Construction of Orthogonal Multiscaling Functions and Multiwavelets
Based on the Matrix Extension Algorithm

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Abstract A algorithm is presented for constructing orthogonal multiscaling functions and
multiwavelets with multiplicity \( r+s (s \geq 1, s \in \mathbb{Z}) \) in terms of any given orthogonal multiscal-
ing functions with multiplicity \( r \). That is, let \( \Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_r(x)]^T \) be an ortho-
gonal multiscaling functions with multiplicity \( r \), with two-scale matrix symbol \( P(z) \). Then
a new orthogonal multiscaling functions \( \Phi_{\text{new}}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \ldots, \phi_{r+s}(x)]^T \)
is constructed by applying the matrix extension method. Similarly, suppose that \( \Psi(x) =
[\psi_1(x), \psi_2(x), \ldots, \psi_r(x)]^T \) is an orthogonal multiwavelets corresponding to \( \Phi(x) \), then ex-

cplicit formula for constructing orthogonal multiwavelets associated with \( \Phi_{\text{new}}(x) \) is ob-
tained. In particular, the spacial case of \( r = s \) is discussed. Finally, we give some examples
illustrating how to use our method to construct orthogonal multiscaling functions and the
corresponding multiwavelets.

Key words Orthogonal, Scaling function, Wavelet, Multiscaling functions, Multiwavelets

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1 Introduction

Several types of uniwavelet are constructed base on multiresolution analysis, such as
the well-known Daubechies’ orthogonal wavelets\(^1\)[1,2] and semi-orthogonal spline wavelets by
Chui and Wang \[^3\] et al. However, multiwavelets can have some features that uniwavelet
cannot. Thus, multiwavelets provide interesting applications in signal processing and some
other fields(See \[^4\][5]). In recent years, multiscaling functions and multiwavelets have been
studied extensively\[^6\]–\[^14\]. Similar to the construction of the uniscaling function, multiscaling
functions with multiplicity \( r \) also can be constructed base on multiresolution analysis. But

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the main difficulties in construction of multiwavelets are verification the convergence of the infinite product of two-scale matrix symbol (See [15][16]). Is there more easier method to construct orthogonal multiwavelets? Can multiwavelets be constructed based on uniwavelet? Generally, can multiwavelets with multiplicity $r+s$ be constructed based on multiwavelets with multiplicity $r$? Aim for this question, in this paper, we present a algorithm for constructing orthogonal multiscaling functions and multiwavelets.

Let $\Phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_r(x)]^T$, satisfying the following two-scale matrix equation:

$$\Phi(x) = \sum_k P_k \Phi(2x - k), \quad (1)$$

for some $r \times r$ matrices sequence $\{P_k\}_{k \in \mathbb{Z}}$ called the two-scale matrix sequence, $\Phi(x)$ is called multiscaling functions with multiplicity $r$.

By taking Fourier transform the both sides of (1), we have

$$\hat{\Phi}(w) = P(e^{-iw/2}) \hat{\Phi}(\frac{w}{2}), \quad (2)$$

where $P(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} P_k z^k$ called the two-scale matrix symbol of the two-scale matrix sequence $\{P_k\}_{k \in \mathbb{Z}}$ of $\Phi(x)$.

By repeated applications of (2), we have

$$\hat{\Phi}(w) = \left( \prod_{j=1}^{\infty} P(e^{-iw/2^j}) \right) \hat{\Phi}(0),$$

If the infinite product $\prod_{j=1}^{\infty} P(e^{-iw/2^j})$ converges, then the $\Phi(w)$ is well-defined and we will say that $\hat{\Phi}(w)$ is generated by $P(w)$.

In [17], Cabrelli, et al gave the following criteria to ensure the convergence of the above infinite product.

The infinite matrix product $\prod_{j=1}^{\infty} P(e^{-iw/2^j})$ converges uniformly on compact sets to a continuous matrix-valued function if and only if the eigenvalues $\lambda_i, i = 1, 2, \cdots, r$ of the matrix $P(1)$ satisfy $\lambda_1 = 1, |\lambda_i| < 1, i = 2, 3, \cdots, r$ (See [4][17]).

