Adaptive Trajectory Tracking of Quantum Systems
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Abstract: A control strategy for tracking the trajectory of a time-variant objective function in closed quantum systems is proposed in this paper. The Lyapunov stability theorem is used to design the control law. Meanwhile, the issue of singularity is discussed and the problem of large value of control field occurred during the process of tracking are solved by using an adaptive regulation algorithm of objective function. Simulation results demonstrate the effectiveness and feasibility of the method proposed. We also explore the relationship of effect between control parameters and the error of the control system.

Keywords: quantum systems, Liouville equation, adaptive tracking algorithm, state trajectory tracking, singularities

1. INTRODUCTION
The quantum computing and quantum communication require active manipulation of various quantum states. In recent years, control methods and technologies in quantum field have been studied widely and considerable progress has also been made[1][2]. However, there are still many problems to be resolved. One of the most important problems in quantum controls is state steering. One needs to design a set of control laws to steer initial state of the system to a desired target state. If the target state is stationary, the problem above mentioned can be called state regulation or state-transfer control of quantum systems, which has been researched widely in the recent years[3][7]. If the target state is time-dependent function, then it becomes a tracking problem from the control systems perspective point. Due to its complexity, the trajectory tracking control of quantum systems has had less studied so far. [8] analyzed theoretically state tracking based on Lyapunov method in quantum systems. [9] and [10] studied the tracking problem of free-evolutionary Liouville equation. [11] combined optimal control and adaptive algorithm to discuss this topic.

In quantum systems, there are several kinds of states such as superposition-state, mixed-state, entangled state and so on, which are very different from the situation in classical control systems. These differences and the particularity of quantum systems make it not so easy to track a given trajectory as that in classical engineering community. In this paper, we’ll study the trajectory tracking for a closed quantum system to a time-dependent function. The control law is designed based on Lyapunov stability theorem. Adaptive algorithm is also used to settle the problem of singularities or bigger control values. Another advantage of our method proposed is that we do not request the initial states of the controlled system and target system are the same. The control strategy proposed here can adaptively track a time-dependent function from an arbitrary initial state of the controlled system.

The paper is organized as follows. The system model and control task are described in section 2. In section 3, we design control law and analysis the issues which may be appeared in the control systems and gives out the solution of the problems. Simulation experiment is given in section 4. Section 5 concludes the paper.

2. DESCRIPTION OF THE SYSTEM MODEL
The main task of this work will focus on the pure-states trajectory tracking of closed quantum systems. A quantum system with wave function \( \psi(t) \) is described by the Schrödinger equation:

\[
\frac{i\hbar}{\partial t} \psi(t) = H \psi(t), \psi(0) = \psi_0
\]

where \( H = H_0 + \sum_{m=1}^{M} u_m(t)H_m \), \( H_0 \) is the internal (free) Hamiltonian of the system and \( H_m \) is the external (control) one, both of which are Hermitian and assumed to be independent of time. We choose Plank constant \( \hbar = 1 \) for convenience. For a closed quantum system, the measurable quantities or observables are represented by linear, self-adjoint operator \( P(t) \) in the Hilbert space. If \( P(t) \) is measured, the outcome is an eigenvalue of \( P(t) \). Let
be a measurable subset of the spectrum of \( P(t) \), if the state at the moment of the measurement is \( |\psi\rangle \), the probability of obtaining a value \( \gamma \) in \( S \), as the result of the measurement is given by
\[
p_{\gamma} (\gamma \in S) = \langle \psi | P_j | \psi \rangle = |P_j |\psi\rangle|^2,
\]
where \( P_j \) is the projector of \( P(t) \), viz. \( P(t) = \sum \gamma_j P_j \) in the discrete spectrum case. So when the system is in the state \( |\psi\rangle \), the expectation of measurable operator \( P(t) \), denoted \( Y(t) \), is represented by:
\[
Y(t) = \langle P(t) | \psi(t) \rangle = \sum_j \gamma_j \langle \psi(t) | P_j | \psi(t) \rangle = \langle \psi(t) | P(t) | \psi(t) \rangle.
\]

Equation (1) is the controlled system and (2) is the controlled variable which is also the output of controlled system (1).

