Certain Operators of Fractional Calculus and Their Applications to Differential Equations

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Abstract—In many recent works, one can find remarkable demonstrations of the usefulness of certain fractional calculus operators in the derivation of (explicit) particular solutions of a number of linear ordinary and partial differential equations of the second and higher orders. Our main objective in the present sequel to these earlier works, is to show how readily and systematically some recent contributions on this subject, involving linear ordinary and partial differential equations of order m (m \in \mathbb{N}), can be obtained (in a unified manner) by suitably appealing to some general theorems on (explicit) particular solutions of a certain family of linear ordinary fractional differintegral equations. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION AND PRELIMINARIES

The subject of fractional calculus (that is, calculus of derivatives and integrals of any arbitrary real or complex order) has gained considerable importance and popularity during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse fields.
of science and engineering (see, for details, [1–3]). In recent years, by making use of the following
definition, properties, and characteristics of a fractional differintegral operator (that is, fractional
derivative operator and fractional integral operator) of order \( \nu \in \mathbb{R} \), many authors have explicitly
obtained particular solutions of a number of families of homogeneous (as well as nonhomogeneous)
linear ordinary and partial fractional differintegral equations (see, for details, [4–8], and indeed
also the various references cited in each of these earlier works).

**Definition.** (See [9–14].) If the function \( f(z) \) is analytic (regular) inside and on \( C \), where

\[
C := \{C^-, C^+\},
\]

\( C^- \) is a contour along the cut joining the points \( z \) and \( -\infty + i\mathbb{R}(z) \), which starts from the point
at \( -\infty \), encircles the point \( z \) once counter-clockwise, and returns to the point at \( -\infty \), \( C^+ \) is a
contour along the cut joining the points \( z \) and \( \infty + i\mathbb{R}(z) \), which starts from the point at \( \infty \),
encircles the point \( z \) once counter-clockwise, and returns to the point at \( \infty \),

\[
f_\nu(z) = (f(z))_\nu := \frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta, \quad (\nu \in \mathbb{R} \backslash \mathbb{Z}^-; \mathbb{Z}^- := \{-1, -2, -3, \ldots\})
\]

and

\[
f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\}, \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}),
\]

where \( \zeta \neq z \),

\[
-\pi \leq \arg(\zeta - z) \leq \pi, \quad \text{for } C^-,
\]

and

\[
0 \leq \arg(\zeta - z) \leq 2\pi, \quad \text{for } C^+,
\]

then \( f_\nu(z) (\nu > 0) \) is said to be the fractional derivative of \( f(z) \) of order \( \nu \) and \( f_\nu(z) (\nu < 0) \) is
said to be the fractional integral of \( f(z) \) of order \( -\nu \), provided that

\[
|f_\nu(z)| < \infty, \quad (\nu \in \mathbb{R}).
\]

**Remark 1.** Throughout our present investigation, we conveniently write simply \( f_\nu \) for \( f_\nu(z) \)
whenever the argument of the differintegrated function \( f \) is clearly understood by the surrounding
context. Moreover, in case \( f \) is a many-valued function, we shall tacitly consider the principal
value of \( f \) in our investigation. For the sake of convenience in dealing with their various (known
or new) special cases, we choose also to state each of the fundamental results (Theorem 1 below)
for homogeneous (as well as nonhomogeneous) linear ordinary fractional differintegral equations
of a general order \( \mu \in \mathbb{R} \).

We now recall here, the aforementioned useful properties and characteristics associated with
the fractional differintegral operator which is defined above (cf., e.g., [9–13]).

**Lemma 1. Linearity Property.** If the functions \( f(z) \) and \( g(z) \) are single-valued and analytic
in some domain \( \Omega \subseteq \mathbb{C} \), then

\[
(k_1 f(z) + k_2 g(z))_\nu = k_1 f_\nu(z) + k_2 g_\nu(z), \quad (\nu \in \mathbb{R}; z \in \Omega),
\]

for any constants \( k_1 \) and \( k_2 \).

