A k-factor of a graph is a k-regular spanning subgraph. A Hamilton cycle is a connected 2-factor. A graph $G$ is said to be primitive if it contains no k-factor with $1 \leq k < \Delta(G)$. A Hamilton decomposition of a graph $G$ is a partition of the edges of $G$ into sets, each of which induces a Hamilton cycle. In this paper, by using the amalgamation technique, we find necessary and sufficient conditions for the existence of a 2k-regular graph $G$ on $n$ vertices which:

1. has a Hamilton decomposition, and
2. has a complement in $K_n$ that is primitive.


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This theorem stands at the other end of an avenue of research in the literature, setting limits on results that seek to extend sets of edge-disjoint 2-factors of one kind to a (complete) 2-factorization of $K_n$ in which the added 2-factors have another property. For example, Buchanan [4], in his dissertation written under the supervision of A.J.W. Hilton, used amalgamations to show that $K_n - E(U)$ has a Hamilton decomposition for any odd $n$ and for any 2-factor $U$ of $K_n$. This result and extensions of it have now also been proved using difference methods [3,13].

Another related question in the literature is the Hamilton–Waterloo problem: For which values of $s$, $t$, and $z$ does there exist a 2-factorization of $K_n$ in which $z$ of the 2-factors consist entirely of $s$-cycles, and the rest consist of $t$-cycles? Horak, Nedela, and Rosa recently addressed this problem [9], making progress in the case when $s = n$ (so these 2-factors are Hamilton cycles) and $t = 3$. Results of Adams et al. [1] address the situation where both $s$ and $t$ are small.

In this paper, we continue the tradition begun by A.J.W. Hilton by finding necessary and sufficient conditions on $(x, n)$ to be able to partition $E(K_n)$ into 2 sets, one of which induces a 2x-regular graph that has a Hamilton decomposition, the other of which induces a primitive graph. Not only is this an interesting graph decomposition, but it also has the appeal of setting limits on results like those addressed by Horak, Nedela, and Rosa and by Buchanan. The results proved here show that for any $x \geq \frac{n - \sqrt{n}}{2}$, when $n$ is even, we can select $x$ edge-disjoint Hamilton cycles and be left with no 2-factors of any type in the complement (See [11,5] when the complete graph is replaced by complete multipartite graphs).

In this paper, graphs may have multiple edges and loops, with each loop contributing 2 to the degree of the incident vertex. The number of edges between $u$ and $v$ in $G$ is denoted by $m_G(u, v)$ or simply $m(u, v)$. If $G$ has an edge-coloring, then let $G(i)$ be the subgraph of $G$ induced by the edges colored $i$, and let $\omega(G)$ be the number of components in $G$ (following the notation in [2]).

2. Primitive graphs

If $d$ is even, then Petersen’s Theorem [12] precludes any non-trivial primitive $d$-regular graph. It is known that there exist primitive regular graphs for every odd degree $d$ [8]. In fact, there exists a primitive regular graph of order $n$ and odd degree $d$ if and only if $n \geq (d + 1)^2$ and $n$ is even. We now define a family of such graphs. For each odd $d \geq 3$ and each even $n \geq (d + 1)^2$, define a set of $d$-regular graphs on $n$ vertices $G(n, d)$ as follows:

$G \in G(n, d)$ if and only if

(a) $G$ contains a cut-vertex $v$ such that $G - v$ has a partition into $d$ subgraphs $C_1, C_2, \ldots, C_d$, where $C_1, C_2, \ldots, C_{d-1}$ are components of $G - v$, and $C_d$ is the union of the remaining components of $G - v$.

(b) Each of $C_1, C_2, \ldots, C_{d-1}$ contains exactly $d + 2$ vertices, exactly one of which is adjacent to $v$.

(c) $C_d$ has $n - (d - 1)(d + 2) - 1$ vertices, exactly one of which is adjacent to $v$, and

(d) $G$ is $d$-regular.

