that \( \otimes \) has two different number-covered sets \( D_1 \) and \( D_2 \), which maybe exist for different (or same) proposition–information field, we certainly know that the only potential true number (or the true one) of \( D_1 \) is the same as the one of \( D_2 \) from Definition 2.10, and denote it as \( d^c \). From Definition 2.14, we get that the number-covered set and only potential true number of \( \otimes \otimes \) are \( D_1 \otimes D_2 \) (or \( D_2 \otimes D_1 \)) and \( d^c \otimes d^c \).
respectively. Because $d^r - d^l = 0$ (see Definition 2.8), we know that the true number 0 appears, and the number-covered set $D_1 - D_2$ has evolved into the Evidence Set $\{0\}$, so $\otimes - \otimes$ is not uncertain and equal to 0. It is the same for $\otimes \div \otimes$.

The specific forms of operation between GNS are as below.

**Definition 2.15.** Supposing that $\otimes_i$ and $\otimes_j$ are two DGNs, and their number-covered sets are $D_i = \{d_{ik} | k = 1, 2, \ldots, n\}$ and $D_j = \{d_{lj} | l = 1, 2, \ldots, m\}$, respectively. Let $\otimes_{ij} = \otimes_i \circ \otimes_j$ and $D_{ij} = D_i \circ D_j$, where $\circ \in \{+, -, \times, \div\}$. Then we have that $D_{ij}$ is the number-covered set of $\otimes_{ij}$ and it is as below:

1. $D_{ij} = D_i + D_j = \{d_{ik} + d_{lj} | k = 1, 2, \ldots, n; l = 1, 2, \ldots, m\};$
2. $D_{ij} = D_i - D_j = \{d_{ik} - d_{lj} | k = 1, 2, \ldots, n; l = 1, 2, \ldots, m\};$
3. $D_{ij} = D_i \times D_j = \{d_{ik} \times d_{lj} | k = 1, 2, \ldots, n; l = 1, 2, \ldots, m\};$
4. $D_{ij} = D_i \div D_j = \{d_{ik} \div d_{lj} | k = 1, 2, \ldots, n; l = 1, 2, \ldots, m\}$, where $0 \in D_j$.

**Definition 2.16.** Supposing that $\otimes_i$ and $\otimes_j$ are two CGNs, and their number-covered sets are $D_i = [a_i, b_i]$ and $D_j = [a_j, b_j]$, respectively. Let $\otimes_{ij} = \otimes_i \circ \otimes_j$ and $D_{ij} = D_i \circ D_j$, where $\circ \in \{+, -, \times, \div\}$. Then we have that $D_{ij}$ is the number-covered set of $\otimes_{ij}$ and it is as below:

1. $D_{ij} = D_i + D_j = [a_i + a_j, b_i + b_j];$
2. $D_{ij} = D_i - D_j = [a_i - b_j, b_i - a_j];$
3. $D_{ij} = D_i \times D_j = \min\{a_i a_j, a_i b_j, a_j b_i, b_i a_j, b_i b_j\}, \max\{a_i a_j, a_i b_j, a_j a_i, b_i a_j, b_i b_j\};$
4. $D_{ij} = D_i \div D_j = \min\{a_i / a_j, a_i / b_j, a_j / a_i, b_i / a_j, b_i / b_j\}, \max\{a_i / a_j, a_i / b_j, a_j / a_i, b_i / a_j, b_i / b_j\}$, where $0 \in D_j$.

**Theorem 2.3.** Supposing that $\otimes$ is a GN and $D$ is its number-covered set. Let $\circ \subseteq \{+, -, \times, \div\}$ be a binary operation. If $a$ is a real number, then we have that $a \otimes$ is a GN and its number-covered set is as below:

$$a \circ D = \{a \circ d | d \in D\},$$

where $0 \in D$ when the operator is “$\div$”.

**Example 2.3.** There are some water in a cup. Someone does not know how much it is, and the volume of water is a GN $\otimes$. But he can determine that the volume is between 10 ml and 13 ml based on some information and get that the number-covered set is $[10, 13]$. If he pours the water, then the volume of water in the cup is the GN $\otimes - \otimes$ and its number-covered set is $[10, 13] - [10, 13] = [-3, 3]$ from Definitions 2.14 and 2.16. However, we certainly know that there is no water in the cup. The true number 0 is in $[-3, 3]$ and $[-3, 3] \supseteq \{0\}$ holds. That is to say, $\otimes - \otimes = 0$.

### 2.3. Further study

With the help of the Part II, we have the covered operation between DGN and CGN as below.

**Definition 2.17.** Supposing that $\otimes_i$ is a DGN and $\otimes_j$ is a CGN, and their number-covered sets are $D_i = \{d_{ik} | k = 1, 2, \ldots, n\}$ and $D_j = [a_j, b_j]$, respectively. Let $\otimes_{ij} = \otimes_i \circ \otimes_j$ and $D_{ij} = D_i \circ D_j$ (or $\otimes_{ji} = \otimes_j \circ \otimes_i$ and $D_{ji} = D_j \circ D_i$), where $\circ \in \{+, -, \times, \div\}$. Then we have that $D_{ij}$ (or $D_{ji}$) is the number-covered set of $\otimes_{ij}$ (or $\otimes_{ji}$) and it is as below:

1. $D_{ij} = D_i + D_j = \bigcup_{k=1}^{n} [d_{ik} + a_j, d_{ik} + b_j];$
2. $D_{ij} = D_i - D_j = \bigcup_{k=1}^{n} [d_{ik} - b_j, d_{ik} - a_j]$ or $D_{ji} = D_j - D_i = \bigcup_{k=1}^{n} [a_j - d_{ik}, b_j - d_{ik}];$
3. $D_{ij} = D_i \times D_j = \bigcup_{k=1}^{n} \{\min\{d_{ik} a_j, d_{ik} b_j\}, \max\{d_{ik} a_j, d_{ik} b_j\}\};$
4. $D_{ij} = D_i \div D_j = \bigcup_{k=1}^{n} \left[\min\left\{\frac{d_{ik}}{a_j}, \frac{d_{ik}}{b_j}\right\}, \max\left\{\frac{d_{ik}}{a_j}, \frac{d_{ik}}{b_j}\right\}\right]$, where $0 \in D_j$.
(5) $D_{ij} = D_j \div D_i = \bigcup_{k=1}^{m} \left[ \min \left\{ \frac{1}{d_k} a_j, \frac{1}{d_k} b_j \right\}, \max \left\{ \frac{1}{d_k} a_j, \frac{1}{d_k} b_j \right\} \right]$, where $0 \in D_i$, where $\cup$ is the unity operator among sets.

From **Definition 2.14**, we know that $d''_{ij} = d''_{i} \circ d''_{j}$ is the only potential true number and $d''_{ij} \in D_{ij}$ holds, where $d''_{i}$ and $d''_{j}$ are the only potential true numbers of $\otimes_i$ and $\otimes_j$, respectively.

**Theorem 2.4.** Let $\otimes_1$ and $\otimes_2$ be two GNs, and $d''_1$ and $d''_2$ be their only potential true numbers, respectively. Then we have $\otimes_1 = \otimes_2 \iff d''_1 = d''_2$.

**Proof.** If $\otimes_1 = \otimes_2$, i.e., they are the same GN, then from **Definition 2.10**, a GN has only one potential true number, and we have $d''_1 = d''_2$.

If $d''_1 = d''_2$, i.e., it is only one, then we have that they exist for the same proposition. Otherwise, we could not affirm that $d''_1$ and $d''_2$ are identical as they are PNs. Because the GN is based on the proposition, we have that the GN is only one under the uncertain situation even if the proposition–information fields are different, so $\otimes_1 = \otimes_2$. □

Denoting that $\otimes^k = \otimes \times \otimes \times \ldots \times \otimes$ and $k\otimes = \otimes + \otimes + \ldots + \otimes$, where $k$ is the sum of $\otimes$, we have the following theorem.

**Theorem 2.5.** For any GN $\otimes$, we have

\[
\otimes^k - \otimes^k = 0 \quad \text{and} \quad k\otimes - k\otimes = 0;
\]

and if 0 is not the only potential true number (or the true number) of $\otimes$, then

\[
\otimes^k \div \otimes^k = 1 \quad \text{and} \quad (k\otimes) \div (k\otimes) = 1.
\]

**Proof.** Supposing that $D$ and $d^\otimes$ are the number-covered set and the only potential true number of $\otimes$, respectively, we have that $D^k = D \times D \times \ldots \times D$ and $(d^\otimes)^k = d^\otimes \times d^\otimes \times \ldots \times d^\otimes$ are the number-covered set and the only potential true number of $\otimes^k$ from **Definition 2.14** (or the (3) of **Definitions 2.15, 2.16**), respectively. Then $0 = (d^\otimes)^k - (d^\otimes)^k \in D^k - D^k$ is the true number of $\otimes^k - \otimes^k$. It means that $D^k - D^k \Rightarrow \{0\}$ has been realized and $\otimes^k - \otimes^k = 0$ holds. The proofs of $k\otimes - k\otimes$, $\otimes^k \div \otimes^k$ and $(k\otimes) \div (k\otimes)$ are similar. □

**Theorem 2.6.** Supposing that $\otimes$ is a GN and $d^\otimes$ is its only potential true number. If there exists a set of real numbers $D$ and $d^\otimes \in D$ holds, then $D$ is the number-covered set of $\otimes$.

**Proof.** For $d^\otimes \in D$, we know that the obtained information is the proposition–information field. On the other hand, $D$ is grey as the specific value of $d^\otimes$ is unknown. Then $D$ is a Grey Hazy Set, and we have that it is a number-covered set of $\otimes$ from **Definition 2.10**. □

**Theorem 2.7.** Supposing that $\otimes_1$, $\otimes_2$ and $\otimes_3$ are three GNs, we get that the operation $\circ \in \{+, \times\}$ among them satisfies the following rules:

1. the commutative law of addition and multiplication, i.e., $\otimes_1 + \otimes_2 = \otimes_2 + \otimes_1$ and $\otimes_1 \times \otimes_2 = \otimes_2 \times \otimes_1$;
2. the combination law of addition and multiplication, i.e., $(\otimes_1 \times \otimes_2) \times \otimes_3 = \otimes_1 \times (\otimes_2 \times \otimes_3)$ and $(\otimes_1 + \otimes_2) + \otimes_3 = \otimes_1 + (\otimes_2 + \otimes_3)$;
3. the distribution law, i.e., $\otimes_1(\otimes_2 + \otimes_3) = \otimes_1\otimes_2 + \otimes_1\otimes_3$.

