10 Triangular norm-based mathematical fuzzy logics

Siegfried Gottwald\(^1\) and Petr Hájek\(^2\)

\(^1\) Institute of Logic and Philosophy of Science, Leipzig University, Leipzig (Germany)
\(\text{gottwald@uni-leipzig.de}\)

\(^2\) Institute of Computer Science, Academy of Sciences, Prague (Czech Republic)
\(\text{hajek@cs.cas.cz}\)

Abstract In this chapter, we consider particular classes of infinite-valued propositional logics which are strongly related to t-norms as conjunction connectives and to the real unit interval as set of their truth degrees, and which have their implication connectives determined via an adjointness condition. Such systems have in the last ten years been of considerable interest, and the topic of important results. They generalize well-known systems of infinite-valued logic, and form a link to as different areas as e.g. linear logic and fuzzy set theory. We survey the most important ones of these systems, always explaining suitable algebraic semantics and adequate formal calculi, but also discussing complexity issues. Finally we mention a type of extension which allows for graded notions of provability and entailment.

10.1 Introduction

Fuzzy sets, i.e. sets with a graded notion of membership, have been used—in the last decades—quite successfully in engineering applications.

It is natural to interpret the membership degrees, which graduate the property of being a member of some (fuzzy) set, as degrees of truth of a sentence expressing this property of being a member. Of course, these truth degrees then vary between the degree 0 (for “false”) and the degree 1 (for “true”). Accepting this point of view makes it natural to try to develop the theory of fuzzy sets within the framework of a suitable many-valued logic which has as its set of truth degrees just the set of membership degrees of the fuzzy sets. The formalized language of this many-valued logic then, besides variables \(x, y, z\),... and constants \(a, b, c\),... for elements of the universe of discourse, uses symbols to denote fuzzy sets and a binary many-valued membership predicate \(\varepsilon\) such that for the membership degree \(\mu_A(a)\) of some object \(a\) with respect to a fuzzy set \(A\)

\[
\mu_A(a) = \|x \varepsilon A\|
\]

holds true, using \(\|H\|\) for the truth degree of a well-formed formula \(H\) of that language.\(^1\)

\(^1\) Of course this has to be done with respect to some evaluation of all the nonlogical symbols of \(H\).
But the (often cumbersome) explicit reference to this evaluation is unimportant here.
From this point of view fuzzy sets become the sets of a generalized (naïve or even axiomatized) set theory based upon a suitable many-valued logic. This type of interpretation proves quite useful: it opens the doors to clarify far reaching analogies between notions and results related to fuzzy sets and those ones related to usual sets.

It is a kind of standard choice in fuzzy set applications, to consider the real unit interval as the class of membership degrees. Furthermore, in discussions about the mathematical foundations for the set algebra of fuzzy sets a kind of agreement has been reached to consider e.g. intersection and cartesian product operations for fuzzy sets which are based upon t-norm combinations of the membership degrees.

Having in mind the standard relationships between set algebraic operations and logical connectives this means that—from the logical point of view—such a generalized intersection operation for fuzzy sets should be based upon a propositional connective which has a t-norm as its truth degree function.

10.2 T-norm-based connectives

For the definition and properties of t-norms see [49] or also [48]. From the point of view of many-valued logic, a t-norm is a suitable candidate for a truth degree function of some generalized conjunction connective. Accepting this, one essentially is concerned with systems of many-valued logic with infinite truth degree set $[0, 1]$. And additionally one prefers to consider such systems which have the truth degree 1 as the only designated truth degree.\(^2\)

Such a system of many-valued logic is called t-norm-based (upon some particular t-norm $T$) if and only if all the other connectives of it have associated truth degree functions which are defined from this t-norm $T$, using possibly some truth degree constants. Usually one considers together with the conjunction connective $\&$ with truth degree function $T$ an implication connective $\to$ with truth degree function $I_T$ characterized by

$$I_T(u, v) = \text{df} \sup \{ z \mid T(u, z) \leq v \}, \quad (10.1)$$

the so-called $R$-implication connected with $T$, and a standard negation connective $\neg$ with truth degree function $n_T$, given as

$$n_T(u) = \text{df} I_T(u, 0). \quad (10.2)$$

The definition (10.1) determines a reasonable implication function just in the case that the t-norm $T$ is left-continuous. Here “reasonable” essentially means that $\to_T$ satisfies a suitable version of the rule of detachment.

In more technical terms it means that for left-continuous t-norms $T$ condition (10.1) defines a residuation operator $I_T$, previously sometimes also called $\varphi$-operator, cf. [29].

\(^2\)This means e.g. that a formula of the language of such a system counts as logically valid just in case it always assumes this designated truth degree 1. This notion, as well as the other notions from many-valued logic are explained in detail e.g. in [30].
And it means also, under this assumption of left-continuity of $T$, that condition (10.1) is
equivalent to the \textit{adjointness condition}
\[ T(u, w) \leq v \iff w \leq I_T(u, v), \]
(10.3)
i.e. that the operations $T$ and $I_T$ form an \textit{adjoint pair}.

Forced by these results one usually restricts, in this logical context, the considerations
to left-continuous—or even to continuous—t-norms.

But together with this restriction of the t-norms, a generalization of the possible truth
degree sets sometimes is useful: one may accept each subset of the unit interval $[0, 1]$ as a
truth degree set which is closed under the particular t-norm $T$ and its residuum.

The restriction to \textit{continuous} t-norms enables even the definition of the operations
\text{max} and \text{min}, which make $[0, 1]$ into an (linearly) ordered lattice. On the one hand one
has from straightforward calculations that always
\[ \min\{u, v\} = T(u, I_T(u, v)). \]
(10.4)
And on the other hand one gets always (cf. e.g. [32, 33])
\[ \max\{u, v\} = \min\{I_T(I_T(u, v), v), I_T(I_T(v, u), u)\}. \]
(10.5)

The systems of fuzzy logic we discuss here are sometimes also called \textit{R-fuzzy logics},
stressing the fact that our implication connectives $\rightarrow$ have as truth degree functions the
residuation operations, characterized by (10.1) or (10.3). Besides these R-fuzzy logics one
occasionally, e.g. in [11, 44], discusses so-called \textit{S-fuzzy logics} which are also based upon
some t-norm, but additionally take the Łukasiewicz negation $n_L(u) = 1 - u$ as a basic
connective. These systems define their implication connective like material implication in
classical logic. They in general lose the rule of detachment as a sound rule of inference if
they have 1 as the only designated truth degree, or they allow all positive reals from $[0, 1]$ as
designated truth degrees.

\section*{10.3 Extracting an algebraic framework}

For the problem of adequate axiomatization of (classes of) t-norm-based systems of many-valued logic there is an important difference to the standard approach toward semantically based systems of many-valued logic: here there is no single, “standard” semantical matrix
for the general approach.

The most appropriate way out of this situation seems to be: to find some suitable
class(es) of algebraic structures which can be used to characterize these logical systems,
and which preferably should be algebraic varieties, i.e. equationally definable.

As it seems, however, the notions of left-continuity and of continuity need the reference
to the notion of limit. And this is not (genuinely) an algebraic notion.

Hence it would either be nice to find—algebraically characterizable—classes of algebraic structures which “approximate” well the class of algebraic structures determined by
the continuous, or by the left-continuous t-norms, or even to find such algebraic structures which grasp well these classes.

From an algebraic point of view, the following conditions seem to be structurally important for t-norms:

- \([0, 1], T, 1\) is a commutative semigroup with a neutral element, i.e. an abelian monoid,
- \(\leq\) is a (lattice) ordering in \([0, 1]\) which has a universal lower bound and a universal upper bound,
- both structures “fit together”: \(T\) is non-decreasing with respect to this lattice ordering.

Thus it seems reasonable to consider abelian lattice-ordered monoids as the truth degree structures for the t-norm-based systems.

In general, however, abelian lattice-ordered monoids may have different elements as the universal upper bound of the lattice and as the neutral element of the monoid. This is not the case for the t-norm-based systems, they make \([0, 1]\) into an integral abelian lattice-ordered monoid as truth degree structure, i.e. one in which the universal upper bound of the lattice ordering and the neutral element of the monoidal structure coincide.