**Lemma 2. Index Law.** If the function \( f(z) \) is single-valued and analytic in some domain \( \Omega \subseteq \mathbb{C} \), then

\[
(f_\mu(z))_\nu = f_{\mu+\nu}(z) = (f_\nu(z))_\mu,
\]

\[
(f_\mu(z) \neq 0; f_\nu(z) \neq 0; \mu, \nu \in \mathbb{R}; z \in \Omega).
\]
LEMMA 3. GENERALIZED LEIBNIZ RULE. If the functions \( f(z) \) and \( g(z) \) are single-valued and analytic in some domain \( \Omega \subseteq \mathbb{C} \), then

\[
(f(z) \cdot g(z))_\nu = \sum_{n=0}^{\infty} \binom{\nu}{n} f_{\nu-n}(z) \cdot g_n(z), \quad (\nu \in \mathbb{R}; \ z \in \Omega),
\]

where \( g_n \) is the ordinary derivative of \( g(z) \) of order \( n \) \((n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})\), it being tacitly assumed (for simplicity) that \( g(z) \) is the polynomial part (if any) of the product \( f(z) \cdot g(z) \).

PROPERTY 1. For a constant \( \lambda \),

\[
(e^{\lambda z})_\nu = \lambda^n e^{\lambda z}, \quad (\lambda \neq 0; \ \nu \in \mathbb{R}; \ z \in \mathbb{C}).
\]

PROPERTY 2. For a constant \( \lambda \),

\[
(e^{-\lambda z})_\nu = e^{i\pi \nu} \lambda^n e^{\lambda z}, \quad (\lambda \neq 0; \ \nu \in \mathbb{R}; \ z \in \mathbb{C}).
\]

PROPERTY 3. For a constant \( \lambda \),

\[
(z^\lambda)_\nu = e^{-i\pi \nu} \frac{\Gamma(\nu - \lambda)}{\Gamma(-\lambda)} z^{\lambda - \nu},
\]

\[
\left( \nu \in \mathbb{R}; \ z \in \mathbb{C}; \ \left| \frac{\Gamma(\nu - \lambda)}{\Gamma(-\lambda)} \right| < \infty \right).
\]

Some of the most recent contributions on the subject of particular solutions of linear ordinary and partial fractional differintegral equations are those given by Tu et al. [4] who presented unification and generalization of a significantly large number of widely scattered results on this subject. We begin by recalling here one of the main results of [4], involving a family of linear ordinary fractional differintegral equations, as Theorem 1 below.

THEOREM 1. (See [4, p. 295, Theorem 1; p. 296, Theorem 2].) Let \( P(z;p) \) and \( Q(z;q) \) be polynomials in \( z \) of degrees \( p \) and \( q \), respectively, defined by

\[
P(z;p) := \sum_{k=0}^{p} a_k z^{p-k} = a_0 \prod_{j=1}^{p} (z - z_j), \quad (a_0 \neq 0; \ p \in \mathbb{N}),
\]

and

\[
Q(z;q) := \sum_{k=0}^{q} b_k z^{q-k}, \quad (b_0 \neq 0; \ q \in \mathbb{N}).
\]

Suppose also that \( f_{-\nu} (\neq 0) \) exists for a given function \( f \).

Then the nonhomogeneous linear ordinary fractional differintegral equation

\[
\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_k(1)(z;q) \phi_{\mu-k}(z)
\]

\[
+ \binom{\nu}{q} \frac{q!}{\mu_0} \phi_{\mu-q-1}(z) = f(z),
\]

\((\mu, \nu \in \mathbb{R}; \ p, q \in \mathbb{N}),
\]

has a particular solution of the form:

\[
\phi(z) = \left( \frac{f_{-\nu}(z)}{P(z;p)} e^{H(z;p,q)} \right)^{\mu-1} e^{-H(z;p,q)} e^{-H(z;p,q)}_{\nu-\mu+1},
\]

\((z \in \mathbb{C} \setminus \{z_1, \ldots, z_p\}),
\)
where, for convenience,

\[ H(z; p, q) := \int z \frac{Q(z, q)}{P(z; p)} \, dz, \quad (z \in \mathbb{C} \setminus \{z_1, \ldots, z_n\}), \quad (1.17) \]

provided that the second member of (1.16) exists.

