Abstract

For classical sets one has their cumulative hierarchy, and also the category \(\text{SET}\) of all sets and mappings as standard approaches toward the universe of all sets.

We discuss the corresponding situation for fuzzy set theory and give is a (concise) survey of a lot of such approaches which have been offered in the past approximately 35 years.

1 Problems

For classical set theory one has a quite satisfactory naive foundation in the idea of the cumulative hierarchy of sets which forms the standard model for the common axiomatizations of set theory, like the systems ZFC or NBG.

To have a similar situation in the fuzzy field, one is interested to find some kind of cumulative universe of fuzzy sets which should be closed under the formation of fuzzy (sub-)sets, and which can serve as a kind of standard model for axiomatizations of fuzzy set theory.

So one immediately faces at least the following problems:

(i) What are fuzzy sets of higher level?
(ii) Is there a kind of standard cumulative universe of fuzzy sets?

Of course, closely related to these problems one has the further problem:

(iii) What are suitable axiomatizations of fuzzy set theory?

But there is still more in the background here: the intuition of graduation of membership seems to call for a graded identity between fuzzy sets too. But is this intuition a reasonable one?

Thus it is important to discuss also the problems

(iv) What about the behavior of identity between fuzzy sets: graded or two-valued?
(v) And how is identity related to comprehension?

2 Classifying the Approaches

The approaches which have been presented toward the main problems of this paper are rather inhomogeneous. So it is not suitable to discuss them in chronological order. Instead we classify the approaches into the following types:

- “naive” constructions of cumulative universes of fuzzy sets;
- model theoretical constructions of a (kind of) standard model of the universe of fuzzy sets;
- category theoretic approaches.

And we disregard a series of only axiomatic approaches which do not introduce some kind of standard universe of fuzzy sets.

3 Naive Constructions

In 1965 D. Klaua presented two versions for a cumulative hierarchy of many-valued sets.\(^1\) In both cases he restricted the membership degrees to a finite set of equidistant points of the unit interval \([0,1]\) understood as the truth degrees of a finitely-valued Łukasiewicz system.

All our “naive” approaches construct their universes \(V\) by induction through the ordinals. Each such construction yields an expanding hierarchy of partial universes \(V_\alpha\) with \(V\) as the global union, and unions also at limit stages. So there is a natural notion of rank, and constructions at successor stages become the crucial steps.

3.1 The very first universe

For the first one of these hierarchies, D. Klaua \([20,23]\) started from some infinite (crisp) set \(U\) of urelements with a graded identity relation \(=\), i.e. a relation which is reflexive, symmetric and \&\text{\(L\)}-transitive

\(^1\)This happened without any influence by or relationship to L.A. Zadeh and his seminal paper \([36]\).
for the ŁUKASIEWICZ t-norm $\&_L$ and its associated implication $\neg_L$. Then he forms, with reference to the standard (crisp) power set operation $\mathcal{P}$, the hierarchy $V^* = \bigcup_{n<\omega} V^*(n)$ with $V^*(0) = U \times \{0\}$ and with
\[
V^*(n+1) = \mathcal{P}(V^*(n)) \times \{1\},
\]
and introduces a graded identity $\equiv_\omega$, a graded membership $\in_\omega$, and a graded inclusion $\subseteq_\omega$ as:
\[
x \equiv_\omega y = \begin{cases} \text{pr}_1(x) \equiv \text{pr}_1(y) , \\ 0 , \\ x \subseteq_\omega y \wedge y \subseteq_\omega x ,
\end{cases}
\]
depending on whether $x, y$ are urelements, of different rank, or of equal rank, and as
\[
x \in_\omega y = \sup_{v \in \text{pr}_1(y)} (x =_\omega v),
\]
\[
x \subseteq_\omega y = \inf_{u \in \text{pr}_1(x)} u \in_\omega y.
\]
The authors main results are:
- There is a natural embedding of the $\omega$-th level of the standard hierarchy of crisp sets into $V^*$.
- Suitable (graded) versions of the axioms of extensionality and of comprehension hold true in this universe.
- Some elementary set algebra is developed.