Let $\Psi(x) = [\psi_1(x), \psi_2(x), \cdots, \psi_r(x)]^T$ be an orthogonal multiwavelets corresponding to $\Phi(x)$, satisfying the following equation:

$$\Psi(x) = \sum_{k \in \mathbb{Z}} Q_k \Phi(2x - k). \quad (3)$$
for some \( r \times r \) matrices sequence \( \{ Q_k \}_{k \in \mathbb{Z}} \). (3) can be rewritten as
\[
\hat{\Psi}(w) = Q(e^{-iw/2}) \hat{\Phi}(w/2),
\]
where \( Q(z) = \frac{1}{2} \sum_{k \in \mathbb{Z}} Q_k z^k \).

Let \( \Phi(x) \) be an orthogonal multiscaling functions, and \( \Psi(x) \) be an orthogonal multiwavelets corresponding to \( \Phi(x) \), with two scale matrix symbol \( P(z) \), and \( Q(z) \), respectively. Then \( P(z) \), and \( Q(z) \) satisfy the following equations (See [13][14])
\[
\begin{align*}
P(z)P(z)^* + P(-z)P(-z)^* &= I_{r \times r}, \\
P(z)Q(z)^* + P(-z)Q(-z)^* &= O_{r \times r}, \\
Q(z)Q(z)^* + Q(-z)Q(-z)^* &= I_{r \times r},
\end{align*}
\]
where \( O \) and \( I_r \) denote the zero matrix and unity matrix, respectively. Here and throughout, the asterisk denotes conjugate transpose of matrix.

### 2 Construction of orthogonal multiscaling functions

In this section, we will introduce a procedure of constructing orthogonal multiscaling functions with multiplicity \( r + s \) in terms of any given orthogonal multiscaling functions with multiplicity \( r \).

To construct orthogonal multiscaling functions, we need the following lemma.

**Lemma 1** Let \( \Phi(x) \) be an orthogonal multiscaling functions, and \( \Psi(x) \) be an orthogonal multiwavelets corresponding to \( \Phi(x) \), with two-scale matrix symbol \( P(z) \), and \( Q(z) \), respectively. Suppose \( Q^k(z), k = 1, 2, \ldots, r \) is the \( k \)-th row of \( Q(z) \). Then
\[
\begin{align*}
P(z)Q^k(z)^* + P(-z)Q^k(-z)^* &= O_{r \times 1}, \quad k = 1, 2, \ldots, r, \\
Q^j(z)Q^k(z)^* + Q^j(-z)Q^k(-z)^* &= \delta_{j,k}, \quad j, k = 1, 2, \ldots, r.
\end{align*}
\]

**Proof:** In terms of orthogonality of \( \Phi(x) \) and \( \Psi(x) \), then \( P(z) \), and \( Q(z) \) satisfy (4). Substituting \( Q(z) = [Q^1(z)^*, Q^2(z)^*, \ldots, Q^r(z)^*]^* \) into (4), we obtain
\[
\begin{align*}
P(z) [Q^1(z)^*, \ldots, Q^r(z)^*] + P(-z) [Q^1(-z)^*, \ldots, Q^r(-z)^*] &= O_{r \times r}, \\
[Q^1(z)^*, \ldots, Q^r(z)^*]^* [Q^1(z)^*, \ldots, Q^r(z)^*] + [Q^1(-z)^*, \ldots, Q^r(-z)^*]^* [Q^1(-z)^*, \ldots, Q^r(-z)^*] &= I_{r \times r}.
\end{align*}
\]
This means that (5) holds.
Let \( h_{i,j}(z), i = 1, 2, \cdots, r; j = 1, 2, \cdots, s; |z| = 1 \) satisfy the conditions: (1): \( h_{i,j}(z) = h_{i,j}(-z), i = 1, 2, \cdots, s; j = 1, 2, \cdots, r \); (2): For any integer \( i, 1 \leq i \leq s, \sum_{j=1}^{r} |h_{i,j}(z)|^2 = a \), where \( 0 < a < 1 \); (3): For any integer \( i, k, 1 \leq i < k \leq s, \sum_{j=1}^{r} h_{i,j}(z)h_{k,j}(z)^* = 0 \). Construct \( s \times r \) matrix \( A(z) \) as follow

\[
A(z) = \begin{bmatrix}
  h_{1,1}(z) & h_{1,2}(z) & \cdots & h_{1,r}(z) \\
  h_{2,1}(z) & h_{2,2}(z) & \cdots & h_{2,r}(z) \\
  \vdots & \vdots & \ddots & \vdots \\
  h_{s,1}(z) & h_{s,2}(z) & \cdots & h_{s,r}(z)
\end{bmatrix}
\begin{bmatrix}
  Q^1(z) \\
  Q^2(z) \\
  \vdots \\
  Q^r(z)
\end{bmatrix} = \begin{bmatrix}
  \sum_{j=1}^{r} h_{1,j}(z)Q^j(z) \\
  \sum_{j=1}^{r} h_{2,j}(z)Q^j(z) \\
  \vdots \\
  \sum_{j=1}^{r} h_{s,j}(z)Q^j(z)
\end{bmatrix}.
\]  

(6)

Lemma 2 Let \( A(z) \) defined in (6) be \( s \times r \) matrix. Then \( A(z)A(z)^* + A(-z)A(-z)^* = aI_{s \times s} \).