Furthermore, the homogeneous linear ordinary fractional differintegral equation

\[ P(z; p) \phi(z) + \sum_{k=1}^{p} \binom{\nu}{k} P_k(z; p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_k(z; q) \phi_{-k}(z) + \binom{\nu}{q} b_0 \phi_{-q}(z) = 0, \quad (\mu, \nu \in \mathbb{R}; \ p, q \in \mathbb{N}), \]

has solutions of the form

\[ \phi(z) = K \left( e^{-H(z; p, q)} \right)^{\nu - \mu + 1}, \quad (1.19) \]

where \( K \) is an arbitrary constant and \( H(z; p, q) \) is given by (1.17), it being provided that the second member of (1.19) exists.

Next, for a function \( u = u(z, t) \) of two independent variables \( z \) and \( t \), we make use of the obviously convenient notation:

\[ \frac{\partial^{\mu + \nu} u}{\partial z^\mu \partial t^\nu} \quad (1.20) \]

to abbreviate the partial fractional differintegral of \( u(z, t) \) of order \( \mu \) with respect to \( z \) and of order \( \nu \) with respect to \( t \) \((\mu, \nu \in \mathbb{R})\). Thus, out of a total of six seemingly different theorems of [4] on the solutions of various families of linear partial fractional differintegral equations, we can recall the following result which happens to be the most relevant for the purpose of this paper.

**Theorem 2.** (See [4, p. 297, Theorem 3].) Let the polynomials \( P(z; p) \) and \( Q(z; q) \) be defined by (1.13) and (1.14), respectively. Suppose also that the function \( H(z; p, q) \) is given by (1.17).

Then the linear partial fractional differintegral equation

\[ P(z; p) \frac{\partial^{\mu + \nu} u}{\partial z^\mu \partial t^\nu} + \sum_{k=1}^{p-1} \binom{\nu}{k} P_k(z; p) + \sum_{k=1}^{q-1} \binom{\nu}{k-1} Q_k(z; q) \frac{\partial^{\mu - k} u}{\partial z^{\mu - k}} + C \frac{\partial^{\mu - p} u}{\partial z^{\mu - p}} - A \frac{\partial^{\mu - p+2} u}{\partial z^{\mu - p} \partial t^2} + B \frac{\partial^{\mu - p+1} u}{\partial z^{\mu - p} \partial t}, \quad (\mu, \nu \in \mathbb{R}; \ p, q \in \mathbb{N}), \]

has solutions of the form

\[ u(z, t) = \begin{cases} 
K_1 \left( e^{-H(z; p, q-1)} \right)^{\nu - \mu + 1} \cdot e^{\xi t}, & (A \neq 0), \\
K_2 \left( e^{-H(z; p, q-1)} \right)^{\nu - \mu + 1} \cdot e^{\eta t}, & (A = 0; \ B \neq 0), 
\end{cases} \quad (1.22) \]

where \( K_1 \) and \( K_2 \) are arbitrary constants, \( A, B, \) and \( C \) are given constants, and \( \xi \) and \( \eta \) are defined by

\[ \xi := \frac{-B \pm \sqrt{B^2 + 4(C - D)A}}{2A}, \quad (A \neq 0), \quad \text{and} \quad \eta := \frac{C - D}{B}, \quad (A = 0; \ B \neq 0), \quad (1.23) \]

with

\[ D := \binom{\nu}{p} p! a_0, \quad (1.24) \]

provided that the second member of (1.22) exists in each case.
REMARK 2. As already noted in conclusion by Tu et al. [4, p. 301], it is fairly straightforward to observe that either or both of the polynomials $P(z; p)$ and $Q(z; q)$, involved in Theorem 1, can be of degree 0 as well. Definitions (1.13) and (1.14) do serve the main purpose of this paper.