Furthermore one also likes to have the t-norm \(T\) combined with another operation, its R-implication operator, which forms together with \(T\) an adjoint pair: i.e. the abelian lattice-ordered monoid formed by the truth degree structure has also to be a residuated one.

Summing up, hence, we are going to consider residuated lattices, i.e. algebraic structures \(\langle L, \cap, \cup, \ast, \rightarrow, 0, 1 \rangle\) such that \(L\) is a lattice under \(\cap, \cup\) with universal lower bound 0 and universal upper bound 1, and an abelian lattice-ordered monoid under \(\ast\) with neutral element 1, and such that the operations \(\ast\) and \(\rightarrow\) form an adjoint pair, i.e. satisfy

\[ x \ast z \leq y \iff z \leq (x \rightarrow y). \]

In this framework one additionally introduces, following the understanding of the negation connective given in (10.12), a further operation \(-\) by

\[-x =_{df} x \rightarrow 0.\]

It is nice to recognize that the adjointness condition is in the present setting the suitable algebraic equivalent of the analytical notion of left-continuity, if one specifies the residuated lattice to be a t-norm algebra

\[ [0, 1]_T = \langle [0, 1], \min, \max, T, I_T, 0, 1 \rangle. \]  (10.6)

10.3.1 Proposition
For any t-norm \(T\) it holds that \(T\) and its R-implication \(I_T\) form an adjoint pair if and only if \(T\) is left-continuous (in both arguments).

Proof. Because of the commutativity condition (T1) it is sufficient to discuss the left-continuity of \(T\) only for one of its arguments, say the second one.
Suppose first that $T$ is left-continuous. Then one has to prove that for the R-implication (10.1) connected with the t-norm $T$ the adjointness condition (10.3) holds true for all $u, v, w \in [0, 1]$. Firstly let $T(u, w) \leq v$. Then one has $w \in \{ z \mid T(u, z) \leq v \}$ and hence $w \leq I_T(u, v)$. If otherwise $w < I_T(u, v)$ holds true, then there exists some $y > w$ such that $T(u, y) \leq v$, which gives $T(u, w) \leq v$ by (T3). If otherwise, however, $w = I_T(u, v)$ holds true, there is either $w \in \{ z \mid T(u, z) \leq v \}$ and thus immediately $T(u, w) \leq v$, or there exists a non-decreasing sequence $(y_i)_{i \geq 0}$ such that always $y_i < w$ and $T(u, y_i) \leq v$ hold true, and such that $w = \lim_{i \to \infty} y_i$. Then one has by the left-continuity of $T$ immediately $T(u, w) = T(u, \lim_{i \to \infty} y_i) = \lim_{i \to \infty} T(u, y_i) \leq v$.

Now conversely let $(T, I_T)$ be an adjoint pair. Assume furthermore that $T$ is not left-continuous. Then there exist values $u, w \in [0, 1]$ and a non-decreasing sequence $(x_i)_{i \geq 0}$ such that always $0 \leq x_i < w$ holds true together with $w = \lim_{i \to \infty} x_i$, but such that one has $T(u, \lim_{i \to \infty} x_i) \neq T(u, w)$, i.e. $T(u, \lim_{i \to \infty} x_i) < T(u, w)$. Then one always has $T(u, x_i) \leq T(u, \lim_{i \to \infty} x_i)$. Thus for any $a$ with $T(u, \lim_{i \to \infty} x_i) \leq a < T(u, w)$ one has $T(u, x_i) \leq a$, hence $x_i \leq I_T(u, a)$ and therefore also $w = \lim_{i \to \infty} x_i \leq I_T(u, a)$, because one also has $w = \sup\{ x_i \mid i \geq 0 \}$ by choice of the sequence $(x_i)_{i \geq 0}$. Thus one has $T(u, w) \leq a$ by the adjointness condition, and $a < T(u, w)$ by choice of $a$. A contradiction. Hence the t-norm $T$ has to be left-continuous.

And also the continuity of the basic t-norm has an algebraic equivalent: the property of divisibility for the commutative lattice-ordered monoids.

10.3.2 Definition
A lattice ordered monoid $\langle L, \ast, 1, \leq \rangle$ is divisible if and only if for all $a, b \in L$ with $a \leq b$ there exists some $c \in L$ with $a = b \ast c$.

For residuated lattices one has another nice and useful characterization of divisibility: a residuated lattice $\langle L, \cap, \cup, \ast, \rightarrow, 0, 1 \rangle$ is divisible if and only if one has $a \cap b = a \ast (a \rightarrow b)$ for all $a, b \in L$.

10.3.3 Proposition
A residuated lattice $\langle L, \cap, \cup, \ast, \rightarrow, 0, 1 \rangle$ is divisible, i.e. corresponds to a divisible lattice ordered monoid$^3$ $\langle L, \ast, 1, \leq \rangle$, if and only if one has $a \cap b = a \ast (a \rightarrow b)$ for all $a, b \in L$.

Proof. We first show that one has in each residuated lattice

$$a \ast (a \rightarrow b) = b \iff \exists x (a \ast x = b) \quad (10.7)$$

for all $a, b \in L$. Of course, in the case $a \ast (a \rightarrow b) = b$ there exists an $x$ such that $a \ast x = b$. So suppose $a \ast c = b$ for some $c \in L$. If one then would have $a \ast (a \rightarrow b) \neq b$, this would mean $a \ast (a \rightarrow b) < b = a \ast c$ because one always has $a \ast (a \rightarrow b) \leq b$ by the adjointness condition, and this hence would mean $c \leq a \rightarrow b$ (because otherwise $c \leq a \rightarrow b$ and hence $b = a \ast c \leq a \ast (a \rightarrow b)$ would be the case) and therefore also $a \ast c = c \ast a \neq b$ by the adjointness condition, a contradiction. Thus (10.7) is established.

$^3$ Of course, $\leq$ here is the lattice ordering of the lattice $\langle L, \cap, \cup \rangle$. 
Supposing now the divisibility of \( \langle L, \cap, \cup, *, \rightarrow, 0, 1 \rangle \), then one has for all \( b \leq a \in L \) from the existence of an \( x \) such that \( (b = a \cdot x) \) immediately \( a \cdot (a \rightarrow b) = b = a \cap b \). And for \( a \leq b \) one has \( a \rightarrow b = 1 \) from the adjointness property, and hence also \( a \cdot (a \rightarrow b) = a \cdot 1 = a = a \cap b \).

Assuming on the other hand that one always has \( a \cap b = a \cdot (a \rightarrow b) \); furthermore, for all \( a \leq b \in L \) from \( a = a \cap b = b \cap a \) one gets the equation \( a = b \cdot (b \rightarrow a) \), and hence there is an \( x \) such that \( a = b \cdot x \).

### 10.3.4 Corollary

In a divisible residuated lattice \( \langle L, \cap, \cup, *, \rightarrow, 0, 1 \rangle \) one has for all \( a, b, c \in L \):

(i) \( a \geq b \iff a \cdot (a \rightarrow b) = b \)

(ii) \( b \geq a, c \Rightarrow a \cdot (b \rightarrow c) = c \cdot (b \rightarrow a) \).

Proof. Because of \( a \geq b \iff a \cap b = b \), claim (i) is an immediate consequence of the last proposition. So assume \( b \geq a, c \). Then one gets from an iterated application of (i) \( a \cdot (b \rightarrow c) = b \cdot (b \rightarrow a) \cdot (b \rightarrow c) = (b \rightarrow a) \cdot c \)

and thus (ii).

### 10.3.5 Proposition

A t-norm algebra \( \langle [0, 1], \min, \max, T, I_T, 0, 1 \rangle \) is divisible if and only if the t-norm \( T \) is continuous.