**Proof.** From **Definition 2.14**, we know that $d''_1 + d''_2$, $d''_3 + d''_4$, $d''_1 \times d''_2$ and $d''_3 \times d''_4$ are the only potential true numbers of $\otimes_1 + \otimes_2$, $\otimes_1 + \otimes_3$, $\otimes_1 \times \otimes_2$ and $\otimes_2 \times \otimes_1$, respectively. Because $d''_1$ and $d''_2$ are real numbers, and the commutative law of addition and multiplication holds for real numbers, we have $d''_1 + d''_2 = d''_2 + d''_1$ and $d''_1 \times d''_2 = d''_2 \times d''_1$. Then $\otimes_1 + \otimes_2 = \otimes_2 + \otimes_1$ and $\otimes_1 \times \otimes_2 = \otimes_2 \times \otimes_1$ hold from **Theorem 2.4**. The proofs of others are similar. □
Example 2.4. Let $\otimes_1$, $\otimes_2$ and $\otimes_3$ be three GNS, and their number-covered sets be $D_1 = \{3\}$, $D_2 = \{0, 2, 9\}$ and $D_3 = [-3, 2]$, respectively. Calculate the number-covered sets of the following GNS: $\otimes_2 - \otimes_2$, $\otimes_1 + \otimes_2$, $\otimes_1 + \otimes_3$, $\otimes_2 + \otimes_3$, $\otimes_1 - \otimes_2$, $\otimes_1 - \otimes_3$, $\otimes_2 - \otimes_3$, $\otimes_1 \otimes_2$, $\otimes_1 \otimes_3$, $\otimes_2 \otimes_3$, $\otimes_2 / \otimes_1$, $\otimes_3 / \otimes_1$.

From Theorem 2.2, we have that the number-covered set of $\otimes_2 - \otimes_2$ is the Evidence Set $\{0\}$.

Note. From Definition 2.15, we know that $D_2 - D_2 = \{-9, -7, -2, 0, 2, 7, 9\}$ is also a number-covered set of $\otimes_2 - \otimes_2$, but it is a relatively inferior one. Because $D_2 - D_2 \Rightarrow \{0\}$ has been realized, we omit the number-covered set $\{-9, -7, -2, 0, 2, 7, 9\}$ (see the set $D_j$ of Definition 2.14).

The number-covered sets of other GNS are as follows, respectively:

- $D_1 + D_2 = \{3, 5, 12\}$, $D_1 + D_3 = \{0, 5\}$, $D_2 + D_3 = [-3, 2] \cup [-1, 4] \cup [6, 11] = [-3, 4] \cup [6, 11]$,
- $D_1 - D_2 = \{-6, 1, 3\}$, $D_1 - D_3 = [1, 6]$, $D_2 - D_3 = [-2, 3] \cup [0, 5] \cup [7, 12] = [-2, 5] \cup [7, 12]$,
- $D_1 D_2 = \{0, 6, 27\}$, $D_1 D_3 = [-9, 6]$, $D_2 D_3 = \{0\} \cup [-6, 4] \cup [-27, 18] = [-27, 18]$,
- $D_2 / D_1 = \{0, 2/3, 3\}$, $D_3 / D_1 = [-1, 2/3]$.

2.4. The difference between the number-covered set of GN and the interval number

The number-covered set of GN is greatly different from the interval number and we point out the difference between them below.

Definition 2.18. For an interval $[a, b]$, if $\forall c \in [a, b]$ is a true number, then we call that $[a, b]$ is an interval number.

Definition 2.19. Supposing that $D_1 = [a^-, a^+]$ and $D_2 = [b^-, b^+]$ are two interval numbers, and $\circ$ is an operator. We call that $D = D_1 \circ D_2 = \{a \circ b | \forall a \in D_1 \text{ and } \forall b \in D_2\}$ is the interval operation.

Theorem 2.8. For an interval number $[a, b]$, we have

1. the result of subtracting itself $[a, b] - [a, b] = [a - b, b - a]$ is an interval number;
2. the result of divided by itself $[a, b] / [a, b] = \left[\min\{a/b, b/a\}, \max\{a/b, b/a\}\right]$ is an interval number, where $0 \notin [a, b]$.

Proof. Because all of numbers in $[a, b]$ are true, we have that all of numbers in $[a - b, b - a]$ are also true. Then $[a - b, b - a]$ is an interval number. The proof of $\left[\min\{a/b, b/a\}, \max\{a/b, b/a\}\right]$ is similar. $\square$

Example 2.5. There are two propositions below:

Proposition 1. The stature of the adults is between 140 cm and 300 cm.

Proposition 2. The stature of someone is between 140 cm and 300 cm.

Because the information about the stature in Proposition 1 is not complete, it is not the proposition–information field from Definition 2.1. The range of numbers $[140, 300]$ is an interval number, and we know that it has not information background from Definition 2.10 (note that the information background of a GN is the proposition–information field). In fact, we certainly know that the stature of someone may be a number of $[140, 300]$, then all numbers of it are true.

On the other hand, the information about the stature in Proposition 2 is complete because every estimated number about his/her stature can be convergent to the true one under the correct information. The obtained information is the proposition–information field. The range of numbers $[140, 300]$ is a number-covered set and there is only one that is true in it.

The difference between the number-covered set of GN and the interval number is below:

1. Information. The GN exists for the proposition and its number-covered set is based on the information background, such as $\lambda(\emptyset)$ and $d^\bullet$. When the information is added, the chaos of GN becomes smaller. When
all of the relevant information appears, the GN vanishes and becomes a true number, and its number-covered set evolves into the Evidence Set. The interval number has no information and it is immovable.

(2) The special form of the operation. The result that a GN subtracts itself is 0, and divides itself is 1 if 0 is not its only potential true number. That is to say, we always have that $D_1 - D_2 \Rightarrow \{0\}$ and $D_1/D_2 \Rightarrow \{1\}$ hold even if the GN has two different number-covered sets $D_1$ and $D_2$. But the result that an interval number subtracts or divides itself is also an interval number.

(3) The true number. It is the only one in the number-covered set of GN. But for the interval number, all elements in it are true.

(4) Form. The number-covered set of GN has many expressive forms. Before the true number appears, it can be expressed by different interval numbers, even by a disperse form. Based on the obtained information, the interval form (if the number-covered set of GN is expressed by it) is dynamic and evolutive. The form of the interval number is sole, which is $[a, b]$.

(5) The sealed property. The operation of GN does not seal for the second aspect and the interval number is adverse.

(6) The operational form. For the CGN, the operational form is the same as the interval number.

To sum up, the numbers of grey systems theory include: The conceptional number; The potential number; True number $d^*$: The only potential true number $d^*$; The GN $\otimes$ and the number-covered set $D$.

3. The grey matrix

The matrix is an important mathematical instrument not only for mathematics, but also for other subjects. The creativity of the matrix theory greatly promotes and enriches the development of other subjects. The appearance and development of many new theories, technologies and methods attribute to the creative application and spread of matrix. In this section, we will propose the grey matrix and some basic operations of it.

3.1. The definition of grey matrix

The definition of grey matrix has been discussed by other researchers [45,46] and is similar to ours.

**Definition 3.1.** For the elements of a matrix $A$, if there exists at least a GN among them, then we call the matrix as grey matrix (GM) and denote it as $A(\otimes)$.

**Remark 3.1.** Apparently, the GM has following properties:

1. The elements of it may be GNs or real numbers and every GN must satisfies the definition 2.10;
2. For a GM $A(\otimes) = (\otimes_{ij})_{m \times n}$, the element $\otimes_{ij}$ exists for the appropriate proposition $\lambda_{ij}$ and its number-covered set is based on the proposition–information field $\lambda_{ij} (\theta) \ (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$. So $A(\otimes)$ exists for all of $\lambda_{ij}$ and its matrix-covered set is based on all of $\lambda_{ij} (\theta)$.

**Definition 3.2.** For a GM $A(\otimes) = (\otimes_{ij})_{m \times n}$, if $D_{ij}$ is the number-covered set of $\otimes_{ij}$ $(i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$, then $A(D) = (D_{ij})_{m \times n}$ is called the matrix-covered set of $A(\otimes)$.

**Definition 3.3.** For $A(\otimes) = (\otimes_{ij})_{m \times n}$, if $d^*_{ij}$ is the only potential true number of $\otimes_{ij}$ $(i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$, then $A^* = (d^*_{ij})_{m \times n}$ is the only potential true matrix of $A(\otimes)$ and $A^* \in A(D)$ holds.

**Definition 3.4.** For $A(\otimes) = (\otimes_{ij})_{m \times n}$ if $\otimes_{ij}$ is the whitened number of $\otimes_{ij}$ $(i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)$, then $A(\hat{\otimes}) = (\hat{\otimes}_{ij})_{m \times n}$ is the whitened matrix of $A(\otimes)$ and $A(\hat{\otimes}) \in A(D)$ holds.

**Definition 3.5.** If there exists at least a GN among the elements of a vector $X$, then we call it a grey vector (GV) and denote it as $X(\otimes)$.
Definition 3.6. Let $X(\otimes) = (\otimes_1, \otimes_2, \ldots, \otimes_n)^T$ be a GV, where T denotes the transpose of a matrix or a vector. Supposing that $x^i_j$, $\otimes_j$ and $D_j$ are the only potential true number, the whitened number and the number-covered set of $\otimes_j$ ($j = 1, 2, \ldots, m$), respectively. Then $X^o = (x^o_1, x^o_2, \ldots, x^o_m)^T$, $X(\otimes) = (\otimes_1, \otimes_2, \ldots, \otimes_m)^T$ and $X(D) = (D_1, D_2, \ldots, D_m)^T$ are the only potential true vector, the whitened vector and the vector-covered set of $X(\otimes)$, respectively, and $X^o, X(\otimes) \in X(D)$ hold.

In fact, a GV is a special GM.

Definition 3.7. Let $A(\otimes)$ be a GM/GV,

1. If there exists at least a DPN among the elements of $A(\otimes)$, then $A(\otimes)$ is a disperse grey matrix/vector (DGM/DGV).
2. If all elements of $A(\otimes)$ are CGNs or real numbers, then $A(\otimes)$ is a continuous grey matrix/vector (CGM/CGV).