In this work, let \( |L_j\rangle \) denotes the \( j \)th eigenstate of free Hamiltonian \( H_0 \), it is convenient to assign the projective operator of the first eigenstate \( |L_1\rangle \) as observable operator \( P(t) \), which is time-independent, viz. \( P = |L_1 \rangle \langle L_1| \). As a matter of fact, the tracking is the evolution of population:
\[
Y(t) = \langle \psi(t) | L_1 \rangle \langle L_1 | \psi(t) \rangle = |\langle \psi(t) | L_1 \rangle|^2.
\]
Because of the probability characteristic, \( Y(t) \) is a real-value ranged from 0 to 1.

The desired target system can take different forms only it should be time-dependent function. Take into account that we have studied the tracking problem of free-evolutionary Liouville equation\(^{9} \), we’ll study in this paper a desired target function \( S(t) \) of an exponential function which is a typical function often used to test the performance of control systems in engineering:
\[
S(t) = 1 - e^{-r^2/2\tau^2}, \quad t \geq 0
\]
where \( \tau \) determines the change rate of target output.

In fact, the output of desired target system (3) corresponds to the range of (2). Our control objective is to track the output of desired target system (3) at each moment. Now, the control task is to make the controlled variable (2) track the output of (3), viz. let \( Y(t) \) follow the output \( S(t) \). The error \( e(t) \) between \( S(t) \) and \( Y(t) \) will be a performance index used to measure the tracking effectiveness.

The system control concept of changing the trajectory tracking control into an error regulation control is used to deal with the trajectory tracking problem in this paper.

The error state \( e(t) \) is defined as:
\[
e(t) = S(t) - Y(t)
\]

Then, subtracting (2) from (3) one can obtain:
\[
e(t) = 1 - e^{-r^2/2\tau^2} - \langle \psi(t) | P(t) | \psi(t) \rangle
\]

The first order time derivation of \( e(t) \) is:
\[
\dot{e}(t) = \frac{1}{\tau^2} e^{-r^2/2\tau^2} - \langle \psi(t) | P(t) | \psi(t) \rangle
\]
\[
- \langle \psi(t) | \dot{P}(t) | \psi(t) \rangle - \langle \psi(t) | P(t) | \dot{\psi}(t) \rangle
\]

Placing (1) into (6), one gets:
\[
\dot{e}(t) = \frac{1}{\tau^2} e^{-r^2/2\tau^2} - \langle \psi(t) | i[H, P(t)] + \dot{P}(t) | \psi(t) \rangle
\]

Then control law is designed based on Lyapunov stability theory according to the error state \( e(t) \) and the first order time derivation of \( e(t) \) obtained in (5) and (6). The control goal will achieved by decreasing gradually error \( e(t) \) tends to zero under the action of the control law designed.

### 3. CONTROL SYSTEM DESIGN

#### 3.1 Design of control law

The methods of designing control laws are plentiful in which the Lyapunov-based is adapted to be used in time-varying and nonlinear systems, and by which the control law designed can ensure the stability of the control system. So we use Lyapunov method to design the control law in this paper.

The basic idea of Lyapunov method is to select \( V(x) \) as Lyapunov function, which must satisfy the following three conditions: a) \( V(x) \) is continuous and its first-order partial derivatives is also continuous in its definition; b) \( V(x) \) is positive semi-definite, i.e., \( V(x) \geq 0 \); c) The first order time derivative of the Lyapunov function is negative semi-definite, i.e., \( \dot{V}(x) \leq 0 \).