Proof. Assume first that \( T \) is continuous. Then one has for \( a \leq b \in [0, 1] \) immediately

\[
T(a, I_T(a, b)) = T(a, 1) = a = \min\{a, b\}.
\]

And one has for \( b < a \):

\[
T(a, I_T(a, b)) = T(a, \max\{z \mid T(a, z) \leq b\}) = \max\{T(a, z) \mid T(a, z) \leq b\} \leq b
\]

(10.8)

already by the left-continuity of \( T \). Continuity of \( T \) furthermore gives from \( 0 = T(a, 0) \leq b < a = T(a, 1) \) the existence of some \( c \in [0, 1] \) with \( b = T(a, c) \), and thus \( T(a, I_T(a, b)) = b = \min\{a, b\} \) by (10.8). Hence this residuated lattice is divisible according to Proposition 10.3.3.

Assume conversely that \( \langle [0, 1], \min, \max, T, I_T, 0, 1 \rangle \) is a divisible residuated lattice. Then the adjointness condition forces \( T \) to be left-continuous. Hence for the continuity of \( T \) one has to show that \( T \) is also right-continuous.

Suppose that this is not the case. Then there exist some \( a, b \in [0, 1] \), and also in \([0, 1] \) some decreasing sequence \( (x_i)_{i \geq 0} \) with \( \lim_{i \to \infty} x_i = b \) such that \( T(a, x_i) \neq \inf_i T(a, x_i) \), i.e. such that \( T(a, b) < \inf_i T(a, x_i) \). Consider now some \( d \) with \( T(a, b) < d < \inf_i T(a, x_i) \leq a \). Then there does not exist some \( c \in [0, 1] \) with \( d = T(a, c) \), because otherwise one would have \( d = T(a, c) > T(a, b) \), hence \( c > b \) and thus \( \inf_i T(a, x_i) \leq T(a, c) = d \) from the fact that \( b = \lim_{i \to \infty} x_i \) and there thus exists some integer \( k \) with \( x_k \leq c \), i.e. the lack of right-continuity for \( T \) would cause the lack of divisibility for \( \langle [0, 1], \min, \max, T, I_T, 0, 1 \rangle \).

Therefore divisibility implies the continuity of \( T \).
A further restriction is suitable with respect to the class of residuated lattices because each t-norm algebra \([0,1]_\rightarrow\) is linearly ordered, and thus makes the well-formed formula \((\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)\) valid.

So finally, following Hájek [32, 33], we call **BL-algebras** those divisible residuated lattices which also satisfy the *pre-linearity* condition:

\[(x \Rightarrow y) \cup (y \Rightarrow x) = 1.\]  

(10.9)

It is interesting to notice that (10.9) can equivalently be characterized in another form, which will become important later on.

**10.3.6 Proposition**

In residuated lattices the following two conditions are equivalent:

(i) \((x \Rightarrow y) \cup (y \Rightarrow x) = 1\),

(ii) \(((x \Rightarrow y) \Rightarrow z) * ((y \Rightarrow x) \Rightarrow z) \leq z\).

**Proof.** Let \((L, \cap, \cup, *, \Rightarrow, 0, 1)\) be any residuated lattice. Because \(\cup\) is the supremum with respect to the lattice ordering \(\leq\), one immediately has the inequalities

\[(x \Rightarrow y) \leq (x \Rightarrow y) \cup (y \Rightarrow x),\]

\[(y \Rightarrow x) \leq (x \Rightarrow y) \cup (y \Rightarrow x),\]

i.e. one has the equations

\[(x \Rightarrow y) \Rightarrow ((x \Rightarrow y) \cup (y \Rightarrow x)) = 1,\]

\[(y \Rightarrow x) \Rightarrow ((x \Rightarrow y) \cup (y \Rightarrow x)) = 1.\]

From these equations condition (ii) yields \((x \Rightarrow y) \cup (y \Rightarrow x) = 1\), i.e. condition (i).

On the other hand, if one has condition (i) satisfied, then one has also condition (ii) satisfied as the following elementary calculations show:

\[\begin{align*}
((x \Rightarrow y) \Rightarrow z) * (y \Rightarrow x) \Rightarrow z
&= ((x \Rightarrow y) \Rightarrow z) * (y \Rightarrow x) \Rightarrow z
* ((x \Rightarrow y) \cup (y \Rightarrow x)) \\
&\leq (((x \Rightarrow y) \Rightarrow z) * (x \Rightarrow y)) \cup (((y \Rightarrow x) \Rightarrow z) * (y \Rightarrow x)) \\
&\leq z \cup z = z.
\end{align*}\]  

\(\square\)

**10.4 The logic of continuous t-norms**

Summing up, this whole discussion means that the **language** of a t-norm-based (propositional) logic should be appropriate for a residuated lattice as truth degree structure, i.e. should have connectives & for a strong conjunction determined by the basic t-norm, \(\rightarrow\) for an implication which corresponds to the residual of &., together with connectives \(\wedge\) for a weak (idempotent) conjunction, \(\lor\) for some (also idempotent) disjunction, and a negation
¬, and also a truth degree constant \( \overline{0} \). Of course, depending on definitional possibilities, not all these symbols have to be primitive ones.

The axiomatization of Hájek [33] for the basic t-norm logic \( \text{BL} \) (in [30] denoted \( \text{BTL} \)), i.e. for the class of all well-formed formulas which are valid in all BL-algebras, is given in a language which has as basic vocabulary the connectives \( \to, \& \) and the truth degree constant \( \overline{0} \), taken in each BL-algebra \( \langle L, \cap, \cup, +, \cdot, \overline{0}, 0, 1 \rangle \) as the operations \( \to, + \) and the element \( 0 \). Then this t-norm-based logic has as axiom system \( \text{Ax}_\text{BL} \) the following schemata:

\[
\begin{align*}
(A_{\text{BL}1}) & \quad (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)), \\
(A_{\text{BL}2}) & \quad \varphi \& \psi \to \varphi, \\
(A_{\text{BL}3}) & \quad \varphi \& \psi \to \psi \& \varphi, \\
(A_{\text{BL}4}) & \quad (\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi), \\
(A_{\text{BL}5}) & \quad (\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi)), \\
(A_{\text{BL}6}) & \quad \varphi \& (\varphi \to \psi) \to \psi \& (\psi \to \varphi), \\
(A_{\text{BL}7}) & \quad ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi), \\
(A_{\text{BL}8}) & \quad \overline{0} \to \varphi,
\end{align*}
\]

and has as its (only) inference rule the rule of detachment, or: modus ponens (with respect to the implication connective \( \to \)).

The logical calculus which is constituted by this axiom system and its inference rule, and which has the standard notion of derivation, shall be denoted by \( \mathcal{K}_\text{BL} \) or just by \( \text{BL} \). (Similarly in other cases.)

Starting from the primitive connectives \( \to, \& \) and the truth degree constant \( \overline{0} \) the language of \( \text{BL} \) is extended by definitions of additional connectives \( \land, \lor, \lnot \) defined, in accordance with (10.4), (10.5), and (10.2), as

\[
\begin{align*}
\varphi \land \psi &= \text{df} \varphi \& (\varphi \to \psi), \\
\varphi \lor \psi &= \text{df} (\varphi \to \psi) \land ((\psi \to \varphi) \to \varphi), \\
\lnot \varphi &= \text{df} \varphi \to \overline{0},
\end{align*}
\]

where \( \varphi, \psi \) are formulas of the language of that system. Occasionally also a further truth degree constant \( \overline{1} = \text{df} \overline{0} \to \overline{0} \) is added.

Calculations (in BL-algebras) show that the additional connectives \( \land, \lor \) just have the BL-algebraic operations \( \cap, \cup \) as their truth degree functions.

It is a routine matter, but a bit tedious, to check that this logical calculus \( \mathcal{K}_\text{BL} \), usually called the axiomatic system \( \text{BL} \), is sound, i.e. derives only such formulas which are valid in all BL-algebras. A proof is given in [33].

10.4.1 COROLLARY
The Lindenbaum algebra of the axiomatic system \( \text{BL} \) is a BL-algebra.

10.4.2 THEOREM (GENERAL COMPLETENESS)
For a well-formed formula \( H \) of \( \mathcal{L}_T \) the following are equivalent:
(i) \( H \) is derivable within the axiomatic system \( \text{BL} \).
(ii) \( H \) is valid in all BL-algebras.
(iii) \( H \) is valid in all linearly ordered BL-algebras, i.e. in all BL-chains.
See [33]. But even more is provable: the logical calculus $K_{BL}$ characterizes just these formulas which hold true with respect to all those t-norm-based logics which are determined by a continuous t-norm. This was proved in [12].