Proof: Since \( A(z)A(z)^* \)

\[
\begin{align*}
\sum_{j=1}^{r} h_{1,j}(z)Q^j(z) & = \begin{bmatrix}
  \sum_{k=1}^{r} h_{1,k}(z)^*Q^k(z)^* \\
  \vdots \\
  \sum_{k=1}^{r} h_{s,k}(z)^*Q^k(z)^*
\end{bmatrix} \\
& = \begin{bmatrix}
  \sum_{j,k=1}^{r} h_{1,j}(z)h_{1,k}(z)^*Q^j(z)Q^k(z)^* \\
  \vdots \\
  \sum_{j,k=1}^{r} h_{s,j}(z)h_{s,k}(z)^*Q^j(z)Q^k(z)^*
\end{bmatrix}.
\end{align*}
\]

Consider (5) and \( h_{i,j}(z) \) satisfying the conditions. We have \( A(z)A(z)^* + A(-z)A(-z)^* = \)

\[
\begin{bmatrix}
  \sum_{j=1}^{r} |h_{1,j}(z)|^2 & 0 & \cdots & 0 \\
  0 & \sum_{j=1}^{r} |h_{2,j}(z)|^2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & \sum_{j=1}^{r} |h_{s,j}(z)|^2
\end{bmatrix} = aI_{s \times s}.
\]

Remark 1 In definition of \( A(z), h_{i,j}(z), i = 1, 2, \cdots, r; j = 1, 2, \cdots, s \) are required satisfying conditions (1)-(3). All the same, there exists a lot of functions satisfying the conditions. For example, \( h_{i,i}(z) = \sqrt{a}, i = \min\{r, s\}; h_{i,j}(z) = 0, i \neq j. \)
Lemma 3  Let $P(z)$, and $Q(z)$ be two scale matrix symbol associated with $\Phi(x)$ and $\Psi(x)$, respectively, $A(z)$ defined in (6) be $s \times r$ matrix. Then

$$P(z)A(z)^* + P(-z)A(-z)^* = O_{r \times s},$$

$$A(z)Q(z)^* + A(-z)Q(-z)^* = \begin{bmatrix}
h_{1,1}(z) & h_{1,2}(z) & \cdots & h_{1,r}(z) \\
h_{2,1}(z) & h_{2,2}(z) & \cdots & h_{2,r}(z) \\
\vdots & \vdots & \ddots & \vdots \\
h_{s,1}(z) & h_{s,2}(z) & \cdots & h_{s,r}(z) \\
\end{bmatrix}.$$ 

Theorem 1  Let $P(z)$, and $Q(z)$ be two-scale matrix symbol associated with $\Phi(x)$ and $\Psi(x)$, respectively, $A(z)$ defined in (6) be $s \times r$ matrix, and $B(z)$ be $s \times s$ matrix, satisfying $B(z)B(z)^* + B(-z)B(-z)^* = (1 - a)I_{s \times s}$, where $a \in (0, 1)$. Define

$$P_{\text{new}}(z) = \begin{bmatrix} P(z) & O \\ A(z) & B(z) \end{bmatrix}. \quad (7)$$

Then

$$P_{\text{new}}^*(z)P_{\text{new}}(z) + P_{\text{new}}^*(-z)P_{\text{new}}(-z)^* = I_{(r+s) \times (r+s)}. \quad (8)$$