The Lyapunov function based on error is chosen here as:
\[
V(x) = \frac{1}{2} e^2(t)
\]
where equation (8) meets the required conditions of Lyapunov function. The first order time derivation of (8) is:

\[ \dot{V}(x) = e(t) \cdot \dot{e}(t) \quad (9) \]

Subtract (7) into (9), one obtains:

\[ \dot{V}(t) = e(t) \cdot \left( \frac{T}{\tau} e^{\frac{-\tau}{2}} - 2 \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_0 \cdot |\psi(t)|) \right) \]

\[ -2 \sum_{m=1}^{M} u_m(t) \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_m \cdot |\psi(t)|) \quad (10) \]

It can be seen from (10) that the right side of the equation contains the drifting item \( \frac{T}{\tau} e^{\frac{-\tau}{2}} - 2 \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_0 \cdot |\psi(t)|) \), which makes it difficult to determine whether \( \dot{V}(x) \leq 0 \) holds. To solve this problem, we divide equation (10) into two parts:

\[ \dot{V}(t) = e(t) \cdot \left( \frac{T}{\tau} e^{\frac{-\tau}{2}} - 2 \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_0 \cdot |\psi(t)|) \right) \]

\[ -2u_1(t) \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_1 \cdot |\psi(t)|) \]

\[ -2 \sum_{m=2}^{M} u_m(t) \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_m \cdot |\psi(t)|) \quad (11) \]

Firstly, we let

\[ e(t) \cdot \left( \frac{T}{\tau} e^{\frac{-\tau}{2}} - 2 \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_0 \cdot |\psi(t)|) \right) \]

\[ -2u_1(t) \cdot \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_1 \cdot |\psi(t)|) = 0 \quad (12) \]

The control law \( u_1 \) is derived as:

\[ u_1(t) = \frac{T}{\tau} e^{\frac{-\tau}{2}} \cdot 2 \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_1 \cdot |\psi(t)|) \quad (13) \]

Secondly, we let

\[ -2e(t) \cdot \sum_{m=2}^{M} u_m(t) \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_m \cdot |\psi(t)|) \leq 0 \quad (14) \]

The control law \( u_m \) are derived as:

\[ u_m(t) = k_m e(t) \cdot \text{Im}(\langle \psi(t) \rangle \cdot P \cdot H_m \cdot |\psi(t)|), (m = 2, 3, ..., M) \quad (15) \]

where \( k_m > 0(m = 2, 3, ..., M) \) is control gain.

From the above derivation process one can see that the drifting item comes from \( H_0 \). In order to get rid of the effect of drifting item and at the same time obtain the good tracking performance one needs at least to design two component control laws \( u_1 \) and \( u_m(t)(m \geq 2) \) in which \( u_1 \) is used to remove the drifting item while \( u_m(t)(m \geq 2) \) is the main control law used to track the target function. However, the functions of these two component control laws are not so well-defined. In the following, the interaction relation between them will be analyzed.

### 3.2 Analysis of control performance

According to the expression of \( u_1(t) \) obtained in (13), it is a fractional and its denominator is related with system state which may result in a denominator of zero. This situation implied that during the tracking progress, control law \( u_1(t) \) may be infinite and singular for some states because of a zero denominator. Such a control law might trigger the following two issues: 1) singularity; 2) too large control values.

In fact, in the process of the trajectory tracking, the singularity can be divided into two categories: the removable ones and the intrinsic ones, which are of the forms \( \frac{0}{0} \) and \( \frac{\alpha}{0} (\alpha \neq 0) \)[12], respectively. For the first case, the removable singularities only appear at certain moments of the whole tracking process and can be deleted by some effective method, such as solving these points in the low-order systems and deleting them by a segmented control law. But for four-order or higher order system, because of the complexity of system structure and control law and many singularities, it is difficult to solve directly the singular points. In such a case, L’Hôpital’s rule is used with \( u_1(t) \), in which one changes the control law \( 0/0 \) into \( \alpha/0(\alpha \neq 0) \) and then no singularity exists. Moreover, a small modification of target trajectory, an alternative method, is suitable to avoid the first type of singularity. In a word, this kind of singularity brings few troubles for tracking control. But it is not the case for the second type. The denominator will keep zero at all times for intrinsic singularities, which are the inherent characteristics of the system. The system in the second case is not controllable and no control law is applicable. All singular points in this paper are mathematical but not physical by the
simple analysis, so it does not indicate an uncontrollable system can be appeared.