Definition 3.8. For a GM $A(\otimes) = (\otimes_{ij})_{m \times n}$, if $\Delta D_{ij}$ is the chaos of $\otimes_{ij}$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$), then we call that $\Delta D = \max_{ij}\{\Delta D_{ij}\}$ is the chaos of $A(\otimes)$.

For the element $\otimes_{ij}$ of a GM/GV $A(\otimes) = (\otimes_{ij})_{m \times n}$, because it may has more than one number-covered set, which is based on the different or same proposition–information field $\lambda_{ij}(\theta)$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$), we know that $A(\otimes)$ may has more than one matrix/vector-covered set.

Definition 3.9. For a GM/GV $A(\otimes) = (\otimes_{ij})_{m \times n}$, if there are two matrix/vector-covered sets $A(D)_1 = (D^1_{ij})_{m \times n}$ and $A(D)_2 = (D^2_{ij})_{m \times n}$, and $A(D)_1 \subseteq A(D)_2$ holds, i.e., $D^1_{ij}$ and $D^2_{ij}$ are different number-covered sets of $\otimes_{ij}$ and $D^1_{ij} \subseteq D^2_{ij}$ holds ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$), then $A(D)_1$ is called the relatively superior matrix/vector-covered set of $A(\otimes)$ and $A(D)_2$ is the relatively inferior one.

Theorem 3.1. Supposing that $A(\otimes) = (\otimes_{ij})_{m \times n}$ is a GM and $A^o = (a^o_{ij})_{m \times n}$ is its only potential true matrix. If there exists a set of matrices $A(D) = (D_{ij})_{m \times n}$ and $A^o \in A(D)$ holds, then $A(D)$ is the matrix-covered set of $A(\otimes)$.

Proof. For $A^o \in A(D)$, we have that $a^o_{ij} \in D_{ij}$ holds for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. From Theorem 2.6, we know that $D_{ij}$ is the number-covered set of $\otimes_{ij}$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$). Then $A(D)$ is the matrix-covered set of $A(\otimes)$ from Definition 3.2.

Example 3.1. Let $\otimes_{11}, \otimes_{12}, \otimes_{21}$ and $\otimes_{22}$ be four GNs, and $D_{11} = [0, 2], D_{12} = \{3\}, D_{21} = \{-1\}$ and $D_{22} = \{0\}$ be their number-covered sets, respectively. If the number 1 is the only potential true number of $\otimes_{11}$, then we get the GM $A(\otimes)$, the matrix-covered set $A(D)$, the only potential true matrix $A^o$ and one of the whitened matrices $A(\otimes)$ below:

$A(\otimes) = \begin{pmatrix} \otimes_{11} & \otimes_{12} \\ \otimes_{21} & \otimes_{22} \end{pmatrix} = \begin{pmatrix} \otimes_{11} & 3 \\ -1 & 0 \end{pmatrix}, \quad A(D) = \begin{pmatrix} [0, 2] & 3 \\ -1 & 0 \end{pmatrix}, \quad A^o = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}, \quad A(\otimes) = \begin{pmatrix} 0.5 & 3 \\ -1 & 0 \end{pmatrix}.$

From Definition 3.7, we know that $A(\otimes)$ above is a CGM. If the number-covered set of $\otimes_{11}$ is $\{1, 2\}$, then $A(\otimes)$ is a DGM and its matrix-covered set is the relatively superior one.

3.2. The covered operation of GM

With the help of Definitions 2.14–2.17 and the matrix theory, we propose the covered operation between GMs in this part.

Definition 3.10. Let $A(\otimes)$ and $B(\otimes)$, $A^o$ and $B^o$ as well as $A(D)$ and $B(D)$ be the whitened matrices, the only potential true matrices and the matrix-covered sets of the GMs $A(\otimes)$ and $B(\otimes)$, respectively. $C(\otimes)$ is the result
between $A(\otimes)$ and $B(\otimes)$ by the operator $\odot$, and $C(D)$ is the matrix-covered set of $C(\otimes)$. If the following equations:

$$A(\otimes) \odot B(\otimes) = C(\otimes); \quad A(D) \odot B(D) = C(D); \quad \forall A(\otimes) \text{ of } A(\otimes) \text{ and } B(\otimes) \text{ of } B(\otimes) \Rightarrow A(\otimes) \odot B(\otimes) \in C(D)$$

hold, where

$$C(D) = \{ C(\otimes) \forall A(\otimes) \text{ of } A(\otimes) \text{ and } B(\otimes) \Rightarrow A(\otimes) \odot B(\otimes) = C(\otimes); \exists A^0 \in A(D) \text{ and } B^0 \in B(D) \Rightarrow C^0 = A^0 \odot B^0 \in A(D) \odot B(D); \quad C^* = A^* \odot B^*; \quad \text{If } C^* \text{ occurs, then } C^0 = C^* \text{, and } C(\otimes) \text{ and } C^0 \text{ vanish at the same time}\},$$

then we call “$\odot$” the covered operation between $A(\otimes)$ and $B(\otimes)$, and denote it as

$$A(\otimes) \odot B(\otimes) \Rightarrow \{ A(D) \odot B(D) \}.$$ 

The definition above shows that the result $C(\otimes)$ is also a GM, and $C^0 \in C(D)$ is its only potential true matrix.

From Definitions 2.15–2.17 and 3.10, we have the specific covered operations of GM below.

**Definition 3.11.** Let $A(\otimes) = (\otimes_{ij})_{m \times n}$ and $B(\otimes) = (\otimes'_{ij})_{m \times n}$ be GMs. $A(D) = (D_{ij})_{m \times n}$ and $B(D) = (D'_{ij})_{m \times n}$ are the matrix-covered sets of $A(\otimes)$ and $B(\otimes)$, respectively. Supposing that $A(D) \odot B(D)$ is the matrix-covered sets of $A(\otimes) \odot B(\otimes)$, where $\odot \in \{+, -, \}$, we have

1. $A(D) + B(D) = (D_{ij} + D'_{ij})_{m \times n}$, where $D_{ij} + D'_{ij}$ \((i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n)\) is just as the (1) of Definitions 2.15–2.17;
2. $A(D) - B(D) = (D_{ij} - D'_{ij})_{m \times n}$, where $D_{ij} - D'_{ij}$ \((i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n)\) is just as the (2) of Definitions 2.15–2.17.

**Definition 3.12.** Let $A(\otimes) = (\otimes_{ij})_{m \times l}$ and $B(\otimes) = (\otimes'_{ij})_{l \times n}$ be GMs. $A(D) = (D_{ij})_{m \times l}$ and $B(D) = (D'_{ij})_{l \times n}$ are the matrix-covered sets of $A(\otimes)$ and $B(\otimes)$, respectively. Then $A(D) \times B(D)$ is the matrix-covered set of $A(\otimes) \times B(\otimes)$ and it is as below:

$$A(D) \times B(D) = \left( \sum_{k=1}^{l} (D_{ik} \times D'_{kj}) \right)_{m \times n},$$

where $D_{ik} \times D'_{kj}$ \((i = 1, 2, \ldots, m; \quad k = 1, 2, \ldots, l; \quad j = 1, 2, \ldots, n)\) is just as the (3) of Definitions 2.15–2.17.

**Remark 3.2.** Supposing that $A^0$ and $B^0$ are the only potential true matrices of $A(\otimes)$ and $B(\otimes)$, respectively. It is apparent that

1. $A^0 + B^0$, $A^0 - B^0$ and $A^0 \times B^0$ are the only potential true matrices of $A(\otimes) + B(\otimes)$, $A(\otimes) - B(\otimes)$ and $A(\otimes) \times B(\otimes)$, respectively;
2. $A^0 + B^0 \in A(D) + B(D)$, $A^0 - B^0 \in A(D) - B(D)$ and $A^0 \times B^0 \in A(D) \times B(D)$ hold.

**Theorem 3.2.** Supposing that $A(\otimes)$ is a GM, we have

$$A(\otimes) - A(\otimes) = O,$$

where $O$ is a matrix that all elements of it are 0.

**Proof.** Supposing that $A(\otimes) = (\otimes_{ij})_{m \times n}$, we have $A(\otimes) - A(\otimes) = (\otimes_{ij} - \otimes_{ij})_{m \times n} = (0)_{m \times n} = O$ from Theorem 2.2.

Specially, supposing that $A(\otimes)$ is a square GM, and denoting

$$A(\otimes)^k = A(\otimes) \times A(\otimes) \times \cdots \times A(\otimes) \quad \text{and} \quad kA(\otimes) = A(\otimes) + A(\otimes) + \cdots + A(\otimes),$$


where \( k \) is the sum of \( A(\otimes) \), we have that

\[
A(D)^k = A(D) \times A(D) \times \cdots \times A(D) \quad \text{and} \quad kA(D) = A(D) + A(D) + \cdots + A(D)
\]

are their matrix-covered sets, respectively. \( \Box \)

**Theorem 3.3.** Supposing that \( A(\otimes) \) is a square GM, we have

\[
A(\otimes)^k - A(\otimes)^k = O \quad \text{and} \quad kA(\otimes) - kA(\otimes) = O.
\]

**Proof.** It is apparent from Theorems 2.4, 2.5, 2.7, 3.2 and Definition 3.10. \( \Box \)

**Remark 3.3.** Let \( A(\otimes) = (\otimes_{ij})_{m \times n} \) be a GM. \( A^o = (a^o_{ij})_{m \times n} \) and \( A(D) = (D_{ij})_{m \times n} \) are the only potential true matrix and matrix-covered set of \( A(\otimes) \), respectively. If \( d \) is a real number, then \( dA(\otimes) = (d \otimes_{ij})_{m \times n} \) is a GM, and \( dA(D) = (DD_{ij})_{m \times n} \) and \( dA^o = (da^o_{ij})_{m \times n} \) are its matrix-covered set and only potential true matrix, respectively.

On the other hand, supposing that \( \otimes \) is a GN. \( D \) and \( d^o \) are the number-covered set and only potential true number of \( \otimes \), respectively. We have that \( \otimes A(\otimes) \) is a GM, and \( DA(D) = (DD_{ij})_{m \times n} \) and \( d^o A^o = (da^o_{ij})_{m \times n} \) are its matrix-covered set and only potential true matrix, respectively.