10.4.3 Theorem (Standard Completeness)
The class of all well-formed formula which are provable in basic logic coincides with the class of all well-formed formulas which are logically valid in all t-norm-based residuated lattices with a continuous t-norm.

The main steps in the proof are to show (i) that each BL-algebra is a subdirect product of BL-chains, i.e. of linearly ordered BL-algebras, and (ii) that each BL-chain can be embedded into the ordinal sum of some BL-chains which are either trivial one-element BL-chains, or linearly ordered MV-algebras, or linearly ordered product algebras, such that (iii) each such ordinal summand is locally embeddable into a t-norm-based residuated lattice with a continuous t-norm, cf. [12, 32] and again [30].

And another generalization of Theorem 10.4.2 deserves to be mentioned. To state it, let us call schematic extension of BL every extension which consists in an addition of finitely many axiom schemata to the axiom schemata of BL. And let us denote such an extension by $BL(C)$. And call $BL(C)$-algebra each BL-algebra $A$ which makes $A$-valid all formulas of $C$.

Then one can even prove (see again [33]):

10.4.4 Theorem (Extended General Completeness)
For each finite set $C$ of axiom schemata and any well-formed formula $H$ of $LT$ there are equivalent:

(i) $H$ is derivable within $BL(C)$,
(ii) $H$ is valid in all $BL(C)$-algebras,
(iii) $H$ is valid in all linearly ordered $BL(C)$-algebras, i.e. in all $BL(C)$-chains.

10.5 The logic of left-continuous t-norms

The fact that for each residuated t-norm lattice $\langle [0, 1], \min, \max, T, I_T, 0, 1 \rangle$ its divisibility is equivalent to the continuity of the determining t-norm $T$ supports the guess that renunciation of this divisibility condition, i.e. the consideration of pre-linear residuated lattices may provide the right algebraic semantics for a logic of all left-continuous t-norms.

This was the idea of Esteva/Godo’s paper [22]. Therefore we now consider pre-linear residuated lattices, and call them MTL-algebras.

Following this idea, one has to modify the previous axiom system in a suitable way. And one has to delete the definition (10.10) of the connective $\land$, because this definition (together with suitable axioms) essentially codes the divisibility condition.\footnote{Definition (10.11) of the connective $\lor$ remains unchanged.}

As a result one now considers a new system MTL of mathematical fuzzy logic, also called monoidal t-norm logic, characterized semantically by the class of all MTL-algebras. It is connected with the axiom system
\( (A_{\text{MTL}1}) \ (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \),
\( (A_{\text{MTL}2}) \ \varphi \& \psi \to \varphi \),
\( (A_{\text{MTL}3}) \ \varphi \& \psi \to \psi \& \varphi \),
\( (A_{\text{MTL}4}) \ (\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi) \),
\( (A_{\text{MTL}5}) \ (\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi)) \),
\( (A_{\text{MTL}6}) \ \varphi \land \psi \to \varphi \),
\( (A_{\text{MTL}7}) \ \varphi \land \psi \to \psi \land \varphi \),
\( (A_{\text{MTL}8}) \ \varphi \& (\varphi \to \psi) \to \varphi \land \psi \),
\( (A_{\text{MTL}9}) \ \top \to \varphi \),
\( (A_{\text{MTL}10}) \ ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \),

Together with the rule of detachment (with respect to the implication connective \( \to \)) as the only inference rule.

It is a routine matter, but again a bit tedious, to check that this axiomatic system MTL is sound, i.e. derives only such formulas which are valid in all MTL-algebras. And, more important, one has also general completeness for the class of all MTL-algebras. Proofs are given in [22].

10.5.1 Corollary
The Lindenbaum algebra of the axiomatic system MTL is an MTL-algebra.

10.5.2 Theorem (General Completeness)
For a well-formed formula \( H \) of \( L_T \) there are equivalent:

(i) \( H \) is derivable within the axiomatic system MTL.
(ii) \( H \) is valid in all MTL-algebras.
(iii) \( H \) is valid in all linearly ordered MTL-algebras, i.e. in all MTL-chains.

But even more is provable: the axiomatic system MTL characterizes just these formulas which hold true with respect to all those t-norm-based logics which are determined by a left-continuous t-norm. A proof is given in [46].

10.5.3 Theorem (Standard Completeness)
The class of all formulas which are provable in the axiomatic system MTL coincides with the class of all formulas which are logically valid in all t-norm-based residuated lattices with a left-continuous t-norm.

Because of the fact that the BL-algebras are the divisible MTL-algebras, one gets another adequate axiomatization of the basic t-norm logic BL if one extends the axiomatic system MTL with the additional axiom schema\(^5\)

\[ \varphi \land \psi \to \varphi \& (\varphi \to \psi). \]

Also for MTL an extended completeness theorem similar to Theorem 10.4.4 remains true if one considers schematic extensions \( \text{MTL}(\mathcal{E}) = \text{MTL} + \mathcal{E} \).

\(^5\) The simplest way to prove that this implication is sufficient is to show that \( x \ast (x \to y) \leq x \cap y \) which corresponds to the converse implication holds true in each MTL-algebra. Similar remarks apply to further extensions of MTL we are going to mention.
10.5.4 Theorem (Extended General Completeness)
For each finite set $\mathcal{C}$ of axiom schemata and any well-formed formula $H$ of $\mathcal{L}_T$ the following are equivalent:

(i) $H$ is derivable within the axiomatic system MTL + $\mathcal{C}$.
(ii) $H$ is valid in all MTL($\mathcal{C}$)-algebras.
(iii) $H$ is valid in all linearly ordered MTL($\mathcal{C}$)-algebras, i.e. in all MTL($\mathcal{C}$)-chains.

10.6 Particular cases of t-norm-based systems

10.6.1 The infinite-valued Łukasiewicz logic
The standard semantics of the infinite-valued Łukasiewicz logic $L_\infty$ (as a t-norm-based logic) is given by the algebra

$$M_L = ([0, 1], \min, \max, T_L, \rightarrow_L, 0, 1)$$

with the residuation operation

$$\rightarrow_L (x, y) = \min \{1, 1 - x + y\},$$

and hence with the Łukasiewicz negation operation $\neg_L(x) = 1 - x$, and has the truth degree 1 as the (only) designated one. A general semantics is given by the class of all MV-algebras, cf. e.g. [13, 30, 33, 52].

These MV-algebras are, however, just those BL-algebras which satisfy the extra condition

$$x = \neg x, \quad \text{i.e.} \quad x = (x \rightarrow 0) \rightarrow 0.$$ 

Therefore, having in mind that the inequality $x \leq \neg x$ is satisfied in each BL-algebra, one gets an adequate axiomatization of $L_\infty$ if one enriches the axiom system of $K_{BL}$ with the axiom schema

$$\neg \neg \phi \rightarrow \phi.$$ 

These implications suffice because the converse ones are BL-provable.

There are simpler formulations of this logic which use a more restricted primitive vocabulary. But this is not of importance for the present considerations. The interested reader may consult [30].

10.6.2 The infinite-valued Gödel logic
The standard semantics of the infinite-valued Gödel logic $G_\infty$ (as a t-norm-based logic) is given by the algebra

$$M_G = ([0, 1], \min, \max, T_M, \rightarrow_G, 0, 1)$$

with the residuation operation

$$\rightarrow_G (x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y, \end{cases}$$
and has the truth degree 1 as the (only) designated one. The corresponding negation operation, known as \( \text{Gödel negation} \), is thus determined as

\[
-_{G}(x, y) = \begin{cases} 
1 & \text{if } x = 0, \\
0 & \text{if } x \neq 0.
\end{cases}
\]

A general semantics for \( \mathbb{G}_{\omega} \) is given by the class of all Gödel algebras, i.e. all those Heyting-algebras (pseudo-complemented lattices) satisfying the pre-linearity condition, cf. e.g. [30, 33]. These algebras are, however, just those BL-algebras which satisfy the extra condition

\[ x = x \cdot x. \]

Therefore one gets an adequate axiomatization of \( \mathbb{G}_{\omega} \) if one enriches the axiom system of BL with the axiom schema

\[ \varphi \to \varphi \& \varphi. \]

Again there are simpler formulations of this logic which use a more restricted primitive vocabulary. Particularly one has that \( \mathbb{G}_{\omega} \) is the intuitionistic logic extended by the schema of pre-linearity. The interested reader may consult [19, 30].