Proof: Since $P(z)$, and $Q(z)$ are two scale matrix symbol associated with $\Phi(x)$ and $\Psi(x)$, respectively, then $|P(z)|^2 + |P(-z)|^2 = I_{r \times r}$. By Lemma 1 and Lemma 2, we have

$$P_{\text{new}}^*(z)P_{\text{new}}(z) + P_{\text{new}}^*(-z)P_{\text{new}}(-z)^* = \begin{bmatrix} P(z) & 0 \\ A(z) & B(z) \end{bmatrix} \begin{bmatrix} P(z)^* & A(z)^* \\ 0 & B(z)^* \end{bmatrix}$$

$$+ \begin{bmatrix} P(-z) & 0 \\ A(-z) & B(-z) \end{bmatrix} \begin{bmatrix} P(-z)^* & A(-z)^* \\ 0 & B(-z)^* \end{bmatrix}$$

$$= \begin{bmatrix} |P(z)|^2 + |P(-z)|^2 & P(z)A(z)^* + P(-z)A(-z)^* \\ A(z)P(z)^* + A(-z)P(-z)^* & A(z)A(z)^* + A(-z)A(-z)^* + B(z)B(z)^* + B(-z)B(-z)^* \end{bmatrix}$$

$$= \begin{bmatrix} I_{r \times r} & O_{r \times s} \\ O_{s \times r} & I_{s \times s} \end{bmatrix} = I_{(r+s) \times (r+s)}.$$ 

Remark 2  There exist a lot of $B(z)$ satisfying the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1 - a)I_{s \times s}$. For example, let $\mathcal{P}(z)$ be two scale matrix symbol associated with orthogonal multiscaling functions with multiplicity $s$. Then $\mathcal{P}(z)\mathcal{P}(z)^* + \mathcal{P}(-z)\mathcal{P}(-z)^* =
Take $B(z) = (1 - a)^{-\frac{1}{2}}P(z)$. It is easy to verify that $B(z)$ satisfies the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1 - a)I_{s \times s}$. Additionally, let $p^i(z), i = 1, 2, \ldots, s$ be two scale symbol associated with orthogonal uniscaling function $\phi^i(x)$. Take $B(z) = (1 - a)^{-\frac{1}{2}}\text{diag}[p^1(z), p^2(z), \ldots, p^s(z)]$. Then $B(z)$ satisfies the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1 - a)I_{s \times s}$.

**Theorem 2** Let $P^{new}(z)$ defined in (7) be a lower triangle matrix. Under the conditions of Theorem 1, if the eigenvalues $\lambda_i, i = 1, 2, \ldots, r$ of the matrix $P(1)$ satisfy $\lambda_1 = 1, |\lambda_i| < 1, i = 2, 3, \ldots, r$, and the eigenvalues $\mu_i, i = 1, 2, \ldots, s$ of the matrix $B(1)$ satisfy $|\mu_i| < 1, i = 1, 2, \ldots, s$, then 1 must be a simple eigenvalue of the matrix $P^{new}(1)$, and all other eigenvalues $\lambda$ of $P^{new}(1)$ must have $|\lambda| < 1$.

**Proof:** Since $P^{new}(1) = \begin{bmatrix} P(1) & O \\ A(1) & B(1) \end{bmatrix}$, then $|\lambda E_{r+s} - P^{new}(1)| = |\lambda E_r - P(1)||\lambda E_s - B(1)|$. Obviously, all the eigenvalues of the matrices $P(1)$ and $B(1)$ must be the eigenvalues of the matrix $P^{new}(1)$. This completes the proof of Theorem 2.

According to [17], the infinite matrix product $\prod_{j=1}^{\infty} P^{new}(e^{-iw/2})$ converges. Thus an orthogonal multiscaling functions $\Phi^{new}(x)$ with multiplicity $r + s$ is well-defined, in terms of Fourier transform, by

$$\hat{\Phi}^{new}(w) = [\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w), \hat{\phi}_{r+1}(w), \ldots, \hat{\phi}_{r+s}(w)]^T$$

$$= \begin{bmatrix} P(e^{-iw/2}) & 0 \\ A(e^{-iw/2}) & B(e^{-iw/2}) \end{bmatrix} [\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w), \hat{\phi}_{r+1}(w), \ldots, \hat{\phi}_{r+s}(w)]^T.$$  

We can see easily that $\phi_1(x), \phi_2(x), \ldots, \phi_r(x)$, the first $r$ components of the orthogonal multiscaling functions $\Phi^{new}(x)$, are old functions generated by $P(z); \phi_{r+1}(x), \phi_{r+2}(x), \ldots, \phi_{r+s}(x)$, the final $s$ components of the orthogonal multiscaling functions $\Phi^{new}(x)$, are new functions that we construct. That is

$$\hat{\Phi}(w) = [\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w)]^T = P(e^{-iw/2})[\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w)]^T,$$

$$[\hat{\phi}_{r+1}(w), \ldots, \hat{\phi}_{r+s}(w)]^T = A(e^{-iw/2})[\hat{\phi}_1(w), \ldots, \hat{\phi}_r(w)]^T + B(e^{-iw/2})[\hat{\phi}_{r+1}(w), \ldots, \hat{\phi}_{r+s}(w)]^T.$$  