However, even if there is no singularity or we have deleted all singularities in the control law, there are problems that remain. In fact, \( u(t) \) not only removes drifting items, but also plays a control role. For example, a smaller denominator of \( u(t) \) will lead to a too large control value which make stable tracking be impossible. But for the most cases, error is permitted in a range. In the case of tracking control the system state tracks its target function is more important thing. Therefore, we can give a limitation of control amplitude and permit a tolerance of the error in tracking performance. When control amplitude exceeds the given limited value, appropriate modification of target trajectory is added.

The concrete process for adjusting target trajectory is as follows:

It is supposed that \( u(t) \) or \( u_m(t) \) is bigger than the boundary value at \( t = t_0 \), then target function \( S(t) \) at \( t = t_1 \) becomes

\[
S(t_1) = S(t_0) + (1 - S(t_0))(1 - e^{-u(t-t_0)/2\tau^2})
\]  

(16)

Now, \( S(t_1) = S(t_0) \) holds at \( t_1 = t_0 \) and \( S(t_1) \to 1 \) when \( t_1 \to \infty \). So the adjusted trajectory \( S(t_1) \) displays the same trend as the former one. \( S(t_0) = S(t_0) \) is almost right on the condition that the time interval is supposed small enough, but it may slightly decrease the control accuracy of tracking performance, so we also need to give a error tolerance permitted. Similarly, the general amended expression of tracking trajectory is:

\[
S_{k+1}(t) = S_k(t) + (1 - S_k(t))(1 - e^{-u(t-t_k)/2\tau^2})
\]  

(17)

That is to say, the trajectory amendment is applied when any one of the control amplitudes exceeds the given boundary value. Furthermore, the control gain of \( u_m(t) \) is reduced at the same time to avoid large control value.

4. NUMERICAL SIMULATION AND RESULT ANALYSIS

In order to verify the effectiveness of the method proposed, numerical simulation will be performed in this section for a four-level quantum system with free control Hamiltonian:

\[
H_0 = \sum_{i=1}^{4} E_i |j\rangle\langle j|
\]  

(17)

where \( E_1 = 0.4948 \), \( E_2 = 1.4529 \), \( E_3 = 2.3691 \), \( E_4 = 3.2434 \), and one has

\[
H_0 = \text{diag}(0.4948, 1.4529, 2.3691, 3.2434)
\]  

(18)

For this four-level system, a full-connected control Hamiltonian is given for convenience, which means that interactions between any two levels are allowable. So the control Hamiltonian is

\[
H_c = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

Based on the discussion in section 2, two control laws are needed at least, so two control Hamiltonians. We assumed

\[
H_1 = \begin{bmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix},
H_2 = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]

Then the total Hamiltonian of the system is

\[
H = H_0 + u_1(t)H_1 + u_2(t)H_2
\]  

(19)

The eigenvalues of the matrix \( H_0 \) is denoted by \( \lambda_i(i=1,2,3,4) \) with \( \lambda_1 = 0.4948 \), \( \lambda_2 = 1.4529 \), \( \lambda_3 = 2.3691 \), \( \lambda_4 = 3.2434 \), and the corresponding eigenvectors are \( |\lambda_1\rangle = |0,0,1,0\rangle^T \), \( |\lambda_2\rangle = |0,1,0,0\rangle^T \), \( |\lambda_3\rangle = |0,0,1,0\rangle^T \), \( |\lambda_4\rangle = |0,0,0,1\rangle^T \). In this experiment, the observable operator is

\[
P = |\lambda_1\rangle \langle \lambda_1| = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Time step is chosen as \( \Delta t = 0.01 \) and \( \tau = 20 \).