### 3.2 A naive fuzzy power set iteration

The same author D. KLAUZA almost immediately modified his first approach and considered in [21] for some crisp set $U$ a transfinite hierarchy starting from $V_0 = U$ with the particular extension steps determined by the sets of all functions $f : V_\alpha \to [0,1]$ which satisfy the condition that their support $f^{-1}([0,1])$ is rank-cofinal in $V_\alpha$.

This condition intends to avoid the “doubling” of objects—in the sense that it forbids to add to $V_\alpha$ membership functions which have the same support as membership functions in $V_\alpha$, but only a larger domain.

The graded membership predicate $\in_\omega$, for some object $a$, is defined as the value of the membership function at $a$. Contrary to the first approach, there is no graded identity inside the set of urelements. But there is again a graded inclusion and a graded identity:
\[
x \subseteq_\omega y = \forall z (z \in_\omega x \rightarrow_L z \in_\omega y),
\]
\[
x =_\omega y = x \subseteq_\omega y \wedge y \subseteq_\omega x.
\]

As indicated here, the author refers to a language of many-valued logic. Its main connectives are minimum-conjunction, max-disjunction, $(1 - \ldots)$-negation, and Łukasiewicz implication.

The considerations, continued in [22, 24, 25], have as main results:
- The graded inclusion and identity have suitable properties.
- Suitable (many-valued) versions of the axioms of extensionality, of comprehension and of separation hold true.
- A lot of set algebra is developed, essentially only in an elementary way, up to the basic notions of cardinals and integers.

### 3.3 Taking graded identity more seriously

The idea that for a graded identity $\equiv$ the fuzzy sets of the hierarchy should satisfy an extensionality condition like
\[
\models x =_\omega y \& y \in_\omega z \rightarrow x \in_\omega z
\]
was the starting point for the modification of that approach by GOTTWALD [8, 9]. The idea was to have KLAUZA’s simple graded inclusion and identity (1), (2) and also condition (3).

So the hierarchy of fuzzy sets was determined by extension steps determined by the sets of all functions $f : V_\alpha^\ast \to [0,1]$ with their support rank-cofinal in $V_\alpha^\ast$, satisfying the additional condition
\[
\forall x \forall y (f(x) \&_L [y \equiv^\ast x] \leq f(y))
\]
which uses the identity (2), and a fuzzified rank equality $\equiv^\ast$, to give $\equiv^\ast$ by
\[
y \equiv^\ast x = x =_\omega y \& y \equiv^\ast x.
\]

The main results of this approach are:
- Within this cumulative universe of fuzzy sets suitable versions of all the basic $\mathbf{ZF}$ axioms hold.
- Natural many-valued generalizations of the basic laws of set and relation algebra hold true.
- One can even define the uniqueness of relations in an argument, and extend this approach up to a suitable notion of equipotency, and thus of cardinality, cf. [10].

### 4 Model Theoretic Constructions

The basic background idea comes from the Boolean valued models for $\mathbf{ZF}$ set theory, cf. e.g. [1, 11].

This means that one forms for a given complete Boolean algebra $\mathcal{B}$ a hierarchy $V_\alpha^\mathcal{B}$ which starts from the empty set and extends the partial universes $V_\alpha^\mathcal{B}$ by all partial functions from $V_\alpha^\mathcal{B}$ into $\mathcal{B}$. And the union of all these sets becomes the universe $V^\mathcal{B}$ of an $\mathcal{B}$-interpretation of $\mathbf{ZF}$ set theory.

For the Boolean valued models $V^\mathcal{B}$ the Boolean algebra $\mathcal{B}$ acts as set of truth degrees.
4.1 Adapting Boolean valued sets

The first such model theoretic approach toward a universe of fuzzy sets was sketched by ZHANG JIN-WEN [37], and given in more detail in [38].

He considers some complete “quasi Boolean” lattice $G$, i.e. a complete distributive lattice satisfying the inequality $x \leq (x^*)^*$ and the equations $0^* = 1, 1^* = 0$.

In [37] only the case that $G$ is a Boolean algebra is treated in some detail. And only routine results are mentioned or proved, which do not explicitly involve the implication operator.

Much more extended is the later paper [38] which, however, restricts the degree structure $G$ to the standard structure $G_Z = \langle [0, 1], \max, \min, 1 - \ldots, 0, 1 \rangle$ enriched with the Gödel implication.