**Theorem 3.4.** Supposing that \( A(\otimes) = (\otimes_{ij})_{m \times n} \) and \( B(\otimes) = (\otimes_{ij})_{m \times n} \) are GMs. \( A^o = (a^o_{ij})_{m \times n} \) and \( B^o = (b^o_{ij})_{m \times n} \) are the only potential true matrices of \( A(\otimes) \) and \( B(\otimes) \), respectively. We have

\[
A(\otimes) = B(\otimes) \iff A^o = B^o.
\]

**Proof.** From Theorem 2.4, we have \( A(\otimes) = B(\otimes) \iff \otimes_{ij} = \otimes'_{ij} \) for all \( i \in \{1,2,\ldots,m\} \) and \( j \in \{1,2,\ldots,n\} \iff d^o_{ij} = (d^o)_{ij}' \) for all \( i \in \{1,2,\ldots,m\} \) and \( j \in \{1,2,\ldots,n\} \iff A^o = B^o. \)

**Theorem 3.5.** Supposing that \( A_1(\otimes), A_2(\otimes) \) and \( A_3(\otimes) \) are three GMs, we have the following rules:

1. **the commutative law of addition**, i.e., \( A_1(\otimes) + A_2(\otimes) = A_2(\otimes) + A_1(\otimes) \);
2. **the combination law of addition and multiplication**, i.e., \( [A_1(\otimes) + A_2(\otimes)] + A_3(\otimes) = A_1(\otimes) + [A_2(\otimes) + A_3(\otimes)] \) and \( [A_1(\otimes) \times A_2(\otimes)] \times A_3(\otimes) = A_1(\otimes) \times [A_2(\otimes) \times A_3(\otimes)] \);
3. **the distribution law**, i.e., \( A_1(\otimes)[A_2(\otimes) + A_3(\otimes)] = A_1(\otimes)A_2(\otimes) + A_1(\otimes)A_3(\otimes) \) and \( [A_1(\otimes) + A_2(\otimes)]A_3(\otimes) = A_1(\otimes)A_3(\otimes) + A_2(\otimes)A_3(\otimes) \).

**Proof.** Let \( A_1^o \) and \( A_2^o \) be the only potential true matrices of \( A_1(\otimes) \) and \( A_2(\otimes) \), respectively. From Definition 3.10, we know that \( A_1^o + A_2^o \) and \( A_1^o + A_2^o \) are the only potential true matrices of \( A_1(\otimes) + A_2(\otimes) \) and \( A_1(\otimes) + A_2(\otimes) \), respectively. Because \( A_1^o \) and \( A_2^o \) are real matrices, and the commutative law of addition holds for real ones, we have \( A_1^o + A_2^o = A_2^o + A_1^o \). Then \( A_1(\otimes) + A_2(\otimes) = A_2(\otimes) + A_1(\otimes) \) holds from Theorem 3.4. The proofs of others are similar. \( \Box \)

### 3.3. The inverse grey matrix

The inverse matrix has an important role in matrix theory, so does the inverse grey matrix in grey matrix theory. And we discuss it below.

**Definition 3.13.** Supposing that \( A(\otimes) \) is a square GM and its only potential true matrix \( A^o \) is inverse. If there exists a square GM \( B(\otimes) \) and the inverse matrix \( (A^o)^{-1} \) is its only potential true matrix, then we call \( B(\otimes) \) as the inverse grey matrix of \( A(\otimes) \) and denote it as \( (A(\otimes))^{-1} \).

**Remark 3.4.** Apparently, the inverse grey matrix \( (A(\otimes))^{-1} \) must exists because the specific value of \( (A^o)^{-1} \) is unknown. That is to say, if the only potential true matrix \( A^o \) of \( A(\otimes) \) is inverse, then \( A(\otimes) \) is inverse and the only potential true matrix of \( A(\otimes)^{-1} \) is \( (A^o)^{-1} \).
Theorem 3.6. Supposing that $A(\boxtimes)$ is a square GM and $A(D)$ is its matrix-covered set. If $\forall A \in A(D)$ is inverse, then $A(\boxtimes)$ is inverse.

Proof. The only potential true matrix $A^\circ$ is inverse for $A^\circ \in A(D)$, so $A(\boxtimes)$ is inverse. □

Remark 3.5. In fact, we do not know the value of the only potential true matrix of $A(\boxtimes)$, so the theorem is an effective method to justify whether $A(\boxtimes)$ is inverse or not.

Definition 3.14. Let $A(\boxtimes)$ be a square GM and $A(D)$ be its matrix-covered set. If $A(\boxtimes)$ is inverse, then we denote the matrix-covered set of $A(\boxtimes)^{-1}$ as $A(D)^{-1}$.

Theorem 3.7. Supposing that the square GM $A(\boxtimes)$ is inverse and $A^\circ$ is its only potential true matrix. If there exists a set of matrices $A'(D)$ and $(A^\circ)^{-1} \in A'(D)$ holds, then $A'(D)$ is the matrix-covered set of $A(\boxtimes)^{-1}$.

Proof. From Definition 3.13, we know that $(A^\circ)^{-1}$ is the only potential true matrix of $A(\boxtimes)^{-1}$. From Theorem 3.1, we have that $A'(D)$ is the matrix-covered set of $A(\boxtimes)^{-1}$ for $(A^\circ)^{-1} \in A'(D)$. □

Note. It is not always true that $\forall A(\boxtimes)'$ is the inverse matrix of $A(\boxtimes)$, where $A(\boxtimes)' \in A(D)^{-1}$ and $A(\boxtimes) \in A(D)$ are the whitened matrices of $A(\boxtimes)$ and $A(\boxtimes)^{-1}$, respectively.

Theorem 3.8. If the square GM $A(\boxtimes)$ is inverse, then we have

$$
A(\boxtimes) \times A(\boxtimes)^{-1} = I,
$$

where $I$ is the identity matrix.

Proof. Let $A(\boxtimes) \times A(\boxtimes)^{-1} = C(\boxtimes)_A$. $C(D)$ is the matrix-covered set of $C(\boxtimes)_A$. Supposing that $A^\circ$ and $(A^\circ)^{-1}$ are the only potential true matrices of $A(\boxtimes)$ and $A(\boxtimes)^{-1}$, respectively.

From Definition 3.10, we have

$$
A^\circ \times (A^\circ)^{-1} = I \in C(D),
$$

i.e., the true matrix $I$ appears, then $A(\boxtimes) \times A(\boxtimes)^{-1} = I$ holds. □

Example 3.2. Let $A$, $B$ and $C$ be matrix-covered sets of $A(\boxtimes)$, $B(\boxtimes)$ and $C(\boxtimes)$, respectively, where

$$
A = \begin{pmatrix} 
0 & 2 & 4 \\
-1 & 2 & 3 \\
-1 & 5 & 0 \\
\end{pmatrix}, \quad 
B = \begin{pmatrix} 
3 & 4 & -2 & 3 & 2 \\
0 & 1 & 9 & 0 & 1 \\
10 & 15 & -3 & 0 & 4 \\
\end{pmatrix}, \quad 
C = \begin{pmatrix} 
1 & 2 & 1 & 4 & -2 & 7 \\
0 & 3 & 4 & 5 & 6 \\
-1 & 0 & -4 & -3 & -4 & -1 \\
\end{pmatrix}.
$$

Calculate the matrix-covered sets of the following GMs:

(1) $A(\boxtimes) - A(\boxtimes)$; (2) $A(\boxtimes) - B(\boxtimes)$; (3) $A(\boxtimes) + C(\boxtimes)$; (4) $A(\boxtimes) \times B(\boxtimes)$.

From Theorem 3.2, we have that the matrix-covered set of $A(\boxtimes) - A(\boxtimes)$ is $\{0\}$.

Note. From Definition 3.11, we know that

$$
A - A = \begin{pmatrix} 
-2 & 0 & 2 & -6 & 6 \\
-3 & 3 & -1 & 1 \\
-6 & 6 & -2 & 0 & 2 \\
\end{pmatrix}
$$

is also a matrix-covered set of $A(\boxtimes) - A(\boxtimes)$, but it is a relatively inferior one. Because the true matrix $O$ appears, we omit $A - A$ from Definition 3.10.
The matrix-covered sets of others are as follows:

\[ A - B = \begin{pmatrix}
-4, -3 \cup [-2, -1] & [0, 11] & [1, 2] \\
-1, 2 & [6, 7] \cup [-2, -1] & [2, 3] \cup [3, 4] \\
-16, -5 & [0, 3] \cup [-2, 1] & [-1, 2]
\end{pmatrix} \]

\[ A + C = \begin{pmatrix}
(1, 2, 3, 4) & [4, 13] & [2, 11] \\
[-1, 2] & [10, 12] & [8, 9] \cup [9, 10] \\
[-2, 5] & (-4, -3, -6, -5) & [-1, 5]
\end{pmatrix} \]

\[ A \times B = \begin{pmatrix}
(0) \cup [6, 8] + 0 + [40, 60] & (0) \cup [-4, 6] + [3, 9] \cup [27, 81] + [-12, 0] & (0) \cup [4, 6] + [0, 9] + 16 \\
[-4, 8] + 0 + [30, 45] \cup [40, 60] & [-4, 6] + [7, 8] \cup [63, 72] + [-9, 0] \cup [-12, 0] & [-3, 6] + [0, 8] + [12, 16] \\
[-4, 20] + 0 + [30, 90] & [-10, 15] + \{0, -2, -18\} + [-18, 0] & [-3, 15] + \{0\} \cup [-2, 0] + [12, 24]
\end{pmatrix} \]

\[ A \times B = \begin{pmatrix}
\end{pmatrix} \]

\[ A \times B = \begin{pmatrix}
[40, 60] & [-13, 87] & [16, 31] \\
[26, 68] & [-9, 14] \cup [47, 78] & [22, 56] \\
\end{pmatrix} \]

### 3.4. The interval matrix and the difference between it and the matrix-covered set of GM

Just as the fact that the interval number is different from the number-covered set of GN, the interval matrix is also greatly different from the matrix-covered set of GM. In this part, we will give the definition of the interval matrix and discuss the difference between them.

**Definition 3.15.** If \([A = (a_{ij}, b_{ij})]_{m \times n}\), where \([a_{ij}, b_{ij}]\) is an interval number that lies in low \(i\) and column \(j\) of \([A]\) \((i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)\), then \([A]\) is an interval matrix.