According to the above discussion, we have the following construction theorem.
Theorem 3  Let $\Phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_r(x)]^T$ be an orthogonal multiscaling functions, and $\Psi(x)$ be an orthogonal multiwavelets corresponding to $\Phi(x)$, with two-scale matrix symbol $P(z)$, and $Q(z)$, respectively. Under the condition of Theorem 1 and Theorem 2, then there are $\phi_{r+1}, \phi_{r+2}, \cdots, \phi_{r+s}$ such that $\Phi_{\text{new}}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \cdots, \phi_{r+s}(x)]^T$ is an orthogonal multiscaling functions with multiplicity $r+s$ and two scale matrix symbol $P_{\text{new}}(z)$ given by (7).

3 Explicit formula for constructing orthogonal multiwavelets

In the above section, we give the methods of constructing orthogonal multiscaling function. In this section, we will discuss the construction of the corresponding multiwavelets.

Construct the matrices $Q_{\text{new}}(z), M(z)$ respectively by

$$Q_{\text{new}}(z) = \begin{bmatrix} XQ(z) & YB(z) \\ O & (1-a)^{-\frac{1}{2}} z^k B(-z)^* \end{bmatrix}, M(z) = \begin{bmatrix} P_{\text{new}}(z) & P_{\text{new}}(-z) \\ Q_{\text{new}}(z) & Q_{\text{new}}(-z) \end{bmatrix}. \tag{9}$$

where $X$ is $r \times r$ matrix, $Y$ is $r \times s$ matrix, and $k$ is odd number.

Next we will give the explicit construction formula of orthogonal multiwavelets corresponding to $\Phi_{\text{new}}(x)$.

Theorem 4  Let $\Phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_r(x)]^T$ be an orthogonal multiscaling functions, and $\Psi(x)$ be an orthogonal multiwavelets corresponding to $\Phi(x)$, with two-scale matrix symbol $P(z)$, and $Q(z)$, respectively, and $B(z)$ be $s \times s$ diagonal matrix, satisfying $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s \times s}$, where $0 < a < 1$. If matrices $X, Y$ satisfy the following conditions:

$$\begin{cases} \\
\begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,r} \\
 h_{2,1} & h_{2,2} & \cdots & h_{2,r} \\
 \cdots & \cdots & \cdots & \cdots \\
 h_{s,1} & h_{s,2} & \cdots & h_{s,r} \end{bmatrix} X^* + (1-a)Y^* = O_{s \times r}, \\
 XX^* + (1-a)YY^* = I_{r \times r},
\end{cases} \tag{10}$$

then under the condition of Theorem 1 and Theorem 3, $M(w)$ defined in (9) is a unitary matrix. Further, suppose $\Phi_{\text{new}}(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \cdots, \phi_{r+s}(x)]^T$ is orthogonal
multiscaling function generated by $P^{new}(z)$, $Q^{new}(z)$ defined in (9) is a upper triangle matrix. Then orthogonal multiwavelets corresponding to $\Phi^{new}(x)$ is given, in terms of Fourier transform, by
\[\hat{\Psi}^{new}(w) = Q^{new}(e^{-iw/2})\hat{\Psi}^{new}(w/2).\]

**Proof:** According to wavelet construction theorem, we only need prove $M(w)$ is a unitary matrix. That is, $P^{new}(z)$ and $Q^{new}(z)$ must satisfy the following equations:

\[P^{new}(z)P^{new}(z)^* + P^{new}(-z)P^{new}(-z)^* = I_{(r+s)\times(r+s)},\]

(11)

\[P^{new}(z)Q^{new}(z)^* + P^{new}(-z)Q^{new}(-z)^* = O_{(r+s)\times(r+s)},\]

(12)

\[Q^{new}(z)Q^{new}(z)^* + Q^{new}(-z)Q^{new}(-z)^* = I_{(r+s)\times(r+s)}.\]

(13)