Now a hierarchy $F = \bigcup_{\xi \in \mathbb{N}} F_\xi$ of fuzzy sets over some (crisp) set $U$ of urelements is determined with reference to the fuzzy power set operator $\mathcal{F}(\mathcal{X})$, which assigns to each crisp set $\mathcal{X}$ the class of all fuzzy subsets of $\mathcal{X}$. The crucial difference to the approach of [21] is that inside $F$ a graded membership predicate $\varepsilon$ and a graded identity $\equiv$ are defined as by SCOTT/SOLOVAY, of course with minimum and Gödel implication instead of lattice meet and implication operation in the Boolean algebras.

The main results are that this universe of fuzzy set is a model of essentially all the $ZF$-axioms, with the exception that in the axiom schema of replacement only negation-free formulas are allowed.

4.2 A Heyting algebra valued approach

G. TAKEUTI/S. TITANI [30, 31] started their construction of the same type of universe (without urelements) from a complete HEYTING algebra (cHA) $H$.

Their main argument in favor of this intuitionistic case is that

$$\varphi \land (\varphi \rightarrow \psi) \rightarrow \psi$$

does not hold for min-conjunction and Łukasiewicz implication (the original ZADEH case), which would mean that extensionality fails for the resulting (fuzzy) sets.

This argument is not convincing because one has

$$\models \varphi \rightarrow_L (\varphi \rightarrow_L \psi),$$

a case, the authors do not consider. And because both these formulas have equivalent classical counterparts, there is no reason to prefer the first one over the second in the more general context here.

As usual in the intuitionistic context, implication is residuation, i.e. the relative pseudo-complement.

As main result the authors give an axiomatic set theory, based on a sequent calculus, which has besides axioms for equality suitable versions of all the $ZF$ axioms, together with two technical axioms concerning the embedding of the $ZF$ universe into the actual cHA valued universe.

The case of intuitionistic logic is extended by these authors in [32]. In that paper they include, over the set $[0, 1]$ of truth degrees, also the connectives for Łukasiewicz’s negation and conjunction, for product conjunction, and a truth degree constant for $\frac{1}{2}$.

To find a suitable logical calculus, they extend a standard intuitionistic sequent calculus by two further inference rules and as much as 46 (!) special axioms.

The crucial implication connective remains, however, the intuitionistic implication.

The set theoretic axioms are chosen mainly as in the authors previous approaches. And also the standard model construction follows the same ideas as in the cHA valued case.

4.3 A further lattice valued generalization

Another generalization of the cHA valued approach is given by S. TITANI [33] in which the truth degree structure is a complete lattice $L$ with an implication operation $\rightarrow^*$ characterized by the properties

$$a \rightarrow^* b = 1 \iff a \leq b,$$

$$a \land (a \rightarrow^* b) \leq b.$$

The formalized language has also a connective $\Rightarrow$ for the “smallest” implication

$$a \Rightarrow b = \begin{cases} 1, & \text{if } a \leq b \\ 0, & \text{otherwise} \end{cases},$$

and the $L$-valued universe is built “as usual”, but using the particular implication $\Rightarrow$ in the definition of the $L$-valued interpretation of equality.

The main results of the author are:

- A Gentzen type sequent calculus is given for the $L$-valued logic.
- A list of axioms is given for an $L$-valued set theory consisting of axioms for equality, giving the “extensionality” of the generalized sets, together with $ZF$-like axioms, and together with two further technical axioms.
- A Completeness Theorem saying that provability in this axiomatic set theory coincides with validity in all $L$-valued universes.

4.4 A BL-algebra valued universe

The most recent(?) approach along these lines is by P. HÁJEK/Z. HANIKOVÁ [13] and based upon the basic


t-norm logic $\mathbf{BL}$ of Hájek enriched with the “globalization” operator $\Delta$, as explained in [12].