For the controlled system (1), the initial state is the superposition of \( |\lambda_1\rangle \), \( |\lambda_2\rangle \), \( |\lambda_3\rangle \) and \( |\lambda_4\rangle \), it is

\[
|\psi_0\rangle = (1/2)|\lambda_1\rangle + (1/2)|\lambda_2\rangle + (1/2)|\lambda_3\rangle + (1/2)|\lambda_4\rangle
\]  

(20)
The initial output variable becomes \( Y(0) = \langle \psi_0 | P | \psi_0 \rangle = 0.25 \). For the target system (3), at \( t = 0 \), initial state is \( S(0) = 0 \). Then the initial error can be gotten: \( e(0) = S(0) - Y(0) = -0.25 \), which is placed into (13) and (15) to get the control laws. We choose control gain \( k_2 = 220 \) for the second control law \( u_2(t) \). If the control value exceeds an boundary value, the control gain will be changed as \( k_2 = 40 \). The initial control value is \( u_1(0) = u_2(0) = 0.005 \). The boundary values for \( u_1(t) \) is 2 and 5 for \( u_2(t) \).

The simulation results are as follows. Figure 1 shows the output evolution of the controlled system and target system. Figure 2 is the record of error. Figure 3 depicts control law \( u_1(t) \) and \( u_2(t) \) is in Figure 4.

From Figure 1 one can see that tracking procedures performs after state transferring, where the initial value 0.25 of the controlled system is transferred to the one of the target system during the first 5 seconds. After that, system output tracks stably target output over time. Between 26 seconds and 26.4 seconds, the controlled state produces a too small value of \( \text{Im}(\langle \psi(t) | P \cdot H_1 | \psi(t) \rangle) \) which leads to a large control value of \( u_1(t) \) than its boundary 2. At this moment, a slight modification of target trajectory and decreasing control gain to 40 of control law \( u_2(t) \) are carried on simultaneously. The two control amplitudes have been limited in their upper boundary values 2 and 5 in Figures 3 and 4. From Figure 2 one can see that at the first 5 seconds, error fast decreased from its initial value 0.25 to 0, which just indicates the system is able to track the system from its initial state to the target system’s function under the action of the control law proposed. An enlarged error of 0.01 appeared at around 26 seconds denotes a too big control value. But during the whole tracking process, error remains within 0.01 and tends to 0 after 40 seconds. The control accuracy of control system reaches 99%. The output of system can track the given time-variant objective trajectory fairly well. Figure 5 is
the error surface of the control system under the relation between control gain $k_2$ and the boundary of control $u_1$ in which $k_2$ ranges 5:5:100; $u_1$ ranges 0.5:0.5:8. From figure 5 one can see that the situation is complicated and there is no simple law to follow between the relationship of the error and parameters, however, Figure 5 can still provide us with some perceptual knowledge about the effects of the control parameters on the error of the control system.

![Figure 5](image_url)

Figure 5  Error surface of the control system under the relation between control gain $k_2$ and the boundary of control $u_1$.

5. CONCLUSION

A method based on Lyapunov stability theorem has been used to design control law for quantum state trajectory tracking. The drifting item in the first order derivation of Lyapunov function causes special fractional forms in control laws. Too large value or singularity in control laws makes it difficult to track the target system stably. We have proposed an adaptive tracking algorithm to avoid this case. A new modified target trajectory has been adopted when singularity arises. At last, the control strategy proposed has been verified in a four-level quantum system. The results of experiment have showed that for a quantum system with any initial state, trajectory tracking task could well completed. The strategy we have proposed may be adapted to other general situations of Lyapunov-based control method.

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REFERENCES