The main object of the present paper is to show how readily some recent contributions on this subject by Tu et al. [15], involving some general families of linear ordinary and partial differential equations of order $m$ ($m \in \mathbb{N}$), can be derived (in a unified manner) by appropriately applying Theorems 1 and 2 above.

2. APPLICATIONS OF THEOREM 1 TO A GENERAL FAMILY OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

In order to apply Theorem 1 of the preceding section to a general family of (nonhomogeneous and homogeneous) linear fractional differintegral equations, we begin by setting, in Theorem 1,

$$p = q + r - 1 = rn, \quad (r, n \in \mathbb{N}), \quad a_0 = 1, \quad b_0 = \rho, \quad (\rho \neq 0),$$

and choosing the remaining coefficients

$$a_1, \ldots, a_r \text{ and } b_1, \ldots, b_{r(n-1)+1},$$

in such a way that

$$P(z; rn) = (z - \kappa_1)^n \ldots (z - \kappa_r)^n$$

and

$$Q(z; r(n - 1) + 1) = (\rho z + \sigma)(z - \kappa_1)^{n-1} \ldots (z - \kappa_r)^{n-1},$$

where the parameters $\rho(\neq 0)$, $\sigma$, and $\kappa_1, \ldots, \kappa_r$ are unrestricted, in general. Then, clearly, definition (1.17) yields

$$H(z; rn, r(n - 1) + 1) = \int^z \frac{\rho \zeta + \sigma}{(\zeta - \kappa_1) \ldots (\zeta - \kappa_r)} d\zeta$$

$$= \begin{cases} 
\log \left[ (z - \kappa_1)^{(\rho_1 + \sigma)/((\kappa_1 - \kappa_2) \ldots (\kappa_1 - \kappa_r))} \ldots \\
(z - \kappa_r)^{(\rho_r + \sigma)/((\kappa_r - \kappa_{r-1}) \ldots (\kappa_r - \kappa_{r-1}))}, \quad (\kappa_1 \neq \kappa_2 \neq \ldots \neq \kappa_r) \\
- \rho [(r - 1)z - \kappa] + \sigma(r - 2)/(r - 1)(r - 2)(z - \kappa)^{r-1}, \quad (\kappa_j = \kappa (j = 1, \ldots, r); \ r \in \mathbb{N} \setminus \{1, 2\}).
\end{cases}$$

(2.5)

Upon substituting from (2.1) to (2.5) into Theorem 1, we readily obtain the following relevant (and potentially useful) consequence of Theorem 1.

THEOREM 3. If the given function $f$ satisfies the constraint (1.6) and $f_{-1} \neq 0$, then the nonhomogeneous linear ordinary fractional differintegral equation

$$(z - \kappa_1)^n \ldots (z - \kappa_r)^n \phi_{\mu}(z) + \sum_{k=1}^{rn} \binom{\nu}{k} \left[ (z - \kappa_1)^n \ldots (z - \kappa_r)^n \right]_{k-1}$$

$$+ \rho \sum_{k=1}^{r(n+1)+2} \binom{\nu}{k-1} \left[ z (z - \kappa_1)^{n-1} \ldots (z - \kappa_r)^{n-1} \right]_{k-1}$$

$$+ \sigma \sum_{k=1}^{r(n+1)+1} \binom{\nu}{k-1} \left[ (z - \kappa_1)^{n-1} \ldots (z - \kappa_r)^{n-1} \right]_{k-1} \phi_{\nu-k}(z) = f(z),$$

$$(\mu, \nu \in \mathbb{R}; \ r, n \in \mathbb{N}; \ z \in \mathbb{C} \setminus \{\kappa_1, \ldots, \kappa_r\}),$$

(2.6)
has a particular solution of the form
\[
\phi(z) = \left( (f_{-\nu}(z) \cdot (z - \kappa_1)^{-n} \ldots (z - \kappa_r)^{-n} \cdot e^{H(z; r, r(n-1)+1)})_{-1} \right)
\]
\[
\cdot e^{-H(z; r, r(n-1)+1)} \right)_{\nu-\mu+1},
\]
\[
(z \in \mathbb{C} \backslash \{\kappa_1, \ldots, \kappa_r\}),
\]
where \(H(z; r, r(n-1)+1)\) is given by (2.5), it being provided that the second member of (2.7) exists.