Note that (c) is implied by the other conditions.

In this paper it is easily shown that any $G \in G(n, d)$ is primitive, and then the amalgamation technique is used to show that $K_n - E(G)$ has a Hamilton decomposition for some $G \in G(n, d)$. This shows that the spectrum of edge-disjoint Hamilton cycles that have primitive complements is equal to the spectrum of maximal sets of 2-factors. Computing this first spectrum is our main result.

To see that graphs in $G(n, d)$ are primitive, consider the following result.

Lemma 2.1. For each odd $d \geq 3$, and even $n \geq (d + 1)^2$, any $G \in G(n, d)$ is primitive.

Proof. Note that, by construction, each of $C_1, C_2, \ldots, C_d$ in $G - v$ has an odd number of vertices. Suppose there exists a proper $d'$-factor $F$. Since $d' < d$, there would be two of $C_1, C_2, \ldots, C_d$, say $C_i$ and $C_j$, such that $F$ contains the edge joining $C_i$ to $v$ but does not contain the edge joining $C_j$ to $v$. But then in the components induced by $V(C_i)$ and $V(C_j)$ in $F - v$, the number of vertices of odd degree differs by 1. So, one of the components has an odd number of vertices of odd degree, which leads to a contradiction.

3. Amalgamations and preliminary results

Informally, an amalgamation of a graph $H$ is a new graph $A$, formed by partitioning the vertices of $H$ and representing each element $p$ of the partition $P$ with a single vertex in $A$, where edges incident with this single vertex are in one-to-one correspondence with the edges incident with original vertices of $H$ in $P$; so edges in $H$ joining two vertices in $p$ correspond to loops in $A$. In other words, for each edge $[u, v]$ in $H$, if $u \in p_1$ and $v \in p_2$, then we add an edge $[p_1, p_2]$ in $A$ (edges between two vertices in the same element of the partition correspond to loops in $A$).

Formally, an amalgamation $A$ of a graph $H$ is formed by a graph homomorphism $f : V(H) \to V(A)$, where each vertex $v$ of $A$ represents $\eta(v) = |f^{-1}(v)|$ vertices of $H$. $\eta(v)$ is called the amalgamation number of $v$, and $f$ is called the amalgamation function of $H$. Notice that any edge coloring of $H$ naturally induces an edge-coloring of $A$ under the homomorphism $f$.

So, how do we use amalgamations? Given a graph $A$ with amalgamation numbers, one could try to find graphs which have $A$ as an amalgamation. Conceptually, this could be achieved by taking each vertex $v$ with $\eta(v) > 1$ and “peeling out”
vertices one by one, at each stage producing a graph $H$ for which $A$ is an amalgamation. $H$ is said to be a disentanglement of $A$. So, every disentanglement $H$ of $A$ has an associated amalgamation function $f$ of $H$. Furthermore, if $A$ is edge-colored, then this disentanglement naturally induces an edge-coloring of $H$.

In this paper, we want to color the edges of an amalgamation of $K_n$ so that when we disentangle the amalgamation, color class 0 will induce a primitive graph and each other color class will induce a Hamilton cycle. The crucial tool for this proof is Theorem 3.3, which says that we can disentangle the amalgamation of $K_n$ so that the colored edges incident to a vertex in the amalgamation will split up evenly among the corresponding vertices in the disentanglement. What we need to do is to show that the conditions of Theorem 3.3 hold.

Now, let $n$ and $d$ be fixed. For every graph $G \in G(n, d)$, let $K(G)$ be a 2-edge-colored copy of $K_n$ with colors 0 and $\alpha$ in which the edges colored 0 induce a copy of $G$ (in a later proof, the edges colored $\alpha$ will be partitioned into several color classes where each color class induces a Hamilton cycle).