**Remark 3.6.** Supposing that \([A = (a_{ij}, a_{ij}^-)_{m \times n}\) and \([B = (b_{ij}, b_{ij}^+)_{m \times n}\) are two interval matrices, we have that

1. \([A]\) is a real matrix if and only if \(a_{ij}^- = a_{ij}^+\) holds for all \(i \in \{1, 2, \ldots, m\}\) and \(j \in \{1, 2, \ldots, n\}\);
2. \([A] = [B]\) if and only if \(a_{ij}^- = b_{ij}^-\) and \(a_{ij}^+ = b_{ij}^+\) hold for all \(i \in \{1, 2, \ldots, m\}\) and \(j \in \{1, 2, \ldots, n\}\).

**Theorem 3.9.** Supposing that \([A = (a_{ij}, b_{ij})_{m \times n}\) is an interval matrix, then the result of \([A] - [A]\) is also an interval matrix.

**Proof.** From Theorem 2.8, we have

\([A] - [A] = [(a_{ij} - b_{ij}, b_{ij} - a_{ij})]_{m \times n}\). Because all of \([a_{ij} - b_{ij}, b_{ij} - a_{ij}]\) \((i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, n)\) are interval numbers, the theorem holds. \(\square\)
Example 3.3. Supposing that there are two sectors for the regional economy. We have two propositions in the following:

**Proposition 3.** For anyone of a large of regions, the values of the input/output between the two sectors are in their appropriate intervals below, i.e., $[A]$ (the unite is million dollars).

\[
[A] = \begin{pmatrix}
\text{sector 1} & \text{sector 2} \\
14,78 & 22,56 \\
8,13 & 7,39
\end{pmatrix}.
\]

The information about the values in Proposition 3 is not complete (see Example 2.5), so $[A]$ is an interval matrix. On the other hand, the information in Proposition 4 is complete and $[A]$ is a matrix-covered set.

**Proposition 4.** For a given region, the values of the input/output between the two sectors are in their appropriate intervals, i.e., $[A]$, which is the same as Proposition 3.

The difference between the matrix-covered set of GM and the interval matrix is as follows:

1. Information. The number-covered set of very element in a GM exists for its appropriate proposition–information field $\lambda_i(0)$, so the matrix-covered set of GM also has the dynamic and evolutive property. As the information is added, the chaos of GM becomes smaller. When all of the relevant information appears, the matrix-covered set of GM should evolve into the Evidence Set and the true matrix is obtained. The interval matrix has no information.

2. The special situation of the operation. For the GM $A(\otimes)$ and its matrix-covered sets $A(D)_1$ and $A(D)_2$, we have that the matrix-covered set $A(D)_1 - A(D)_2$ or $A(D)_2 - A(D)_1$ of $A(\otimes) - A(\otimes)$ must evolve into $\{O\}$. That is to say, $A(D)_1 - A(D)_2 \Rightarrow \{O\}$ and $A(\otimes) - A(\otimes) = O$ holds. But the result of an interval matrix subtracting itself is still an interval matrix.

3. The true matrix. It is the only one in the matrix-covered set of GM, but all the matrices in the interval matrix are true.

4. Form. The matrix-covered set of GM has many forms. Based on the proposition–information fields of the GM’s elements, it can be expressed by different interval matrices (if all of its elements are continuous) before the true matrix appears. On the other hand, the matrix-covered set of GM also has disperse forms. It is dynamic and must evolve into the Evidence Set at last. But the form of interval matrix is sole and fixed.

5. The sealed property. The operation of GM does not seal for the second aspect and the interval matrix is adverse.

6. The operational form. For the CGM, the operational form is the same as the interval matrix.

To sum up, the matrices of grey systems theory include: True matrix $A^*$ and the only potential true matrix $A^\circ$; The GM $A(\otimes)$; The whitened matrix $A(\otimes)$ and the matrix-covered set $A(D)$.

4. Grey input–output analysis

The input–output analysis has been a scientific method and effective tool to study the economical balance and forecast the development of future economy. Combining traditional input–output analysis with grey matrix theory given in section above, we propose the grey input–output analysis for uncertainty. There are two parts below. We introduce the traditional value-type input–output analysis and get some results in Part I, and give our method in Part II.

4.1. The introduction of traditional value-type input–output analysis

Suppose that there are $n$ sectors in an economic system, and the unit of statistic datum is money, i.e., the value type (see Table 1).
The table of value-type input–output analysis

<table>
<thead>
<tr>
<th>The value of intermediate input of sector i</th>
<th>The value of final use</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>The value of intermediate output of sector j</td>
<td>Y_i (i = 1, 2, ..., n)</td>
<td>X_i (i = 1, 2, ..., n)</td>
</tr>
<tr>
<td>Total value-added</td>
<td>N_j (j = 1, 2, ..., n)</td>
<td></td>
</tr>
<tr>
<td>Total input</td>
<td>X_j (j = 1, 2, ..., n)</td>
<td></td>
</tr>
</tbody>
</table>

The meaning of the variables in Table 1 is as follows:

- \(x_{ij}\): The value of Intermediate Input that sector \(j\) throws into sector \(i\) or Intermediate Output that sector \(i\) supplies sector \(j\).
- \(Y_i\): The value of Final Use of sector \(i\). It includes the value of Total Final Consumption Expenditure, Capital Formation and Net Export, etc.
- \(N_j\): Total Value-added of sector \(j\). It includes the depreciation of Fixed Assets, Compensation for Laborers, Net Taxes on Production and Operating Surplus, etc.
- \(X_i/X_j\): The value of Total Input/Output of sector \(i/j\) and

\[
X_i = \sum_{j=1}^{n} x_{ij} + Y_i \quad \text{or} \quad X_j = \sum_{i=1}^{n} x_{ij} - N_j \quad (i, j = 1, 2, \ldots, n).
\]  

(4.1)

Denoting \(X = (X_1, X_2, \ldots, X_n)^T\) and \(Y = (Y_1, Y_2, \ldots, Y_n)^T\), we can get the following input–output low-model from the lows of Table 1:

\[
(I - A)X = Y,
\]  

(4.2)

where \(A = (a_{ij})_{n \times n}\) is the direct consumption coefficient matrix and \(a_{ij} = \frac{X_j}{Y_i} (i, j = 1, 2, \ldots, n)\) is the direct consumption coefficient.

For the value-type input–output analysis, the direct consumption coefficient \(a_{ij}\) satisfies

\[
0 \leq a_{ij} < 1 \quad \text{and} \quad \sum_{i=1}^{n} a_{ij} < 1 \quad (i, j = 1, 2, \ldots, n).
\]  

(4.3)

This is the famous Solow condition.

**Theorem 4.1.** The matrix \(I - A\) is inverse if all elements of \(A\) satisfy Eq. (4.3).

From the direct consumption coefficient matrix \(A\), we can get other important coefficients below:

1. \(B = (I - A)^{-1} - I = (b_{ij})_{n \times n}\) is called the complete consumption coefficient matrix, where \(b_{ij} (i, j = 1, 2, \ldots, n)\) is the complete consumption coefficient.
2. \(B = (I - A)^{-1} = (b_{ij})_{n \times n}\) is called the complete required coefficient matrix, where \(b_{ij} (i, j = 1, 2, \ldots, n)\) is the complete required coefficient.
3. \(F_j = \sum_{i=1}^{n} \frac{b_{ij}}{b}\) is the influence coefficient of sector \(j (j = 1, 2, \ldots, n)\), where \(b = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\).
4. \(E_i = \sum_{j=1}^{n} \frac{b_{ij}}{b}\) is the induction coefficient of sector \(i (i = 1, 2, \ldots, n)\).

The following lemma has been proved in matrix theory.

**Lemma 4.1.** For a square matrix \(B\), if \(\|B\|_p < 1\) \((1 \leq p \leq +\infty)\) holds, where \(\|\cdot\|_p\) is the \(p–\text{norm of a matrix}\) vector, then we have

\[
(I - B)^{-1} = I + B + B^2 + \cdots + B^k + \cdots
\]

**Theorem 4.2.** For the direct consumption coefficient matrix \(A = (a_{ij})_{n \times n}\), we have

\[
(I - A)^{-1} = I + A + A^2 + \cdots + A^k + \cdots
\]
Proof. From Eq. (4.3), we have

$$\|A\|_{col} = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} a_{ij} \right\} = a^* < 1,$$

where $\|\cdot\|_{col}$ is the column-sum norm of a matrix/vector. Then the theorem holds from Lemma 4.1. \qed

### 4.2. The grey input–output analysis

When we control or manage an economic system using the input–output analysis, the statistic datum are usually required to be exact and it is difficult to do in reality. For example, when ascertaining Intermediate Output/Input $x_{ij}$, we could only estimate its range, i.e., $[x_{ij}] = [x_{ij}^-, x_{ij}^+]$. But the true number $x_{ij}^*$ must be affirmed in $[x_{ij}^-, x_{ij}^+]$ under the correct information, i.e., the proposition–information field $\lambda_{ij}(\theta)$. Under the situation, we can get a grey number $x_{ij}(\otimes)$ and its number-covered set $[x_{ij}]$. It is the same for Final Use, Total Value-added and Total Input/Output, and we denote these grey numbers as $Y_i(\otimes), N_j(\otimes)$ and $X_i(\otimes)/X_j(\otimes)$, and their number-covered sets as $[Y_i] = [Y_i^-, Y_i^+]$, $[N_j] = [N_j^-, N_j^+]$ and $[X_i] = [X_i^-, X_i^+]$ $(i, j = 1, 2, \ldots, n)$, respectively (see Table 2), where

$$X_i^+ = \sum_{j=1}^{n} x_{ij}^+ + Y_i^- \quad \text{and} \quad X_i^- = \sum_{j=1}^{n} x_{ij}^- + Y_i^+$$

or

$$X_j^- = \sum_{i=1}^{n} x_{ij}^- + N_j^- \quad \text{and} \quad X_j^+ = \sum_{i=1}^{n} x_{ij}^+ + N_j^+.$$

Note that the number-covered sets $[X_i^-, X_i^+]$ $(j = 1, 2, \ldots, n)$ must be identical with the corresponding ones $[X_i^-, X_i^+]$ $(i = 1, 2, \ldots, n)$.

Supposing that $X(\otimes) = (X_1(\otimes), X_2(\otimes), \ldots, X_n(\otimes))^T$ and $Y(\otimes) = (Y_1(\otimes), Y_2(\otimes), \ldots, Y_n(\otimes))^T$ are grey vectors of Total Output/Input and Final Use, and $[X] = ([X_1], [X_2], \ldots, [X_n])^T$ and $[Y] = ([Y_1], [Y_2], \ldots, [Y_n])^T$ are their vector-covered sets, respectively.