In a language with primitive predicates $\in$, $\subseteq$, $=$ of all fuzzy sets too: a point of view never accepted by the authors. The “standard” model for this theory is formed w.r.t. some complete $\mathbf{BL}$-chain $\mathbf{L}$ and given by a hierarchy which starts from the empty set and extends the partial universes $V_\alpha$ by all partial functions from $V_\alpha$ into $\mathbf{L}$. The primitive predicates are interpreted as

$$
[x \in y] = \bigcup_{u \in \text{dom}(y)} ([u = x] \ast y(u)),
$$

$$
[x \subseteq y] = \bigcap_{u \in \text{dom}(x)} (x(u) \Rightarrow [u \in y]),
$$

$$
[x = y] = \Delta [x \subseteq y] \ast \Delta [y \subseteq x].
$$

The last condition forces the equality to be crisp, and makes the authors standard form of the axiom of extensionality trivially true in the model.

The authors main result is that the structure $V^\mathbf{L} = \bigcup_{\alpha \in \text{On}} V_\alpha^{\mathbf{L}}$ is a model of all the axioms given by the authors.

5 Category Theoretic Approaches

The paradigmatic situation for category theoretic characterizations first was the category $\mathbf{SET}$ of (crisp) sets and mappings [26]. With the development of the notion of an elementary topos and the understanding that topoi describe generalized set theories, the situation changed and the paradigm became the Higgs topos $\mathbf{SET}(H)$ of cHA valued sets [14].

Since $H$-valued maps admit an internalization as characteristic morphism in $\mathbf{SET}(H)$, some authors have claimed that the Higgs topos would give the category of all fuzzy sets too: a point of view never accepted by those people which had a closer relationship to (and better knowledge of) fuzzy set theory.

There are two core points for the rejection of this “solution” by the fuzzy people: (i) that the Łukasiewicz negation cannot be internalized as a truth arrow, and (ii) that the internal logic of topos is intuitionistic logic.

5.1 The first approaches

The first who introduced a category of fuzzy sets was J.A. Goguen [6]. A bit later he [7] gave a categorical characterization of fuzzy sets. A rather similar approach was given by M. Eytan [4].

A nice unifying survey of these and other categorical approaches gives O. Wyler [35]. The basic logic remains the intuitionistic one, i.e. reference is only to cHA’s $H$ as value structures. But graded identities come into consideration, as is the case in the Higgs topos.

$H$-valued fuzzy sets are pairs $A = (|A|, \varepsilon_A)$ of a crisp set $|A|$ and a membership function $\varepsilon_A : |A| \rightarrow H$.

$H$-valued totally fuzzy sets (or $H$-sets, for short) are pairs $A = (|A|, \delta_A)$ of a set $|A|$ and a map $\delta_A : |A| \times |A| \rightarrow H$ subjected to the conditions of symmetry and transitivity.

$H$-valued fuzzy set $A = (|A|, \varepsilon_A)$ determine $H$-sets $A_\varepsilon = (|A|, \delta_A)$ by

$$
\delta_{\varepsilon_A}(x, y) = \left\{ \begin{array}{ll}
\varepsilon_A(x), & \text{if } x = y \\
\bot & \text{otherwise.}
\end{array} \right.
$$

Thus, $H$-sets form a refinement of fuzzy set, and the $H$-valued fuzzy sets are the discrete $H$-sets.

After fixing objects describing the intuitive idea of fuzzy sets, one has to specifying morphisms. One considers two fundamentally different types of morphisms: crisp maps and certain binary $H$-valued relations.

We begin with Wyler’s category $\mathbf{Set}_{tc}(H)$: Objects are just the $H$-sets with non-empty support and morphisms $f : A \rightarrow B$ are ordinary maps $f : |A| \rightarrow |B|$ satisfying

$$
\delta_A(x, x') \leq \delta_B(f(x), f(x')).
$$

The subcategory of all discrete $H$-sets (i.e. of all $H$-valued fuzzy sets) coincides with Goguen’s category $\mathbf{Set}(L) (= \mathbf{Set}_{dc}(H))$.

A trouble with $\mathbf{Set}_{tc}(H)$-morphisms is that maps between $H$-sets are not necessarily extensional. This leads to a crisp equivalence relation $\equiv$ in the hom-sets of $\mathbf{Set}_{tc}(H)$:

$$
f \equiv g \Leftrightarrow \delta_A(x, x) \leq \delta_B(f(x), g(x)).
$$

$f$ and $g$ are extensionally equal iff $f \equiv g$.