Point 3.3 Let $\eta$ be the amalgamation formed from $K(G)$ using the partition $\{[v], V(C_1), \ldots, V(C_{d-1})\}$. Properties (a) and (b) imply that for any two graphs $G_1, G_2 \in G(n, d)$, the amalgamations $K'(G_1)$ and $K'(G_2)$ are isomorphic. Thus, we let $K(n, d)$ be the unique edge-colored amalgamated graph formed like this. Note that $V(K(n, d)) = \{a_i \mid 0 \leq i \leq d\}$, and that

$$\eta(a_i) = \left\{ \begin{array}{ll} 1 & \text{if } i = 0, \\
2 & \text{if } 1 \leq i \leq d - 1, \\
n - (d - 1)(d + 2) - 1 & \text{if } i = d. \end{array} \right.$$ 

(\*)

Then $K(n, d)$ has the following properties:

(a') There are no edges colored 0 joining $a_i$ and $a_j$, for $1 \leq i < j \leq d$,
(b') There is exactly one edge colored 0 joining $a_i$ and $a_0$, for $1 \leq i \leq d$,
(c') $d_{A(0)}(a_i) = \eta(a_i)d$, for $0 \leq i \leq d$,
(d') $d_{A(0)}(a_i) = (n - 1 - d)\eta(a_i)$, for $0 \leq i \leq d$.

A disentanglement $H$ of $A$ is said to be regular if each color class of $H$ is regular, and a disentanglement $H$ of $A$ is said to be final if $\eta(h) = 1$ for all $h \in V(H)$.

**Lemma 3.1.** Every regular final disentanglement $H$ of $K(n, d)$ has the property that $H(0) \in G(n, d)$ where $H(0)$ is the subgraph of $H$ induced by the edges colored 0.

**Proof.** Let $H$ be a regular final disentanglement of $K(n, d)$. By (a'), there are no edges colored 0 joining $a_i$ and $a_j$, for $1 \leq i < j \leq d$, so there is a cut-vertex in $H(0)$ and this satisfies (a). By (\*) and (b'), each of $C_1, \ldots, C_{d-1}$ contains exactly $d + 2$ vertices and $C_d$ contains $n - (d - 1)(d + 2) - 1$ vertices; in each case exactly one vertex of which is adjacent to cut-vertex $v$. This proves (b) and (c).

By (c'), $d_{A(0)}(a_i) = \eta(a_i)d$. Since $H$ is regular and it is a final disentanglement, $\eta(h) = 1$ for each $h \in H$. This says that $H(0)$ is $d$-regular, proving (d). Hence, $H(0) \in G(n, d)$. ■

We will use the following two results. We will color the edges with colors 0, 1, $\ldots$, $\ell$, so we state the results here for ($\ell$ + 1)-edge-colorings.

**Lemma 3.2** ([10]). Let $H \cong K_n$ be an ($\ell + 1$)-edge-colored graph where each color class $i$ is $d_i$-regular, and let $f : V(H) \to V(G)$ be an amalgamation function with amalgamation numbers given by the function $\eta : V(G) \to \mathbb{N}$. The following conditions hold for any pair of vertices $w, v \in V(G)$:

1. $d(w) = \eta(w)(n - 1)$,
2. the number of edges between $w$ and $v$ is $m(w, v) = \eta(w)\eta(v)$ if $w \neq v$,
3. $w$ has $\eta(w)(\eta(w) - 1)/2$ loops, and
4. $d_{G(i)}(w) = \eta(w)d_i$ for each color $i \in \{0, 1, \ldots, \ell\}$.