Furthermore, the homogeneous linear ordinary fractional differintegral equation
\[
(z - \kappa_1)^n \ldots (z - \kappa_r)^n \phi_\mu(z) + \left[ \sum_{k=1}^{r} \binom{\mu}{k} [(z - \kappa_1)^n \ldots (z - \kappa_r)^n]_k 
\]
\[
+ \rho \sum_{k=1}^{r(n-1)+2} \binom{\nu}{k-1} [z (z - \kappa_1)^{n-1} \ldots (z - \kappa_r)^{n-1}]_{k-1}
\]
\[
+ \sigma \sum_{k=1}^{r(n-1)+1} \binom{\nu}{k-1} [(z - \kappa_1)^{n-1} \ldots (z - \kappa_r)^{n-1}]_{k-1} \right) \phi_{\mu-k}(z) = 0,
\]
\[
(\mu, \nu \in \mathbb{R}; r, n \in \mathbb{N}; z \in \mathbb{C} \backslash \{\kappa_1, \ldots, \kappa_r\}),
\]
has solutions of the form
\[
\phi(z) = K \left( e^{-H(z; r, r(n-1)+1)} \right)_{\nu-\mu+1},
\]
where \(K\) is an arbitrary constant and \(H(z; r, r(n-1)+1)\) is given by (2.5), it being provided that the second member of (2.9) exists.

A simple special case of Theorem 3 when
\[
r = 2, \quad \kappa_1 = a, \quad \text{and} \quad \kappa_2 = b,
\]
can be stated here as the following.

**Theorem 4.** If the given function \(f\) satisfies the constraint (1.6) and \(f_{-\nu} \neq 0\), then the nonhomogeneous linear ordinary fractional differintegral equation
\[
(z - a)^n (z - b)^n \phi_\mu(z) + \left[ \sum_{k=1}^{2n} \binom{\nu}{k} [(z - a)^n (z - b)^n]_k 
\]
\[
+ \rho \left\{ \binom{\nu}{k-1} [z (z - a)^{n-1} (z - b)^{n-1}]_{k-1} \right\}
\]
\[
+ \sigma \sum_{k=1}^{2n-1} \binom{\nu}{k-1} [(z - a)^{n-1} (z - b)^{n-1}]_{k-1} \right) \phi_{\mu-k}(z) = f(z),
\]
\[
(\mu, \nu \in \mathbb{R}; n \in \mathbb{N}; z \in \mathbb{C} \backslash \{a, b\}),
\]
has a particular solution of the form
\[
\phi(z) = \begin{cases} 
((f_{-\nu}(z) \cdot (z - a)^{-n} + (a+\sigma)/(a-b) \cdot (z - b)^{-n} + (b+\sigma)/(b-a))_{-1} 
\cdot (z - a)^{(a+\sigma)/(b-a)} \cdot (z - b)^{(b+\sigma)/(a-b)})_{\nu-\mu+1}, & (a \neq b), \\
((f_{-\nu}(z) \cdot (z - a)^{2n} \cdot e^{(a+\sigma)/(z-a)})_{-1} 
\cdot (z - a)^{-\rho} \cdot e^{(a+\sigma)/(z-a)})_{\nu-\mu+1}, & (a = b), 
\end{cases}
\]
\[
(z \in \mathbb{C} \backslash \{a, b\}; n \in \mathbb{N}),
\]
provided that the second member of (2.12) exists in each case.
Furthermore, the homogeneous linear ordinary fractional differintegral equation

\[(z-a)^n(z-b)^n\phi_\mu(z) + \sum_{k=1}^{2n} \left( \binom{\nu}{k} \right) [(z-a)^n(z-b)^n]_k + \rho \left( \sum_{k=1}^{2n-1} \left( \binom{\nu}{k-1} \right) [(z-a)^{n-1}(z-b)^{n-1}]_k \right) \]