The category $\mathbf{Set}_{dc}(H)$ of extensional morphisms can be defined as follows: Objects are again $H$-sets with non-empty support set and morphisms are $\equiv$-equivalence classes.

There exists an alternative to express morphisms by binary $H$-valued relations; e.g. the Higgs topos — i.e. the category $\mathbf{Set}_{tf}(H)$—consists of the following data: Objects are again $H$-sets with non-empty support sets, but morphisms are $H$-valued functional relations.

Now the subcategory of discrete $H$-sets of $\mathbf{Set}_{tf}(H)$ coincides with Eytan’s category $\mathbf{Fuz}(H) (= \mathbf{Set}_{df}(H)$ in Wyler’s notation).

Some interesting results are:

- $\mathbf{Set}_{dc}(H)$ is a topological quasitopos over sets, and
hence has “crisp” internal logic.

- Set_{ec}(H) is cartesian closed, but not a quasitopos.
- Set_{if}(H) is a topos with H-valued internal logic, and equivalent to the category sh(H) of sheaves over H, cf. [5, p. 363].
- Set_{ec}(H) is a quasitopos with H-valued internal logic, and equivalent to the category spsh(H) of separated presheaves over H, cf. [3].

The other categories are, in general, neither topoi nor quasitopoi.

But this structural deficiency never was the main point of criticism from fuzzy people: their core objection always was that the intuitionistic context is too restricted by not allowing to discuss non-idempotent conjunctions, cf. [29, 2, 19].

5.2 Categories of monoidal sets

A more general non-intuitionistic, monoidal context together with a graded notion of identity for fuzzy sets is the core problem for the approach of U. Höhle as explained mainly in [15, 16, 17, 18].

Instead of totally fuzzy sets, i.e. instead of H-sets, he considers M-sets, with $M = (L, \leq, *)$ an integral, divisible, commutative completely lattice-ordered monoid with zero, i.e. a complete divisible residuated lattice–called GL-monoid by this author.

The point is to consider global M-sets $A = ([A], \delta_A)$ which are, like the H-sets of the Higgins topos, characterized by an M-valued global equality relation $\delta_A$ satisfying the conditions of reflexivity, symmetry, and *-transitivity. They are called separated iff they additionally satisfy the condition

$$\delta_A(x, y) = 1 \Rightarrow x = y.$$  (6)

The separated, global M-sets are natural generalizations of fuzzy sets (of higher level), because the most natural, naive understanding of a graded identity between fuzzy sets seems to involve the ideas that (i) each fuzzy set is identical with itself to the highest possible degree, and (ii) that two fuzzy sets which are identical to the highest possible degree are truly identical.

The separated, global M-sets become the objects of a category if one takes as morphisms the “structure preserving” maps $f : [A] \to [B]$ which have to satisfy the preservation of equality condition

$$\delta_A(x, y) \leq \delta_B(f(x), f(y)).$$  (7)

For these separated, global M-sets this choice of morphisms gives a category with interesting properties. Particularly this category is complete and cocomplete, i.e. has all limits and colimits. However, it does not allow for a unique classification of (extremal) subobjects, as shown in [18].

The problem is to find a finitely complete category C of M-sets, or of some other (suitably related) objects, which

- has a subobject classifier $\Omega$ and a truth arrow $t$;
- allows unique classification of the $(\Omega, t)$-classifiable subobjects;
- internalizes M-valued maps as C-morphisms with codomain $\Omega$;
- is equivalent with the Higgins topos in the case that the underlying GL-monoid is a cHA.

Höhle has in mind that for each cHA $H$ the Higgins topos SET(H) of H-valued sets (and H-set morphisms) is categorically equivalent to the category sh(H) of sheaves over $H$. Furthermore one knows that Fourman/Scott [5] have shown that every presheaf determines an H-set, and that the sheaves correspond to the complete H-sets (which have all their singleton subsets determined by a single element).

His generalization thus starts with the search for a notion of presheaf in the monoidal context by looking for suitable singletons. And this happens not for global M-sets, but for M-sets, which generalize the global M-sets in that they refer to a notion of local existence as was done for the cHA valued case in [28].

The details are too complex to be explained here in detail. But the approach is successful.

References