### 10.6.3 Product logic

The standard semantics of the (infinite-valued) product logic \( \Pi \) (as a t-norm-based logic) is given by the algebra

\[ M_{p} = ([0, 1], \min, \max, T_{p}, \rightarrow_{p}, 0, 1) \]

with truth degree 1 as the (only) designated one and with its residuum

\[ \rightarrow_{p} (x, y) = \begin{cases} 
1 & \text{if } x \leq y, \\
y & \text{if } x > y,
\end{cases} \]

and has the truth degree 1 as the (only) designated one. Note that this gives the same negation operation as Gödel negation. A general semantics is given by the class of all product algebras, cf. e.g. [33, 30].

These product algebras are just those BL-algebras which satisfy the extra conditions

\[ x \cap -x = 0, \]
\[ (\neg \neg z) \ast (x \ast z \rightarrow y \ast z) \leq (x \rightarrow y), \]

for the standard negation function \( \neg x = x \rightarrow 0 \). Therefore one gets an adequate axiomatization of \( \Pi \) if one enriches the axiom system of \( \mathbb{K}_{BL} \) with the axiom schemata

\[ \varphi \land \psi \to \varphi, \]
\[ \neg \neg \chi \to ((\varphi \& \chi \rightarrow \psi \& \chi) \rightarrow (\varphi \rightarrow \psi)). \]
10.6.4 Extending MTL

In an analogy with Łukasiewicz logic we may extend MTL by asking its negation to be involutive. Because one has in MTL the logical validity

$$\models_{\text{MTL}} \varphi \rightarrow \neg \neg \varphi,$$

satisfaction of the double negation principle for the standard negation $$\neg$$ in the context of MTL amounts to forcing to add

$$\neg \neg \varphi \rightarrow \varphi$$
as an axiom to the list of axioms of $$K_{\text{MTL}}$$. Let us call this extended system IMTL.

As an interesting side remark let us mention that the system IMTL can also be axiomatized in a form which does not refer to the adjointness condition (10.3), which appears in the axioms of $$K_{\text{MTL}}$$ as the schemata (AxMTL4), (AxMTL5) of importation and exportation, but refers to the rotation construction for t-norms. The details are given in [31].

Note that IMTL is weaker (more general) than Łukasiewicz logic; the most famous example of a left-continuous but not continuous t-norm having its standard negation involutive is Fodor’s nilpotent minimum: for an $$a$$ with $$0 < a < 1$$, $$T(x,y) = 0$$ if $$x + y \leq a$$, otherwise $$T(x,y) = \min(x,y)$$. We may extend MTL in a similar way as product logic extends BL, namely adding to it the axioms (10.13), (10.14) of product logic. The resulting logic is called $$\Pi_{\text{MTL}}$$. As shown in [34], $$\Pi_{\text{MTL}}$$ is weaker (more general) as the product logic: there exists a left-continuous t-norm which makes the axioms of $$\Pi_{\text{MTL}}$$ true but which is not continuous.

Finally note that MTL with the axiom of idempotence of conjunction, i.e. with $$\varphi \rightarrow (\varphi \& \varphi)$$ added, is equivalent to the Gödel logic $$G_\infty$$, cf. [34].

10.6.5 Completeness theorems

For all above mentioned logics we get general completeness theorems as corollaries of the general completeness theorem for BL or MTL. For standard completeness the situation is as follows: Each of the logics $$L_\infty, G_\infty, \Pi$$ has a unique continuous t-norm giving its standard semantics. Standard completeness means that a formula is provable in the respective logic if and only if it is a tautology of this standard semantics. See [33] for proofs. The standard semantics of IMTL is given by all left-continuous t-norms with involutive negation, i.e., $$\neg \neg x = x$$ for each $$x$$. The corresponding standard completeness theorem is proved in [20].

In the same paper, the question whether the logic $$\Pi_{\text{MTL}}$$ has the standard completeness property (with respect to all left-continuous t-norms making the axioms of $$\Pi_{\text{MTL}}$$ true), is left as an open problem.\(^6\)

10.7 Computational complexity and other topics

Each continuous t-norm $$T$$ determines four important sets of formulas, $$[0,1]_T$$ being the BL-algebra given by $$T$$ as in (10.6):

\(^6\) Horčík has announced a positive solution and intends to present it in his Ph.D. Thesis.
$\texttt{1TAUT}(T)$ is the set of all $\varphi$ valid in $[0, 1]_T$, i.e. having for each evaluation of propositional variables by elements of $[0, 1]$ the value 1 in $[0, 1]_T$ (1-tautologies);

$\texttt{posTAUT}(T)$ is the set of all positive tautologies (formulas having for each evaluation a positive value in $[0, 1]$);

$\texttt{SAT}(T)$ is the set of 1-satisfiable $\varphi$ (having the $[0, 1]_T$ value 1 for some evaluation);

$\texttt{posSAT}(T)$ is the set of all positively satisfiable formulas (having a positive $[0, 1]_T$-value).

Similarly we can define analogous sets for a set of t-norms; in particular, for BL we take the set of all continuous t-norms, defining

$\texttt{1TAUT}(BL) = \bigcap \{ \texttt{1TAUT}(T) \mid T \text{ a continuous } T\text{-norm} \}$,

$\texttt{posTAUT}(BL) = \bigcap \{ \texttt{posTAUT}(T) \mid T \text{ a continuous } T\text{-norm} \}$,

$\texttt{1SAT}(BL) = \bigcup \{ \texttt{1SAT}(T) \mid T \text{ a continuous } T\text{-norm} \}$,

$\texttt{posSAT}(BL) = \bigcup \{ \texttt{posSAT}(T) \mid T \text{ a continuous } T\text{-norm} \}$.

We present results on computational complexity of these sets.  

10.7.1 Theorem

For the cases that the t-norm $T$ is the Łukasiewicz, the Gödel, or the product t-norm, the sets $\texttt{1TAUT}(T)$ and $\texttt{posTAUT}(T)$ are co-NP-complete and the sets $\texttt{1SAT}(T)$ and $\texttt{posSAT}(T)$ are NP-complete.

This is shown in [33].

10.7.2 Theorem

$\texttt{1TAUT}(BL)$ and $\texttt{posTAUT}(BL)$ are co-NP-complete, and $\texttt{1SAT}(BL)$ as well as $\texttt{posSAT}(BL)$ are NP-complete.

See [8] and [35]. And the next result was given in [42].

10.7.3 Theorem

For each continuous t-norm $T$, $\texttt{1TAUT}(T)$ is co-NP-complete.

Needless to say, these results imply that all the mentioned sets are recursive (computable). Note that there are several results on equality or inequality among the sets involved (see [33, 35]), e.g.

$\texttt{1SAT}(G_\infty) = \texttt{posSAT}(G_\infty) = \texttt{1SAT}(II) = \texttt{posSAT}(II)$,

$\texttt{1SAT}(L_\infty) \neq \texttt{posSAT}(L_\infty)$,

$\texttt{posSAT}(BL) = \texttt{posSAT}(L_\infty)$.

7 The reader is assumed to know basic notions of polynomial complexity. A suitable reference is e.g. [53].
and many others.

To close this section let us mention some other results, just giving relevant references. The papers [4, 5, 3] contain deep results relating $1$-tautologicity of a formula being a $1$-tautology of $[0, 1]_L$, $[0, 1]_G$, $[0, 1]_Π$ to being a $1$-tautology of finitely many finite valued logics of estimated complexity and similar results for $1TAUT(BL)$. For example, $\varphi$ is in $1TAUT(L)$ if and only if it is a $1$-tautology of the finitely valued Łukasiewicz logic $L_m$ for $m$ being $2^{\#(\varphi)}$ where $\#(\varphi)$ is the number of occurrences of variables in $\varphi$. ($L_m$ has the truth values $0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}, 1$.)