**Theorem 3.3** ([10]). Let $A$ be an ($\ell + 1$)-edge-colored graph satisfying conditions (1)–(4) of **Lemma 3.2** for the function $\eta : V(A) \to \mathbb{N}$. Then there exists a disentanglement $H$ of $A$ with amalgamation function $f(H)$ such that $H \cong K_n$ and the following two conditions hold:

(i) For any $z \in V(A)$, degree $d_{H(i)}(u) \in \left\{ \left\lceil \frac{d_{A(i)}(z)}{\eta(i)} \right\rceil, \left\lfloor \frac{d_{A(i)}(z)}{\eta(i)} \right\rfloor \right\}$ for all $i \in \{0, \ldots, \ell\}$ and all $u \in f^{-1}(z)$, and

(ii) If $\frac{d_{A(i)}(z)}{\eta(i)}$ is an even integer for all $z \in V(A)$, then $w(A(i)) = \omega(H(i))$. 
This result will be used in the following way: We will color the edges of \(K(n, d)\) with two colors 0 and \(\alpha\), and then recolor the edges colored \(\alpha\) with \((n - d - 1)/2\) colors in such a way that, each color class produces a Hamilton cycle in \(K_n\) when Theorem 3.3 is applied to the recolored graph. To do this, we will need Lemma 3.4. We will also show that the edges colored 0 in \(K_n\) induce a copy of \(G\) for some \(G \in G(n, d)\).

An edge-coloring of \(G\) is said to be *evenly-equitable* if for each pair of colors \(i\) and \(j\) and for each \(v \in V(G)\), \(d_{G_0}(v)\) is even and \(|d_{G_0}(v) - d_{G_0}(v)| \in \{0, 2\}.

**Lemma 3.4** ([7]). For each \(m \geq 1\), each finite Eulerian graph has an evenly-equitable edge-coloring with \(m\) colors.

**4. Main result**

In this section, we will prove the following theorem.

**Theorem 4.1.** There exists a set \(S\) of \(x\) edge-disjoint Hamilton cycles in \(K_n\) such that \(K_n - E(S)\) is primitive if and only if

\[
\begin{align*}
x &= \frac{n - 1}{2} \quad \text{if } n \text{ is odd, and} \\
x &\geq \frac{n - \sqrt{n}}{2} \quad \text{if } n \text{ is even.}
\end{align*}
\]

We begin by proving the following theorem, which implies the sufficiency of the condition for even \(n\).

**Theorem 4.2.** For each odd \(d \geq 3\) and each even \(n \geq (d + 1)^2\), there exist a \(G \in G(n, d)\) such that \(K_n - E(G)\) has Hamilton decomposition.

**Proof.** We begin with the 2-edge-colored graph \(K(n, d)\) on \(d + 1\) vertices, which is an amalgamation of \(K_n\) and has the amalgamation numbers given in (\(*\)).

Let \(A = K(n, d)\) for convenience. By multiplying the amalgamation numbers in (\(*\)) by \((n - 1)\), we get:

\[
\begin{align*}
d_\alpha(a_0) &= n - 1, \\
d_\alpha(a_i) &= (d + 2)(n - 1) \quad \text{for } 1 \leq i \leq d - 1, \quad \text{and} \\
d_\alpha(a_d) &= (n - 1)(n - d - 1)(d + 2) - 1,
\end{align*}
\]

where \(a_0 \in V(A)\) corresponds to the cut-vertex \(v\) in \(G \in G(n, d)\), and \(a_i \in V(A)\) corresponds to the vertices in \(G\) for \(1 \leq i \leq d\).

Next, we recolor the edges of \(A(\alpha)\) with colors 1, \ldots, \(\ell = (n - d - 1)/2\) in so that, for each color \(k \in \{1, 2, \ldots, \ell\}\) and each vertex \(z \in V(A)\):

(a) \(A(k)\) is connected, and
(b) \(d_{A(k)}(z) = 2\eta(z)\) (we already know \(d_{A(\alpha)}(z) = d\eta(z)\)).