From Eq. (4.2), we get the following grey model:

$$(I - A(\otimes))X(\otimes) = Y(\otimes),$$

where $A(\otimes) = (a_{ij}(\otimes))_{n \times n}$ is the grey direct consumption coefficient matrix, and $a_{ij}(\otimes) = \frac{x_{ij}(\otimes)}{X(\otimes)}$ $(i, j = 1, 2, \ldots, n)$ is the grey direct consumption coefficient.

The Eq. (4.6) is called grey input–output model and its covered model is below:

$$(I - [A])[X] = [Y],$$

where $[A]$ is the matrix-covered set of $A(\otimes)$ and it has following two forms:

1. $[A] \triangleq [A]_1 = ([a_{ij}^*, a_{ij}^+])_{n \times n}$, where

$$a_{ij}^- = \frac{x_{ij}^-}{X_j^-} \quad \text{and} \quad a_{ij}^+ = \frac{x_{ij}^+}{X_j^+} \quad (i, j = 1, 2, \ldots, n).$$

<table>
<thead>
<tr>
<th>Table 2</th>
<th>The number-covered table of grey value-type input–output analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>The number-covered set of intermediate output of sector $j$</td>
<td>The number-covered set of final use</td>
</tr>
<tr>
<td>$[x_{ij}^-, x_{ij}^+]$ $(i = 1, 2, \ldots, n)$</td>
<td>$[Y_i^-, Y_i^+]$ $(i = 1, 2, \ldots, n)$</td>
</tr>
<tr>
<td>The number-covered set of intermediate input of sector $i$</td>
<td></td>
</tr>
<tr>
<td>$[N_j^-, N_j^+]$ $(j = 1, 2, \ldots, n)$</td>
<td></td>
</tr>
<tr>
<td>The number-covered set of total value-added of sector $j$</td>
<td></td>
</tr>
<tr>
<td>$[X_i^-, X_i^+]$ $(j = 1, 2, \ldots, n)$</td>
<td></td>
</tr>
<tr>
<td>The number-covered set of total input</td>
<td></td>
</tr>
</tbody>
</table>
Denote $\Delta x_{ij} = x^+_j - x^-_j$, $a_{ij} = \frac{x^+_j}{X_j^+ - \Delta x_{ij}}$ and $a^+_j = \frac{x^+_j}{X_j^+ + \Delta x_{ij}}$ \((i, j = 1, 2, \ldots, n)\). \hspace{1cm} (4.8)

It is apparent that $[A]_1$ is the matrix-covered set of $A(\otimes)$ from Definition 2.16, and we will verify that $[A]_2$ is also such one in the following.

**Theorem 4.3.** $[A]_2$ is a matrix-covered set of grey direct consumption coefficient matrix $A(\otimes)$.

**Proof.** Denote $\Delta N_j = N^+_j - N^-_j$ \((j = 1, 2, \ldots, n)\). Let $x_{ij}(\theta_i) = x^-_{ij} + \Delta x_{ij}\theta_i$ and $N_{j0}(\theta^\prime_{j0}) = N^-_{j0} + \Delta N_{j0}\theta^\prime_{j0}$, where $\theta_i, \theta^\prime_{j0} \in [0, 1]$ for the given $j_0$ \((i, j_0 = 1, 2, \ldots, n)\). From Eqs. (4.1) and (4.5), we have the total input/output

$$X_{j0}(\theta_1, \theta_2, \ldots, \theta_n, \theta^\prime_{j0}) = \sum_{k=1}^{n} x_{kj0} + N_{j0} = X^-_j + \Delta N_{j0}\theta^\prime_{j0} + \sum_{k=1}^{n}(\Delta x_{kj0}\theta_k),$$

and the direct consumption coefficient

$$a_{ij0}(\theta_1, \theta_2, \ldots, \theta_n, \theta^\prime_{j0}) = \frac{x_{ij0}(\theta_i)}{X_{j0}(\theta_1, \theta_2, \ldots, \theta_n, \theta^\prime_{j0})} = \frac{x^-_{ij0} + \Delta x_{ij0}\theta_i}{X^-_j + \Delta N_{j0}\theta^\prime_{j0} + \sum_{k=1}^{n}(\Delta x_{kj0}\theta_k)}. \hspace{1cm} (4.9)$$

In order to obtain the range of $a_{ij0}(\theta_1, \theta_2, \ldots, \theta_n, \theta^\prime_{j0})$, we should calculate its partial differential.

$$\frac{\partial(a_{ij0}(\theta_1, \theta_2, \ldots, \theta_n, \theta^\prime_{j0}))}{\partial \theta_i} = \begin{cases} \frac{\Delta x_{ij0}(X^-_j - x^-_{ij0}) + \Delta N_{j0}\theta^\prime_{j0} + \sum_{k \neq i}(\Delta x_{kj0}\theta_k)}{(X^-_j + \Delta N_{j0}\theta^\prime_{j0} + \sum_{k=1}^{n}(\Delta x_{kj0}\theta_k))^2} > 0, & l = i; \\ -\frac{\Delta x_{ij0}}{(X^-_j + \Delta N_{j0}\theta^\prime_{j0} + \sum_{k=1}^{n}(\Delta x_{kj0}\theta_k))^2} < 0, & l \neq i. \end{cases}$$

From the equation above, we get the largest value when $\theta_i = 1, \theta_k = 0 (k = 1, 2, \ldots, n; k \neq i)$ and $\theta^\prime_{j0} = 0$ and the smallest one when $\theta_i = 0, \theta_k = 1 (k = 1, 2, \ldots, n; k \neq i)$ and $\theta^\prime_{j0} = 1$ of $a_{ij0}(\theta_1, \theta_2, \ldots, \theta_n, \theta^\prime_{j0})$. Then the number-covered set of $a(\otimes)_{ij0}(i, j_0 = 1, 2, \ldots, n)$ can be obtained from Eq. (4.9),

$$\left[\begin{array}{c} x^-_{ij0} + \Delta x_{ij0} \\ X^-_j + \Delta N_{j0} + \sum_{k \neq i}(\Delta x_{kj0}) \\ X^-_j + \Delta x_{ij0} \end{array}\right] = \left[\begin{array}{c} x^-_{ij0} + \Delta x_{ij0} \\ X^-_j + \Delta N_{j0} + \sum_{k \neq i}(\Delta x_{kj0}) \\ X^-_j + \Delta x_{ij0} \end{array}\right],$$

and $[A]_2$ is the matrix-covered set of grey direct consumption coefficient matrix $A(\otimes)$.

It is apparent that $[A]_2 \subset [A]_1$ holds, i.e., $[A]_2$ is the relatively superior matrix-covered set of $A(\otimes)$ and $[A]_1$ is the relatively inferior one.

Substituting $[A]$ with $[A]_1$ and $[A]_2$, respectively, we get two covered models from Eq. (4.7) below:

$$\begin{align*}
(I - [A]_1)[X] &= [Y] & \text{and} & & (I - [A]_2)[X] &= [Y]. \hspace{1cm} \square
\end{align*}$$

**Remark 4.1.** Although $[A]$ $([A] = [A]_1$ or $[A]_2)$ is the matrix-covered set of $A(\otimes)$, we do not know that $\forall A = (\tilde{a}_{ij})_{n \times n} \in [A]$ is the whitened matrix of $A(\otimes)$, even if $\tilde{a}_{ij}$ satisfies Eq. (4.3) for all $i, j \in \{1, 2, \ldots, n\}$. The reason is that $X_i$ is dependent on the values of $x_{ij}$ and $Y_i$ (or $N_j$) \((i, j = 1, 2, \ldots, n)\). Denoting $A^+ = (a^+_ij)_{n \times n}$ and $A^- = (a^-ij)_{n \times n}$, we do not know that the following equation:

$$\|A^+\|_{col} = \max_{1 \leq i, j \leq n} \left\{ \sum_{i=1}^{n} a^+ij \right\} < 1 \hspace{1cm} (4.10)$$

is always true, but it maybe holds under the given condition.
Lemma 4.2. Supposing that $a$, $b$, $A_1$ and $A_2$ are non-negative real numbers, and $a < b$ holds, we have

$$\frac{a + A_1}{b + A_1} \leq \frac{a + A_2}{b + A_2} \iff A_1 \leq A_2.$$ 

Proof. For $a < b$ and the following equation:

$$\frac{a + A_1}{b + A_1} - \frac{a + A_2}{b + A_2} = \frac{(a + A_1)(b + A_2) - (a + A_2)(b + A_1)}{(b + A_1)(b + A_2)} = \frac{(b - a)(A_1 - A_2)}{(b + A_1)(b + A_2)},$$

we have

$$\frac{a + A_1}{b + A_1} - \frac{a + A_2}{b + A_2} \leq 0 \iff A_1 - A_2 \leq 0,$$

so the lemma holds. \(\Box\)

Remark 4.2. From Lemma 4.2, we have that $a_{ij}^- = \frac{x_{ij}}{X_i}$ and $a_{ij}^+ = \frac{x_{ij}^+}{X_i + \Delta x_{ij}}$ are smaller while $\Delta x_{ij}$ becomes smaller $(i, j = 1, 2, \ldots, n)$. Then we should obtain the statistic datum as accuracy as possible. On the other hand, Eq. (4.10) is usually satisfied in a large economic system as the value of $N_j$ or $Y_i (i, j = 1, 2, \ldots, n)$ is large.

Theorem 4.4. Supposing that $A(\otimes)$ is grey direct consumption coefficient matrix, we have that $I - A(\otimes)$ is inverse and its inverse grey matrix is below:

$$(I - A(\otimes))^{-1} = I + A(\otimes) + A(\otimes)^2 + \cdots$$ (4.11)

Proof. Supposing that $A^0 = (a_{ij}^0)_{n \times n}$ is the only potential true matrix of $A(\otimes)$. Because $a_{ij}^0 (i, j = 1, 2, \ldots, n)$ satisfies Eq. (4.3), we get that $I - A(\otimes)$ is inverse from Theorem 4.2 and Definition 3.13.