These considerations lead to some sequent calculi for the logics involved; other papers on (hyper)sequent calculi for fuzzy logic are [6, 51]. For investigation of compactness of the logic in question (in different senses of the word compactness) see [17]; for interpolation in fuzzy logics see [9]. [23] presents a detailed hierarchy of t-norm-based residuated fuzzy logics, relating them to Hohle’s monoidal logic and to the Lambek calculus.

### 10.8 Expanding the language

Till now we have discussed logics whose connectives were conjunction $\&$, implication $\to$ (binary), truth constant $\overline{0}$ and connectives defined from them. Here we shall survey some systems having more connectives.

#### 10.8.1 $\Delta$-projection

This connective was introduced to the context of fuzzy logic by Baaz [7], but was used in the context of fuzzy sets already e.g. in [10, 55]. Its standard semantics is $\Delta(1) = 1$ and $\Delta(x) = 0$ for $0 \leq x < 1$; a formula $\Delta \varphi$ may be read “$\varphi$ is absolutely true”. The following are axioms for $\Delta$:

- $(\Delta 1)$ $\Delta \varphi \lor \neg \Delta \varphi$,
- $(\Delta 2)$ $\Delta (\varphi \lor \psi) \to (\Delta \varphi \lor \Delta \psi)$,
- $(\Delta 3)$ $\Delta \varphi \to \varphi$,
- $(\Delta 4)$ $\Delta \varphi \to \Delta \psi$,
- $(\Delta 5)$ $\Delta (\varphi \to \psi) \to (\Delta \varphi \to \Delta \psi)$.

A $BL_{\Delta}$-algebra is an algebra $(L, \cap, \cup, *, \Rightarrow, 0, 1, \Delta)$ which is a $BL$-algebra with an additional operation (denoted also by $\Delta$) in which the above axioms are valid.

Let $BL_{\Delta}$ be the corresponding logic with the above axioms and the deduction rules of modus ponens and of $\Delta$-generalization; from $\varphi$ infer $\Delta \varphi$. This logic has both the general and the standard completeness properties, in particular: a formula is provable in $BL_{\Delta}$ if and only if it is valid in each $BL_{\Delta}$-algebra, if and only if it is valid in each $BL_{\Delta}$-chain, if and only if it is valid in each $BL_{\Delta}$-chain given by a continuous t-norm.

#### 10.8.2 Adding an involutive negation

Recall that the truth degree function of the negation in Łukasiewicz logic is $-L x = 1 - x$, hence it is involutive: $-L -L x = x$. The Gödel logic and the product logic are examples
of t-norm-based logics with Gödel negation and thus having $x \cap -x = 0$ for each $x$, i.e. having $\neg(\varphi \land \neg \varphi)$ as a tautology. In [25] one studies the logic $SBL$ (BL plus the axiom schema $\varphi \land \neg \varphi \to \emptyset$) and its extension $SBL_\sim$ by a new (second) negation $\sim$ understood as involutive. Now $\triangle$ becomes a definable connective: $\triangle \varphi$ is $\neg \sim \varphi$. The axioms of $SBL_\sim$ are those of $SBL$ plus
\begin{align*}
(\sim 1) & \quad \sim \varphi \equiv \varphi, \\
(\sim 2) & \quad \neg \varphi \equiv \neg \varphi, \\
(\sim 3) & \quad \triangle(\varphi \to \psi) \to \triangle(\neg \psi \to \neg \varphi).
\end{align*}

The deduction rules are as in $SBL$.\footnote{Originally also $(\Delta 1), (\Delta 2), (\Delta 5)$ had been postulated to be axioms. However, they are redundant as shown in [14].} Note that the remaining $\Delta$-axioms become provable.

The standard semantics interprets $\sim$ as the Łukasiewicz negation. $SBL_\sim$-algebras are defined in the obvious way. General completeness holds true; for $G_\sim$ (the extension of Gödel logic with the involutive negation) we have even the standard completeness property (but not for $\Pi_\sim$, the product logic with involutive negation). Note that according to [43], the sets $TAUT(G_\sim)$ and $TAUT(\Pi_\sim)$ are both co-NP complete.

### 10.8.3 Putting Łukasiewicz and product logic together

The logic combining Łukasiewicz connectives and connectives of product logic has been intensively studied and proved to be very powerful. Relevant references are [21,26,15,14,16]. Since in Łukasiewicz logic the strong conjunction $\&$ is definable from implication and negation, one can take as primitive connectives just two implications $\to_L$, $\to_P$ and one conjunction $\circ$ (plus the constant $\overline{0}$). From these one defines two negations $\neg_L$, $\neg_P$ ($\neg_L \varphi$ is $\varphi \to_L \overline{0}$, similarly for $\neg_P$) the Łukasiewicz conjunction (now denoted $\circ$, $\varphi \circ \psi$ is $\neg_L (\varphi \to_L \neg_P \psi)$), further $\land, \lor, \oplus$ (Łukasiewicz’s strong disjunction), also $\circ (\varphi \circ \psi$ is $\varphi \circ \neg_L \psi$ and finally $\triangle \varphi$ (which is $\neg_L \neg_P \varphi$). The standard semantics is obviously given by the unit interval $[0, 1]$ together with the Łukasiewicz and the product conjunctions and their residua. There are several mutually equivalent axiomatizations, we choose the following:

- Axioms of Łukasiewicz logic,
- Axioms of product logic (without $\Lambda 7$),
- $\neg_P \phi \to \neg_L \phi$,
- $\triangle(\phi \to_P \psi) \equiv_L \triangle(\varphi \to_P \psi)$,
- $\phi \circ (\psi \circ \chi) \equiv_L ((\phi \circ \psi) \circ (\varphi \circ \chi))$. (distributivity)

The deduction rules are modus ponens and $\triangle$-generalization. This logic is called $\mathcal{L}P$; $\mathcal{L}P$-algebras are defined in the obvious way.

$\mathcal{L}P\frac{1}{2}$ is the logic with an additional truth degree constant $h$ (or $\frac{1}{2}$) and the additional axiom $h \equiv_L \neg_L h$. In the standard semantic the value of this constant $h$ is $\frac{1}{2}$.

Both the general and the standard completeness theorems hold true for $\mathcal{L}P$ and $\mathcal{L}P\frac{1}{2}$; a formula is provable if and only if it is a standard tautology if and only if it is a tautology with respect to all $\mathcal{L}P$-algebras, or all $\mathcal{L}P\frac{1}{2}$-algebras, respectively.
ŁΠ (and also ŁΠ₁) extends conservatively both L∞ and Π. Furthermore, the Gödel implication → is definable in ŁΠ and thus the Gödel logic becomes a sublogic of it, and ŁΠ₁ extends G∞ even conservatively.

Even more holds true: For each continuous t-norm T which is the ordinal sum of finitely many copies of L∞, Π, G∞, there is a syntactic transformation ϕ ↦ ϕ′ of formulas making the logic given by T a sublogic of ŁΠ₁. ϕ is a tautology of [0, 1]T if and only if ϕ′ is ŁΠ₁-provable.

As far as computational complexity is concerned, TAUT(ŁΠ₁) can be shown to be in PSPACE by reducing it to the set of universal formulas of the theory of real numbers, cf. [41].

On the other hand, Cintula [14] has shown that ŁΠ is (equivalent to) the extension of Π (product logic with an involutive negation) by the definition of ϕ →L ψ as ¬L(ϕ ⊗ ¬L(ϕ →Π ψ)) (where →Π, ⊗ are connectives of Π and ¬L is the involutive negation) and by the single axioms schema of transitivity of →L, i.e. by

\[(ϕ →L ψ) → ((ψ →L χ) → (ϕ →L χ)).\]

Finally, let us mention the paper [45] studying the extension of Łukasiewicz logic by product conjunction (but not by product implication).