Then, we can apply Theorem 3.3 to obtain the graph \(H \cong K_n\) satisfying

(i) for all \(u \in f^{-1}(z)\), \(d_H(u) = \frac{d_{A(k)}(z)}{\eta(z)} = \begin{cases} 0 & \text{for } 1 \leq k \leq \ell, \\
2 & \text{for } k = 0,
\end{cases}
\]

(ii) for \(1 \leq k \leq \ell\), \(H(k)\) is connected (since \(\frac{d_{A(k)}(z)}{\eta(z)} = 2\) is even for all \(z \in V(A)\)).

Notice that (i) and (ii) imply that, for each color \(k \in \{1, 2, \ldots, \ell\}\), the color class \(H(k)\) induces a Hamilton cycle. By Lemma 3.1, the edges colored 0 in \(H\) induce a primitive graph. We only need to specify the \((\ell + 1)\)-edge-coloring of \(A(\alpha)\).

We now start with recoloring the edges of \(A(\alpha)\). In the first step, we will guarantee the connectivity of each color class. In the second step, we will boost the degree of each vertex \(a_i\) in each color class to \(2\eta(a_i)\).

First, for \(1 \leq i \leq d - 1\) and for \(1 \leq k \leq \ell\), recolor 2 edges joining \(a_i\) to \(a_d\) with color \(k\); to be able to do this, we should check that there are at least \(2\ell = d + 1\) edges colored \(\alpha\) between \(a_i\) and \(a_d\) to ensure this first step is possible. Suppose \(1 \leq i \leq d - 1\). All the edges between \(a_i\) and \(a_d\) are in \(A(\alpha)\). Since \(A\) is an amalgamation of \(K_n\), there are \(\eta(a_i)\eta(a_d) = (d + 2)(n - d^2 - d + 1)\) edges between \(a_i\) and \(a_d\). So, we now show that \((d + 2)(n - d^2 - d + 1) \geq n - d - 1\).

Recall that by the hypothesis, \(n \geq (d + 1)^2\) and \(d \geq 3\). So,

\[
n(d + 1) \geq (d + 1)^3 = d^3 + 3d^2 + 3d + 1 \geq d^3 + 3d^2 - 3 \quad \text{(since } d \geq 3).\]

Therefore,

\[
(d + 2)(n - d^2 - d + 1) = nd - d^3 - d^2 + d + 2n - 2d^2 - 2d + 2 \\
= nd + n - d^3 - 3d^2 - d + 2 \\
= (n - d - 1) + nd + n - d^3 - 3d^2 + 3 \\
= (n - d - 1) + n(d + 1) - (d^3 + 3d^2 - 3) \\
> n - d - 1.
\]
Hence, we have enough edges in $A(\alpha)$ to recolor 2 edges between $a_i$ and $a_d$ with color $k$, for each color $k \in \{1, 2, \ldots, \ell \}$ and each $i \in \{1, 2, \ldots, d - 1\}$.

Now, in our second step, we recolor the remaining edges colored $\alpha$ with the same $\ell$ colors. Let $\tilde{A}(\alpha)$ be a graph induced by these edges. Then, $\tilde{A}(\alpha)$ is connected since for each $i \in \{1, 2, \ldots, d - 1\}$, vertex $a_i$ is joined to $a_d$ with $(d + 2)(n - d^2 - d + 1) - (n - d - 1) > 0$ edges and the degree $d_{\tilde{A}(\alpha)}(a_i)$ is even:

$$d_{\tilde{A}(\alpha)}(a_i) = \begin{cases} 2\ell & \text{for } i = 0, \\ \eta(a_i)2\ell - 2\ell & \text{for } 1 \leq i \leq d - 1, \\ \eta(a_i)2\ell - 2\ell(d - 1) & \text{for } i = d. \end{cases} \quad (**)$$