From Theorems 3.2 and 3.5, we have

$$(I - A(\otimes))(I + A(\otimes) + A(\otimes)^2 + \cdots) = I(I + A(\otimes) + A(\otimes)^2 + \cdots) - A(\otimes)(I + A(\otimes) + A(\otimes)^2 + \cdots)$$

$$= I + A(\otimes) + A(\otimes)^2 + \cdots - A(\otimes) - A(\otimes)^2 - \cdots$$

$$= I + (A(\otimes) - A(\otimes)) + (A(\otimes)^2 - A(\otimes)^2) + \cdots$$

$$= I.$$

Then Eq. (4.11) holds from Theorem 3.8. \(\Box\)

Theorem 4.5. The matrix-covered set $(I - [A])^{-1}$ of the inverse grey matrix $(I - A(\otimes))^{-1}$ is below:

$$(I - [A])^{-1} = I + [A] + [A]^2 + \cdots$$

Proof. Because the matrix-covered set of $(A(\otimes))^k = [A]^k$ $(k = 1, 2, \ldots)$, we have that $I + [A] + [A]^2 + \cdots$ is the matrix-covered set of $I + A(\otimes) + (A(\otimes))^2 + \cdots$, i.e., $(I - A(\otimes))^{-1}$. \(\Box\)

Denoting $A^- = (a_{ij}^-)_{n \times n}$, $A^+ = (a_{ij}^+)_{n \times n}$, $A^-(\otimes)^k = (a_{ij}^{-(\otimes)}^k)_{n \times n}$ and $(A^+)^k = (a_{ij}^{+(\otimes)}^k)_{n \times n}$ $(k = 2, 3, \ldots)$, we get the following theorems to calculate the matrix-covered set $(I - [A])^{-1}$ of $(I - A(\otimes))^{-1}$.

Theorem 4.6. If Eq. (4.10) holds, then $\forall \epsilon > 0$, there must exists an positive integer $k_0$, and the matrix-covered set $(I - [A])^{-1}$ of $(I - A(\otimes))^{-1}$ is as below:

$$(I - [A])^{-1} \triangleq (I - [A])^{-1} = I + [A] + [A]^2 + \cdots + [A]^{k_0} + ([0, \epsilon])_{n \times n}.$$ (4.12)

Proof. If $(A^+)^{k_0} = O$ holds for a real number $k_0 > 0$, then the theorem is true.
Otherwise, because the equations
\[ 0 \leq a_{ij}^+ \leq a^* = \| A^+ \|_{\text{col}} < 1 \quad \text{and} \quad \|(A^+)^k\|_{\text{col}} \leq (\| A^+ \|_{\text{col}})^k = (a^*)^k \]
hold, we have
\[ 0 \leq a_{ij}^+ \leq (a^*)^k \quad (i, j = 1, 2, \ldots, n; k = 2, 3, \ldots), \]
and then
\[ 0 \leq a_{ij}^+ + a_{ij}^{(k+1)+} + \cdots \leq (a^*)^k + (a^*)^{k+1} + \cdots = \frac{(a^*)^k}{1 - a^*} < \varepsilon \iff k > \frac{\ln(1 - a^*)}{\ln a^*}. \]

Supposing that \( k_0 = \text{int}\left(\frac{\ln(1 - a^*)}{\ln a^*}\right) + 1 \), where the value of \( \text{int}(x) \) is an integer which is not larger than \( x \). From the equation above, we have
\[ (A^+)^{k_0+1} + (A^+)^{k_0+2} + \cdots \in ([0, \varepsilon])_{n \times n}, \]
For \( O \leq (A^*)^k \leq (A^+)^k \) \( (k = 2, 3, \ldots) \), we also have
\[ (A^+)^{k_0+1} + (A^+)^{k_0+2} + \cdots \in ([0, \varepsilon])_{n \times n}, \]
so the following equation:
\[ (I - A^+)^{-1} \in I + [A] + [A]^2 + \cdots + [A]^{k_0} + ([0, \varepsilon])_{n \times n} \]
holds. From Theorem 3.7, we have
\[ (I - [A])^{-1} = I + [A] + [A]^2 + \cdots + [A]^{k_0} + ([0, \varepsilon])_{n \times n}. \]

\section*{Theorem 4.7}
Denote \((I - A^-)^{-1} = (b_{ij}^1)_{n \times n}\) and \((I - A^+)^{-1} = (b_{ij}^2)_{n \times n}\). If Eq. (4.10) holds, then we have the matrix-covered set \((I - A(\otimes))^{-1}\) below:
\[ (I - [A])^{-1} \triangleq (I - [A])^{-1} = ([b_{ij}^1, b_{ij}^2])_{n \times n}. \]

\section*{Proof}
For \(a_{ij} \geq 0 \; (i, j = 1, 2, \ldots, n)\), we have
\[ (A^-)^k = (a_{ij}^{-k})_{n \times n} = \left(\sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \cdots \sum_{l_k=1}^{n} a_{l_1}^- a_{l_1}^- a_{l_2}^- \cdots a_{l_k-1}^- a_{l_k}^-\right)_{n \times n} \]
and
\[ (A^+)^k = (a_{ij}^{+k})_{n \times n} = \left(\sum_{l_1=1}^{n} \sum_{l_2=1}^{n} \cdots \sum_{l_k=1}^{n} a_{l_1}^+ a_{l_1}^+ a_{l_2}^+ \cdots a_{l_k-1}^+ a_{l_k}^+\right)_{n \times n}. \]
From Definition 3.12, we have
\[ [A]^k = \left(\sum_{l_{k-1}=1}^{n} \sum_{l_{k-2}=1}^{n} \cdots \sum_{l_1=1}^{n} [a_{l_1}^+, a_{l_1}^-] [a_{l_1}^+, a_{l_1}^-] \cdots [a_{l_{k-1}}^+, a_{l_{k-1}}^-] [a_{l_{k-1}}^+, a_{l_{k-1}}^-]\right)_{n \times n} \]
\[ = \left(\sum_{l_{k-1}=1}^{n} \sum_{l_{k-2}=1}^{n} \cdots \sum_{l_1=1}^{n} a_{l_1}^- a_{l_1}^- a_{l_2}^- \cdots a_{l_{k-1}}^- a_{l_{k-1}}^- a_{l_k}^- a_{l_k}^- a_{l_{k+1}}^- a_{l_{k+1}}^- \right)_{n \times n} \]
\[ = \left(\sum_{l_{k-1}=1}^{n} \sum_{l_{k-2}=1}^{n} \cdots \sum_{l_1=1}^{n} a_{l_1}^- a_{l_1}^- a_{l_2}^- \cdots a_{l_{k-1}}^- a_{l_{k-1}}^- a_{l_k}^- a_{l_k}^- a_{l_{k+1}}^- a_{l_{k+1}}^- \right)_{n \times n} \]
As Eq. (4.10) holds, we have that $I - A^+$ and $I - A^-$ are inverse and denote them as below:

$$(I - A^*)^{-1} = I + A + (A^*)^2 + \cdots = I + (a^+_{ij})_{n \times n} + (a^{2+}_{ij})_{n \times n} + \cdots = (b^2_{ij})_{n \times n}$$

and

$$(I - A^-)^{-1} = I + A^- + (A^-)^2 + \cdots = I + (a^-_{ij})_{n \times n} + (a^{2-}_{ij})_{n \times n} + \cdots = (b^1_{ij})_{n \times n}.$$  

Then Eq. (4.13) holds from Theorem 4.5. □

It is apparent that $(I - [A])^{-1}$ is the relatively superior matrix-covered set of $I - A(\otimes)$ and $(I - [A])^{-1}$ is the relatively inferior one.

Eq. (4.10) shows that the chaos of $A(\otimes)$ should be small enough, i.e., the information background of every element of $A(\otimes)$ may be more.

From Eqs. (4.12) and (4.13), we get the matrix/number-covered sets of other grey matrices/numbers below:

1. $[B] = ([b_{ij}])_{n \times n} = ([I - A])^{-1} - I$ is the matrix-covered set of grey complete consumption coefficient matrix $B(\otimes)$.
2. $[\bar{B}] = ([\bar{b}_{ij}])_{n \times n} = ([I - A])^{-1}$ is the matrix-covered set of grey complete required coefficient matrix $\bar{B}(\otimes)$.
3. $[F]_j = \sum_{i=1}^{n} [b_{ij}]$ is the number-covered set of grey influence coefficient $F(\otimes)_j$ $(j = 1, 2, \ldots, n)$, where $[b] = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} [b_{ij}]$.
4. $[E]_i = \sum_{j}[b_{ij}]$ is the number-covered set of grey induction coefficient $E(\otimes)_i$ $(i = 1, 2, \ldots, n)$.

From Eq. (4.6), we have following grey model and its covered form:

$$X(\otimes) = (I - A(\otimes))^{-1}Y(\otimes) \quad \text{and} \quad [X] = (I - [A])^{-1}[Y],$$

so the economic system can be controlled.

The following algorithm is to calculate the matrix-covered set $(I - [A])^{-1}$.

**Algorithm:**

**Step 1:** Do statistics and get the number-covered sets $[x_{ij}]$, $[Y_j]$ and $[N_j]$ $(i, j = 1, 2, \ldots, n)$.

**Step 2:** Calculate $[X_i]$ $(i = 1, 2, \ldots, n)$ using Eqs. (4.4) and (4.5) and get the matrix-covered set $[A] = ([a^+_{ij}, a^{2+}_{ij}])_{n \times n}$ using Eq. (4.8).

**Step 3:** Let $A^+ = (a^+_{ij})_{n \times n}$ and $A^- = (a^-_{ij})_{n \times n}$. If Eq. (4.10) fails, then go to step 1 and make the chaos of $x_{ij}(\otimes)$, $Y_j(\otimes)$ and $N_j(\otimes)$ be smaller.

**Step 4:** Calculate $(I - A^+)^{-1} = (b^2_{ij})_{n \times n}$ and $(I - A^-)^{-1} = (b^1_{ij})_{n \times n}$, and get $(I - [A])^{-1} = ([b^1_{ij}, b^2_{ij}])_{n \times n}$ from Eq. (4.13).

5. Case study

In this section, we will use our methodology to study a modified case.

The national economy is a complex system and is made of many sectors. For the sake of simplicity, we suppose that it has only six sectors, which are the agriculture (sector 1), the industry (sector 2), the construction (sector 3), the transportation-post (sector 4), the business and catering trade (sector 5), and the other service departments (sector 6).