### 10.9 Further generalizations

The basic fuzzy logic BL has been generalized to the monoidal t-norm logic MTL by giving up divisibility. Here we discuss generalizations giving up the constant 0 (falsity free logics) or the commutativity of conjunction.

#### 10.9.1 Hoop logics

Hoops were studied in algebra originally without any relation to fuzzy logic; the relation of hoops to BL-algebras was studied in [1]. A full treatment of hoop logic is presented in [24].

#### 10.9.1 Definition

A hoop is an algebra \(H = (H, *, \rightarrow, 1)\) such that * is commutative with 1 as a unit element such that

\[
\begin{align*}
x \rightarrow x &= 1, \\
x * (x \rightarrow y) &= y * (y \rightarrow x), \\
(x * y) \rightarrow z &= x \rightarrow (y \rightarrow z)
\end{align*}
\]

for all \(x, y, z\). (Thus axioms (A4), (A5) of BL are valid.) Define \(x \leq y\) to mean \(x \rightarrow y = 1\).

It follows that \(\leq\) is an ordering, * is associative and monotone (non-decreasing) and 1 is the greatest element.
One further defines: a hoop $H$ is \textit{bounded} if and only if it has a least element; and $H$ is \textit{basic} if and only if (A6) is valid, i.e. if and only if
\[(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z\]
holds true for all $x, y, z$.

It turns out that the BL-algebras are just bounded basic hoops. The \textit{hoop logic} BLH has the axiom (A1)–(A6) of basic logic, i.e. all axioms of BL except (A7). Its general semantics is formed by the variety of all basic hoops.

The \textit{general completeness theorem} says that BLH proves $\varphi$ if and only if $\varphi$ is valid in all basic hoops if and only if it is valid in all hoop chains.

Each continuous t-norm $T$ defines a basic hoop (in fact a BL-chain) on $[0,1]$; for $T$ being Gödel or product, the restriction of $[0,1]$ to the half-open interval $(0,1]$ is a basic hoop without a least element.

Both BL and SBL are conservative extensions of BLH; in other words, a formula $\varphi$ not containing the constant $\emptyset$ (and thus not containing negation) is provable in BLH if and only if it is provable in BL. This implies the \textit{standard completeness} theorem: BLH proves $\varphi$ if and only if $\varphi$ is a tautology of each $[0,1]$, $T$ being a continuous t-norm.

Using hoops one can prove an elegant representation theorem of BL-chains as ordinal sums of Wajsberg hoop chains [2]. A hoop chain is \textit{Wajsberg} if and only if the identity $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ is valid in it. Assume that $\{H_i\}_{i \in I}$ is a linearly ordered system of basic hoops which all have the same maximal element 1, and which otherwise are pairwise disjoint. Let $H = \bigcup H_i$ and let
\[x \leq y \quad \text{if and only if for some } i: x, y \in H_i \text{ and } x \leq_i y, \text{ or else } x \in H_i, y \in H_j \text{ and } i < j;
\]
\[x * y = z \quad \text{if and only if for some } i: x, y \in H_i \text{ and } z = x *_i y, \text{ or else } x * y = \min(x, y).
\]
Finally let $x \rightarrow y$ be the residuum of $*$. (It is uniquely determined.)

Then $H = (H, *, \rightarrow, 1)$ is a basic hoop denoted by $\bigoplus_{i \in I} H_i$ and called the ordered sum of the hoops $H_i, i \in I$. It can be proved that each basic hoop chain is the ordered sum of a system of Wajsberg hoop chains. And each BL-chain is an ordered sum of a system $\{H_i\}_{i \in I}$ of Wajsberg hoop chains having a minimal member $H_0$, which is bounded.

In [24] one finds a definition and properties of stronger logics LH, GH, ΠH generalizing Łukasiewicz, Gödel and product logic. Furthermore, one has \textit{semigroups} which are a falsity-free generalization of MTL-algebras and the corresponding MTLH-logic whose axioms are all axioms of MTL except (A7). One gets general completeness, conservativeness of MTL over MTLH and hence standard completeness of MTLH with respect to algebras on $[0,1]$ given by left-continuous t-norms.

10.9.2 Fuzzy logics with a non-commutative conjunction

The papers [18] present an extensive study of pseudo-BL-algebras, a generalization of BL-algebras not assuming the commutativity of the $*$-operation. The definition reads as follows:
A pseudo-BL-algebra, or: psBL-algebra for short, is an algebraic structure

\[ A = \langle A, \cap, \cup, *, \rightarrow, \leadsto, 0, 1 \rangle \]

such that \( \langle A, \cap, \cup, 0, 1 \rangle \) is a lattice with universal bounds 0, 1; * is associative with 1 as a unit element, and the following double residuation, double divisibility and double pre-linearity conditions hold:

\[ x * y \leq z \iff x \leq y \rightarrow z \iff y \leq x \leadsto z, \]
\[ x \cap y = (x \rightarrow y) * x = x * (y \rightarrow y), \]
\[ (x \rightarrow y) \cup (y \rightarrow x) = (x \leadsto y) \cup (y \leadsto x) = 1. \]

If one drops the double divisibility one gets the definition of pseudo-MTL-algebras (first studied in [27] under the name “weak pseudo-BL-algebras”); see also [38]).

A pseudo-t-norm is a binary operation \( T \) on \([0,1]\) satisfying all conditions put on a t-norm except commutativity; in particular, a pseudo-t-norm is associative, monotone in both arguments, and has 0 and 1 as zero and unit element. Each continuous pseudo-t-norm is commutative, hence a t-norm (see [27]); but there are non-commutative left-continuous pseudo-t-norms. (Mesiar’s example: given \( 0 < a < b < 1 \), let \( x * y = 0 \) for \( x \leq a, y \leq b \) and \( x * y = \min(x,y) \) elsewhere.) For a sort of “semistandard” non-commutative psBL-algebras see [36].

The corresponding logics psBL, psMTL are developed in [37, 38]; their axiom systems are careful analogies of the axioms of BL and MTL, e.g. \((Ax_{BL}1)\) (cf. p. 282) splits into the two axioms

\[ (\psi \rightarrow \chi) \rightarrow ((\phi \rightarrow \psi) \rightarrow \phi \rightarrow \chi), \]
\[ (\psi \leadsto \chi) \rightarrow ((\phi \leadsto \psi) \rightarrow (\phi \leadsto \chi). \]

And similarly \((Ax_{BL}4), (Ax_{BL}5)\) become

\[ (\phi \rightarrow (\psi \rightarrow \chi)) \iff ((\phi & \psi) \rightarrow \chi), \]
\[ (\phi \leadsto (\psi \leadsto \chi)) \iff ((\psi \& \phi) \rightarrow \chi), \]

etc. (Here the formula \( \phi_1 \iff \phi_2 \) is used as shorthand for the two formulas \( \phi_1 \rightarrow \phi_2 \) and \( \phi_2 \rightarrow \phi_1 \).)

The deduction rules are modus ponens for \( \rightarrow \) and two new rules for implications: from \( \phi \rightarrow \psi \) infer \( \phi \leadsto \psi \), and conversely.

The logics psBL and psMTL have the general completeness property w.r.t. all psBL-algebras, or all psMTL-algebras, respectively. But (surprisingly) these algebras do not satisfy the subdirect representation property. Kühr [50] invented additional axioms characterizing the subdirectly representable psBL-algebras, and his proof works also for psMTL. This leads to the following additional axioms to extend these logics:

\[ (\psi \rightarrow \phi) \lor (\chi \leadsto ((\phi \rightarrow \psi) & \chi)), \]
\[ (\psi \leadsto \phi) \lor (\chi \rightarrow (\chi & (\phi \leadsto \psi))). \]
The extensions of psBL, psMTL by these axioms are called psBLr, psMTLr. These extended logics have the general completeness property with respect to all linearly ordered psBL-algebras, or psMTL-algebras, respectively.

Jenei and Montagna [47] proved the standard completeness theorem for psMTLr: A formula is provable in psMTLr if and only if it is valid in each standard t-norm algebra \([0, 1]_T\) given by a left-continuous pseudo-t-norm and by its residua.