\[+ \sigma \sum_{k=1}^{2n-1} \left( \binom{\nu}{k-1} \right) [(z-a)^{n-1}(z-b)^{n-1}]_k \phi_{\mu-k}(z) = 0, \]

\[(\mu, \nu \in \mathbb{R}; n \in \mathbb{N}; z \in \mathbb{C}\{a, b\}), \quad (2.13)\]

has solutions of the form

\[\phi(z) = \begin{cases} 
K ((z-a)(\rho a + \sigma)/(b-a) \cdot (z-b)(\rho b + \sigma)/(a-b))^{\nu-\mu+1}, & (a \neq b), \\
K ((z-a)^{-\rho} \cdot e^{(\rho a + \sigma)/(z-a)})^{\nu-\mu+1}, & (a = b), \\
\cdot (z \in \mathbb{C}\{a, b\}),
\end{cases} \quad (2.14)\]

where \(K\) is an arbitrary constant, it being provided that the second member of (2.14) exists in each case.

**Remark 3.** An obvious further special case of Theorem 4 when

\[\mu = m, \quad (m \in \mathbb{N}), \quad \nu = \alpha + 1, \quad \rho = -(2\alpha - D + 2), \quad \text{and} \quad \sigma = (a+b)(\alpha+1) + C, \quad (2.15)\]

happens to yield two of the main results of [15, p. 47, Theorem 1; p. 50, Theorem 2]; their third main result [15, p. 51, Theorem 3] is easily derivable, by the Principle of Superposition, from solutions (2.12) and (2.14) under the special case given by (2.15).

### 3. SOLUTIONS OF GENERAL LINEAR PARTIAL FRACTIONAL DIFFERINTEGRAL EQUATIONS

Just as in Section 2 above, by applying Theorem 2 (with \( q \to q + 1 \)) under assumptions (2.1) to (2.4), we can easily deduce the following.

**Theorem 5.** The linear partial fractional differintegral equation:

\[(z-\kappa_1)^n \ldots (z-\kappa_r)^n \frac{\partial^\mu u}{\partial z^\mu} + \sum_{k=1}^{r(n-1)+1} \left( \binom{\nu}{k} \right) [(z-\kappa_1)^n \ldots (z-\kappa_r)^n]_k + \rho \sum_{k=1}^{r(n-1)+1} \left( \binom{\nu}{k-1} \right) [(z-\kappa_1)^n \ldots (z-\kappa_r)^{n-1}]_k \cdot \frac{\partial^{\mu-k} u}{\partial z^{\mu-k}}
\]

\[+ \sigma \sum_{k=1}^{r(n-1)+1} \left( \binom{\nu}{k-1} \right) [(z-\kappa_1)^{n-1} \ldots (z-\kappa_r)^{n-1}]_k \frac{\partial^{\mu-k} u}{\partial z^{\mu-k}} + C \frac{\partial^{\mu-rn} u}{\partial z^{\mu-rn}} = A \frac{\partial^{\mu-rn+2} u}{\partial z^{\mu-rn}} + B \frac{\partial^{\mu-rn+1} u}{\partial z^{\mu-rn}} + \frac{\partial^{\mu-rn} u}{\partial t}, \]

\[(\mu, \nu \in \mathbb{R}; r, n \in \mathbb{N}; z \in \mathbb{C}\{\kappa_1, \ldots, \kappa_r\}), \quad (3.1)\]

has solutions of the form:

\[u(z, t) = \begin{cases} 
K_1 (e^{-H(z;rn,r(n-1)+1)})^{\nu-\mu+1} \cdot e^{\delta t}, & (A \neq 0), \\
K_2 (e^{-H(z;rn,r(n-1)+1)})^{\nu-\mu+1} \cdot e^{\nu t}, & (A = 0; B \neq 0),
\end{cases} \quad (3.2)\]
where $K_1$ and $K_2$ are arbitrary constants, $H(z; r, n, r(n - 1) + 1)$ is given (as also in Theorem 3) by (2.5), $A$, $B$, and $C$ are given constants, and $\xi$ and $\eta$ are defined by (1.23) with

$$D = \binom{\nu}{r_n}(r_n)! + \rho \binom{\nu}{r(n - 1) + 1}(r(n - 1) + 1)!, \quad (r, n \in \mathbb{N}),$$  \hspace{1cm} (3.3)

provided that the second member of (3.2) exists in each case.