So, $\tilde{A}(\alpha)$ is Eulerian and, by Lemma 3.4, we can give $\tilde{A}(\alpha)$ an evenly equitable edge-coloring with $\ell$ colors. So, for each $a_i \in V(\tilde{A}(\alpha))$, and each $1 \leq k \leq \ell$, $d_{\tilde{A}(\alpha)}(a_i)$ is either $2\left\lfloor \frac{d_{\tilde{A}(\alpha)}(a_i)}{2\ell} \right\rfloor$ or $2\left\lceil \frac{d_{\tilde{A}(\alpha)}(a_i)}{2\ell} \right\rceil$. Since $2\ell$ is a factor of $d_{\tilde{A}(\alpha)}(a_i)$ for each vertex $a_i \in V(\tilde{A}(\alpha))$, we have

$$d_{\tilde{A}(\alpha)}(a_i) = 2\left\lfloor \frac{d_{\tilde{A}(\alpha)}(a_i)}{2\ell} \right\rfloor = 2\left\lceil \frac{d_{\tilde{A}(\alpha)}(a_i)}{2\ell} \right\rceil = \frac{d_{\tilde{A}(\alpha)}(a_i)}{\ell}.$$

Substituting from $(**)$, we get

$$d_{\tilde{A}(\alpha)}(a_i) = \begin{cases} 2\ell & \text{for } i = 0, \\ 2(\eta(a_i) - 1) & \text{for } 1 \leq i \leq d - 1, \\ 2(\eta(a_i) - d + 1) & \text{for } i = d. \end{cases}$$

For $1 \leq i \leq d - 1$, and for $1 \leq k \leq \ell$, $a_i$ is incident with 2 edges colored $k$ that were recolored in step 1 and is incident with $2\eta(a_i) - 2$ edges colored $k$ that were recolored in step 2; so, $a_i$ is incident with $2\eta(a_i)$ edges colored $k$, as required by (b).

Similarly, $d_{\tilde{A}(\alpha)}(a_i) = 2\eta(a_i)$ for $i \in [0, d]$.

Hence, we have the desired $(\ell + 1)$-coloring of $K(n, d)$. So, Theorem 3.3 provides an $(\ell + 1)$-edge-coloring of $K_n$ where color 0 induces a primitive graph $G$ and each of colors 1 to $\ell$ induces a Hamilton cycle in $K_n - E(G)$. $\blacksquare$

Now, we prove the converse.

**Theorem 4.3.** If there exists a set $S$ of $x$ edge-disjoint Hamilton cycles such that $K_n - E(S)$ contains no 2-factors, then $x \geq (n - \sqrt{n})/2$ when $n$ is even, and $x = (n - 1)/2$ when $n$ is odd.

**Proof.** If $K_n - E(S)$ contains no 2-factors, then it must be primitive; say it is $d$-regular. We consider the cases when $n$ is odd and when $n$ is even.

If $n$ is odd and $x < (n - 1)/2$, then since $K_n - E(S)$ is regular of even degree, Petersen’s Theorem [12] guarantees that it contains a 2-factor. Hence $K_n - E(S)$ is not primitive.

If $n$ is even, then Hoffman et al. [8] showed that $K_n - E(S)$ can be primitive with degree $d$ if and only if $d$ is odd and $n \geq (d + 1)^2$. So, $\sqrt{n} - 1 \geq d$. Since $S$ contains $x = (n - 1)/2$ edge-disjoint Hamilton cycles, substituting for $d$ gives us

$$x \geq \frac{n - 1 - \sqrt{n} + 1}{2} = \frac{n - \sqrt{n}}{2},$$

as required. $\blacksquare$

Theorems 4.2 and 4.3 together prove Theorem 4.1.

5. Final comment

We conclude this paper with the following avenue for future research. Let $G'(n, d)$ be the more general family of graphs defined by all the properties of graphs in $G(n, d)$, except that properties (b) and (c) are relaxed to allow $C_1, \ldots, C_d$ to contain any odd number of vertices. It is easy to see that graphs in $G'(n, d)$ are primitive.

**Conjecture:** There exists a Hamilton decomposition of $K_n - E(G)$ for all $G \in G'(n, d)$.

References