Finally let us mention the paper [39] which gives a common generalization of all these above mentioned logics, thus giving up divisibility, the falsity constant, and commutativity. The corresponding algebras are called fleas (or flea algebras), and the logic is the flea logic FlL. The extension by Kühr’s axioms is FlLr. There are examples of fleas on \((0, 1]\) not satisfying divisibility, nor commutativity, and having no least element. The axioms are those of psMTL without the axiom which mentions the constant 0. The logic psMTL is a conservative extension of FlL; the same holds true for psMTLr and FlLr. One also has general completeness theorems analogously as for psMTL, psMTLr. The flea logic appears to be the most general t-norm-based (in fact pseudo-t-norm-based) fuzzy logic.

10.10 Allowing graded notions of consequence

The systems of t-norm-based logics discussed up to now have been designed to formalize the logical background for fuzzy sets, and they allow themselves for degrees of truth of their formulas. But they all have crisp notions of consequence, i.e. of entailment and of provability.

It is natural to ask whether it is possible to generalize these considerations to the case that one starts from fuzzy sets of formulas, and that one gets from them as consequence hulls again fuzzy sets of formulas. This problem was first treated by Pavelka [54]. The basic monograph elaborating this approach is Novák et al. [52]. We shall follow their kind of approach, because it uses graded relations of entailment and of provability.

However, it should be mentioned that there is also another, more algebraically oriented approach toward consequence operations for the classical case, originating from Tarski [56] and presented e.g. in [58]. This approach treats consequence operations as closure operations. And this type of approach has been generalized to closure operations in classes of fuzzy sets of formulas by Gerla [28].

The Pavelka-style approach has to deal with fuzzy sets \(\Sigma^-\) of formulas, i.e. besides formulas \(\phi\) also their membership degrees \(\Sigma^- (\phi)\). And these membership degrees are just the truth degrees. We may assume that these degrees again form a residuated lattice \(\Sigma = (\Lambda, \cap, \cup, *, \Rightarrow, 0, 1)\). Thus we (slightly) generalize the naive notion of fuzzy set (with membership degrees from the real unit interval). Therefore the appropriate language has the same logical connectives as in the previous considerations.

The Pavelka-style approach is an easy matter as long as the entailment relationship is considered. An evaluation \(e\) is a model of a fuzzy set \(\Sigma^-\) of formulas if and only if

\[\Sigma^- (\phi) \leq e(\phi)\]
holds for each formula $\varphi$. This immediately yields as definition of the entailment relation that the semantic consequence hull of $\Sigma^\sim$ should be characterized by the membership degrees

$$C^{\text{sem}}(\Sigma^\sim)(\psi) = \bigwedge \{ e(\psi) \mid e \text{ model of } \Sigma^\sim \}$$

for each formula $\psi$.

For a syntactic characterization of this entailment relation it is necessary to have some calculus $K$ which treats formulas of the language together with truth degrees. So the language of this calculus has to extend the language of the basic logical system by having also symbols for the truth degrees. Further on we indicate these symbols by overlined letters like $\overline{a}, \overline{c}$. And we realize the common treatment of formulas and truth degrees by considering evaluated formulas, i.e. ordered pairs $(\overline{a}, \varphi)$ consisting of a truth degree symbol and a formula. This trick transforms in a natural way each fuzzy set $\Sigma^\sim$ of formulas into a (crisp) set of evaluated formulas, again denoted by $\Sigma^\sim$.

So $K$ has to allow to derive evaluated formulas out of sets of evaluated formulas, of course using suitable axioms and rules of inference. Derivations in $K$ out of some set $\Sigma^\sim$ of evaluated formulas are finite sequences of evaluated formulas which either are axioms, or elements of $\Sigma^\sim$, or result from former evaluated formulas by application of one of the inference rules.

Each $K$-derivation of an evaluated formula $(\overline{a}, \varphi)$ counts as a derivation of $\varphi$ to the degree $a \in L$. The provability degree of $\varphi$ from $\Sigma^\sim$ in $K$ is the supremum over all these degrees. This now yields that the syntactic consequence hull of $\Sigma^\sim$ should be the fuzzy set $C^{\text{syn}}_K$ of formulas characterized by the membership function

$$C^{\text{syn}}_K(\Sigma^\sim)(\psi) = \bigvee \{ a \in L \mid K \text{ derives } (\overline{a}, \psi) \text{ out of } \Sigma^\sim \}$$

for each formula $\psi$.

Despite the fact that $K$ is a standard calculus, this is an infinitary notion of provability. For the infinite-valued Łukasiewicz logic $L_\infty$ this machinery works particularly well because it needs in an essential way the continuity of the residuation operation. In this case we can form a calculus $K_L$ which gives an adequate axiomatization for the graded notion of entailment in the sense that one has suitable soundness and completeness results.

This calculus $K_L$ has as axioms any axiom system of $L_\infty$ which provides together with the rule of detachment an adequate axiomatization of $L_\infty$, but $K_L$ replaces this standard rule of detachment by the generalized form

$$(\overline{a}, \varphi) \quad (\overline{c}, \varphi \rightarrow \psi)$$

for evaluated formulas.

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9 Depending upon the truth degree structure, this may mean that the language of this calculus becomes an uncountable one.

10 These axioms are usually only formulas $\varphi$ which, however, are used in the derivations as the corresponding evaluated formulas $(\overline{1}, \varphi)$. 
The soundness result for this calculus $K_L$ yields the fact that the $K_L$-provability of an evaluated formula $(\pi, \phi)$ says that $a \leq e(\phi)$ holds for every valuation $e$, i.e. that the formula $\pi \to \phi$ is valid—however as a formula of an extended propositional language which has all the truth degree constants among its vocabulary.\footnote{Of course, now the evaluations $e$ have additionally to satisfy $e(\pi) = a$ for each $a \in [0, 1]$.}

And the completeness result for $K_L$ says that a strong completeness theorem holds true giving

$$c^{\text{sem}}(\Sigma^-)(\psi) = c^{\text{syn}}(\Sigma^-)(\psi) \quad (10.15)$$

for each formula $\psi$ and each fuzzy set $\Sigma^-$ of formulas.

If one takes the previously mentioned turn and extends the standard language of propositional $L_\infty$ by truth degree constants for all degrees $a \in [0, 1]$, and if one reads each evaluated formula $(\pi, \phi)$ as the formula $\pi \to \phi$, then a slight modification $K_L^+$ of the former calculus $K_L$ again provides an adequate axiomatization: one has to add the bookkeeping axioms

$$(\pi \to \tau) \equiv a \Rightarrow_L c,$$

as explained e.g. in [52]. And if one is interested to have evaluated formulas together with the extension of the language by truth degree constants, one has also to add the degree introduction rule

$$\frac{(\pi, \phi)}{a \to \pi}.$$

However, even a stronger result is available which refers only to a notion of derivability over a countable language. The completeness result (10.15), for $K_L^+$ instead of $K_L$, becomes already provable if one adds truth degree constants only for all the rationals in $[0, 1]$, as was shown in [33]. And this extension of $L_\infty$ is even only a conservative one, cf. [40]. For more details the reader may also consult e.g. [33, 52, 57].

10.11 Conclusion

This surveyed offers the present state of knowledge on fuzzy propositional logics that are \textit{t-norm-based}, i.e. related in one way or another to (left) continuous t-norms (or to pseudo-t-norms) as truth functions for a (usually non-idempotent) conjunction, and which have the corresponding residua as truth functions for implication(s). For each logic we distinguished a standard semantics, provided by t-norm-based structures as just mentioned, and also some general semantics given always by a class, usually a variety, of algebras naturally related to (and generated by) the standard algebras of truth functions. But we did not develop any systematic algebraic theory; this would need another paper. We hope that we have shown that the wide family of t-norm-based fuzzy logics is a rich and interesting one, and that we have collected the relevant references. Needless to say, this survey cannot contain all details and facts; the interested reader will find them (as well as several open problems) in the papers we refer to. In particular, for fuzzy predicate logic see [33].
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References