Under the specialization listed in (2.10), Theorem 5 readily assumes the relatively simpler form given by the following.

**THEOREM 6.** The linear partial fractional differintegral equation:

$$(z - a)^n (z - b)^n \frac{\partial^\nu u}{\partial z^\nu} + \sum_{k=1}^{2n-1} \binom{\nu}{k} \left[(z - a)^n(z - b)^{n-1}\right]_{k-1} \cdot 
$$

$$+ \rho \left\{ \frac{\partial^{\nu-k} u}{\partial z^{\nu-k}} \right\} \frac{\partial^{\nu-k} u}{\partial z^{\nu-k}} + C \frac{\partial^{\nu-2n} u}{\partial z^{\nu-2n}} = A \frac{\partial^{\nu-2n+2} u}{\partial z^{\nu-2n+2}} + B \frac{\partial^{\nu-2n+1} u}{\partial z^{\nu-2n+1}} \frac{\partial^{\nu-1} u}{\partial t^{\nu-1}},$$

$$\left(\mu, \nu \in \mathbb{R}; \ n \in \mathbb{N}; \ z \in \mathbb{C}\setminus\{a, b\}\right),$$

has solutions of the form:

$$u(z, t) = \begin{cases} 
K_1(\Phi(z; a, b; \nu, \rho, \sigma))_{\nu-\mu+1} \cdot e^{\xi t}, & (A \neq 0), \\
K_2(\Phi(z; a, b; \nu, \rho, \sigma))_{\nu-\mu+1} \cdot e^{\eta t}, & (A = 0; B \neq 0), \\
\end{cases}$$

\hspace{1cm} (3.5)

where, for convenience,

$$\Phi(z; a, b; \nu, \rho, \sigma) := \begin{cases} 
(z - a)^{\alpha+\sigma}/(\alpha - b) \cdot (z - b)^{\rho+\sigma}/(\rho - a), & (a \neq b), \\
(z - a)^{-\rho} \cdot e^{(\rho+\sigma)/(\alpha - b)}, & (a = b), \\
\end{cases}$$

\hspace{1cm} (3.6)

$K_1$ and $K_2$ are arbitrary constants, $A$, $B$, and $C$ are given constants, and $\xi$ and $\eta$ are defined (as before) by (1.23) with (cf. equation (3.3) for $r = 2$)

$$D = (\rho + \nu - 2n + 2) \prod_{j=1}^{2n-1} (\nu - j + 1), \quad (n \in \mathbb{N}),$$

\hspace{1cm} (3.7)

provided that second member of (3.5) exists in each case.

The two remaining main results in the aforementioned work of [15, p. 53, Theorem 4; p. 56, Theorem 5] (see also Remark 3 above) correspond to the obvious further special cases of Theorem 6 when (cf. equation (2.15))

$$\mu = 2n, \quad (n \in \mathbb{N}), \quad \nu = \alpha + 1, \quad \rho = -(2\alpha - d + 2), \quad \text{and} \quad \sigma = (a + b)(\alpha + 1) + c, \quad (3.8)$$

and

$$\mu = m, \quad (m \in \mathbb{N}), \quad \nu = \alpha + 1, \quad \rho = -(2\alpha - d + 2), \quad \text{and} \quad \sigma = (a + b)(\alpha + 1) + c, \quad (3.9)$$

respectively.

Many other interesting (known or new) consequences and applications of each of the nine main results (Theorems 1–9) of [4], in addition (of course) to Theorems 3–6 of this paper, can indeed be presented in a similar manner.
REFERENCES