The input–output analysis can reflect the dependence relationship between the input and the output of sectors. It is usually utilized to do economic analysis, policy simulation, forecasting and controlling, etc. The direct consumption coefficient is an important parameter to reflect the technologic interconnection among sectors. However, because the system is complex and the information is inadequate, it is impossible to get the exact value of all statistical datum. They are uncertain, but we can get their range and affirm that the true ones are included using the correct investigation method. Under the situation, the statistic datum are grey numbers, i.e., $x_{ij}(\otimes)$, $Y_j(\otimes)$ and $N_j(\otimes)$, and we can get their number-covered sets $[x_{ij}]$, $[Y_j]$ and $[N_j]$ $(i, j = 1, 2, \ldots, 6)$, respectively (see Table 3 and the unite is 100 million RMB).
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Final use</th>
<th>Total output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[1260, 1264]</td>
<td>[2980, 2985]</td>
<td>[17, 18.5]</td>
<td>[0.28, 0.35]</td>
<td>[204, 207]</td>
<td>[41, 42.5]</td>
<td>[4550, 4560]</td>
<td>[9052.28, 9077.35]</td>
</tr>
<tr>
<td>2</td>
<td>[1420, 1423]</td>
<td>[18,690, 18,700]</td>
<td>[2930, 2933]</td>
<td>[900, 906]</td>
<td>[1527, 1528]</td>
<td>[2092, 2096]</td>
<td>[9490, 9504]</td>
<td>[37,049, 37,090]</td>
</tr>
<tr>
<td>3</td>
<td>[1.1, 1.4]</td>
<td>[16, 17]</td>
<td>[35, 36]</td>
<td>[3.5, 4]</td>
<td>57</td>
<td>115,118</td>
<td>[4968, 4970]</td>
<td>[5195.6, 5203.4]</td>
</tr>
<tr>
<td>4</td>
<td>[100, 102]</td>
<td>[659, 660]</td>
<td>[154, 155]</td>
<td>[37, 38.5]</td>
<td>[700, 704]</td>
<td>[327, 330]</td>
<td>[660, 663]</td>
<td>[2637, 2652.5]</td>
</tr>
<tr>
<td>5</td>
<td>[194, 196]</td>
<td>[2777, 2779]</td>
<td>477</td>
<td>130</td>
<td>[254, 255]</td>
<td>[360, 362]</td>
<td>[2144, 2150]</td>
<td>[6336, 6349]</td>
</tr>
<tr>
<td>6</td>
<td>[243, 245]</td>
<td>[1501, 1507]</td>
<td>[74, 75]</td>
<td>[95, 96]</td>
<td>[689, 691]</td>
<td>[777, 780]</td>
<td>[4530, 4533]</td>
<td>[7909, 7927]</td>
</tr>
</tbody>
</table>

| Total value added | [5834.18, 5845.95] | [10,426, 10,442] | [1508.6, 1508.9] | [1471.25, 1477.65] | [2905, 2907] | [4197, 4198.5] |
| Total input       | [9052.28, 9077.35] | [37,049, 37,090] | [5195.6, 5203.4] | [2637, 2652.5] | [6336, 6349] | [7909, 7927] |
If the national leaders and advisers want to analyze the scientific and technologic interconnection among the six sectors, and to control the Total Output for the given Final Use next year, then how to do that using these inexact numbers?

In order to analyze the technological interconnection among the six sectors, we should get the number-covered sets of grey direct consumption coefficients.

**Step 1:** The number-covered sets of grey statistical datum have been obtained (see Table 3); 

**Step 2:** After calculating $[X_i] = [X_i^+, X_i^-]$ using Eq. (4.5) (see Table 3) and $\Delta x_{ij} = x_{ij}^+ - x_{ij}^-$, we have $a_{ij}^- = \frac{x_{ij}^-}{\Delta x_{ij}}$ and $a_{ij}^+ = \frac{x_{ij}^+}{\Delta x_{ij}}$ from Eq. (4.8). Then the number-covered sets $[a_{ij}^-, a_{ij}^+]$ ($i, j = 1, 2, \ldots, n$) of grey direct consumption coefficients can be obtained (see Table 4).

Table 4 shows that the true direct consumption coefficients must be in their appropriate ranges even if we do not know their values. For example, $a_{12} \in [0.0804, 0.0806]$. It means that the true value of direct consumption coefficient between sectors 1 and 2 must be in the interval $[0.0804, 0.0806]$, so we can justify that the technologic relation between them is not close. In order to speed the development of the agriculture and improve the interconnection between them, the leaders should set down some policies to support the agriculture’s products. On the other hand, $a_{23} \in [0.5634, 0.5643]$ means that the direct consumption coefficient between the industry and the construction is about 0.5638, and the interconnection of the two sectors are very close. Using these ranges, our leaders and advisers can presume the relationship among the six sectors.

On the other hand, how to control the total output if we strengthen the Final Use next year? For example, in order to improve the standard of living, the government wants to produce the Final Use as below:

$$[Y'] = ([5000, 5200], [91, 000, 92, 000], [5200, 5500], [1100, 1300], [3400, 3600], [5900, 6180])^T.$$  

From the following equation:

$$[X'] = (I - [A])^{-1}[Y'],$$

we can get the range of total output under the given technological condition next year.

At first, we should calculate the matrix-covered set $(I - [A])^{-1}$.

**Step 2:** (continue) Denote the matrix-covered set of grey direct consumption coefficient matrix as $[A] = ([a_{ij}^+, a_{ij}^-])_{n \times n}$ (see Table 4). Let $A^+ = (a_{ij}^{+\infty})_{n \times n}$ and $A^- = (a_{ij}^{-\infty})_{n \times n}$. Get $a^+ = \|A^+\|_{col} = 0.7193 < 1$ (If $a^+ \geq 1$, then we should re-obtain the number-covered sets $[x_{ij}]$, $[Y_i]$ and $[N_j]$), and let the chaos of $x_{ij}(\otimes), Y_i(\otimes)$ and $N_j(\otimes)$ be smaller for all $i, j \in \{1, 2, \ldots, 6\}$).

**Step 3:** After calculating $(I - A^+)_{n \times n}^{-1} = (b_{ij}^{+\infty})_{n \times n}$ and $(I - A^-)_{n \times n}^{-1} = (b_{ij}^{-\infty})_{n \times n}$, we obtain the matrix-covered set $(I - [A])^{-1} = ([b_{ij}^+, b_{ij}^-])_{n \times n}$ using Eq. (4.13) (see Table 5).

**Step 4:** Get the vector-covered set $[X'] = (I - [A])^{-1}[Y']$ below:

$$[X'] = ([27,940, 28,910], [226,210, 232,380], [5820, 6220], [9080, 9740], [23,670, 24,690], [17,800, 21,700])^T.$$

Table 4
The matrix-covered set $[A]$ of grey direct consumption coefficient matrix

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0139, 0.1396</td>
<td>0.00804, 0.0806</td>
<td>0.0033, 0.0036</td>
<td>0.0001, 0.0001</td>
<td>0.0321, 0.0327</td>
<td>0.0052, 0.0054</td>
</tr>
<tr>
<td>2</td>
<td>0.1565, 0.1571</td>
<td>0.05043, 0.05047</td>
<td>0.05634, 0.05643</td>
<td>0.3401, 0.3421</td>
<td>0.2405, 0.2411</td>
<td>0.2642, 0.2654</td>
</tr>
<tr>
<td>3</td>
<td>0.0001, 0.0002</td>
<td>0.0004, 0.0005</td>
<td>0.0067, 0.0069</td>
<td>0.0013, 0.0015</td>
<td>0.0090, 0.0090</td>
<td>0.0145, 0.0149</td>
</tr>
<tr>
<td>4</td>
<td>0.0110, 0.0113</td>
<td>0.0178, 0.0178</td>
<td>0.0296, 0.0298</td>
<td>0.0140, 0.0146</td>
<td>0.1103, 0.1110</td>
<td>0.00413, 0.00417</td>
</tr>
<tr>
<td>5</td>
<td>0.0214, 0.0216</td>
<td>0.0749, 0.0750</td>
<td>0.00917, 0.00917</td>
<td>0.0490, 0.0490</td>
<td>0.0490, 0.0490</td>
<td>0.0454, 0.0458</td>
</tr>
<tr>
<td>6</td>
<td>0.0268, 0.0271</td>
<td>0.0284, 0.0407</td>
<td>0.0142, 0.0144</td>
<td>0.0358, 0.0363</td>
<td>0.1086, 0.1090</td>
<td>0.0981, 0.0987</td>
</tr>
</tbody>
</table>
The result \(X'\) means that we should produce the range of products’ value next year. For example, if the product of sector 1 is between 27,940 and 28,910 million RMB, then it can be promised that the Final Use is in \([5000, 5200]\). Otherwise, the national economy maybe greatly lose its balance.

The case study shows that we can analyze and control the national economy using the grey input–output analysis even if the information is inadequate. It is impossible to get the exact statistic datum for the large and complex economy, but the range of them can be obtained under correct investigation. The grey input–output analysis provides a methodology to resolve this problem. We can analyze it using the matrix-covered set of grey direct consumption coefficient matrix, and presume the interconnection among sectors. We also control the production of sectors through Final Use. On the other hand, the range of other coefficients can also be obtained and they are helpful for the analysis of economy.

6. Conclusions

We introduce grey number and its covered operation, and propose grey matrix theory, and get some operational rules about the grey number and grey matrix. Although others have given the definition of grey matrix, they did not give the definition of inverse grey matrix. The inverse one has an important role for the development of grey matrix theory, and it can greatly promote the development of grey systems theory. The re-definition of grey matrix, especially inverse grey matrix, and its calculation formulas in this paper may have positive meaning both the theoretic progress and practical applications. We also propose grey input–output analysis based on grey systems theory and traditional input–output analysis, especially the computational formulas of the matrix-covered set of inverse grey Leontief coefficient’s matrix \((I - A(\oplus))^{-1}\). It is important to analyze and control the economic system and make it possible to estimate the interconnection among sectors under the information-missing situation. The proposed method provides an effective tool to study economic systems by the input–output analysis in uncertain situation.

However, the calculation of inverse grey matrix is particular, i.e., the true elements of grey matrix should satisfy Eq. (4.3). How to get the matrix-covered set of inverse grey matrix under the general situation? On the other hand, how to do for other technical methods of input–output analysis, such as the dynamic one, the enterprise one, and the input-occupancy-output analysis, etc? It may be our next work.

Acknowledgement

We are very grateful to our referees, especially Dr. Gordon Huang. Their suggestion is greatly helpful for improving this paper.

References