By Theorem 1, (11) holds. Next, we only need to prove (12) and (13) hold. In fact

\[P^{new}(z)Q^{new}(z)^* = \begin{bmatrix} P(z) & 0 \\ A(z) & B(z) \end{bmatrix} \begin{bmatrix} Q(z)^*X^* & O \\ B(z)^*Y^* & (1-a)^{-\frac{1}{2}}z^kB(-z) \end{bmatrix} = \begin{bmatrix} P(z)Q(z)^*X^* & O \\ A(z)Q(z)^*X^* + B(z)B(z)^*Y^* & (1-a)^{-\frac{1}{2}}z^kB(z)B(-z) \end{bmatrix} \]

By [14], we have $P(z)Q(z)^* + P(-z)Q(-z)^* = O_{r\times r}$. Hence $[P(z)Q(z)^* + P(-z)Q(-z)^*]X^* = O_{r\times r}$. Using the condition $B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{s\times s}$ and Lemma 3, we obtain

\[A(z)Q(z)^* + A(-z)Q(-z)^*]X^* + [B(z)B(z)^* + B(-z)B(-z)^*]Y^* = \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,r} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ h_{s,1} & h_{s,2} & \cdots & h_{s,r} \end{bmatrix} X^* + (1-a)Y^* = O_{s\times r}. \]

Therefore (12) holds. Again since

\[Q^{new}(z)Q^{new}(z)^* = \begin{bmatrix} XQ(z) & YB(z) \\ O & (1-a)^{-\frac{1}{2}}z^kB(-z) \end{bmatrix} \begin{bmatrix} Q(z)^*X^* & O \\ B(z)^*Y^* & (1-a)^{-\frac{1}{2}}z^kB(-z) \end{bmatrix} = \begin{bmatrix} XQ(z)Q(z)^*X^* + YB(z)B(z)^*Y^* & (1-a)^{-\frac{1}{2}}z^kB(z)B(-z) \\ (1-a)^{-\frac{1}{2}}z^kB(-z)^*B(z)^*Y^* & (1-a)^{-1}z^kB(-z)B(-z) \end{bmatrix} \]

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By (10), we have
\[ Q^{\text{new}}(z)Q^{\text{new}}(z)^* + Q^{\text{new}}(-z)Q^{\text{new}}(-z)^* \]
= \[
\begin{bmatrix}
XX^* + (1-a)YY^* & O \\
O & (1-a)^{-1}[B(z)^*B(z) + B(-z)^*B(-z)]
\end{bmatrix}
\]
= \[
\begin{bmatrix}
I_{r \times r} & O \\
O & I_{s \times s}
\end{bmatrix} = I_{(r+s) \times (r+s)},
\]
which implies that \( M(w) \) is a unitary matrix. This completes the proof of Theorem 4.

Next, we will discuss a special setting: \( r = s \). Similar to Theorem 3, we also have the following theorem.

**Theorem 5** Let \( \Phi(x) = [\phi_1(x), \phi_2(x), \cdots, \phi_r(x)]^T \) be an orthogonal multiscaling functions, and \( \Psi(x) \) be an orthogonal multiwavelets corresponding to \( \Phi(x) \), with two scale matrix symbol \( P(z) \), and \( Q(z) \), respectively, and \( B(z) \) be \( r \times r \) matrix, satisfying \( B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{r \times r} \), where \( 0 < a < 1 \). Suppose that \( H \) is any unitary matrix. Define
\[
P^\#(z) = \begin{bmatrix}
P(z) & O \\
\sqrt{a}HQ(z) & B(z)
\end{bmatrix}.
\]
Then there are \( \phi_{r+1}(x), \cdots, \phi_{2r}(x) \) such that \( \Phi^\#(x) = [\Phi^T(x), \phi_{r+1}(x), \phi_{r+2}(x), \cdots, \phi_{2r}(x)]^T \) is an orthogonal multiscaling functions with multiplicity \( 2r \) and two scale matrix symbol \( P^\#(z) \) given by (14).

To give explicit formula of constructing orthogonal multiwavelets corresponding to \( \Phi^\#(x) \), we suppose that matrix \( B(z) \) satisfies \( B(z)B(-z) = B(-z)B(z) \). Extraordinarily, we suppose that \( B(z) \) is \( r \times r \) diagonal matrix.

**Theorem 6** Suppose that \( B(z) \) is \( r \times r \) diagonal matrix, satisfying \( B(z)B(z)^* + B(-z)B(-z)^* = (1-a)I_{r \times r} \), \( 0 < a < 1 \). Let \( k \) be an odd number. Define
\[
Q^\#(z) = \begin{bmatrix}
\sqrt{1-a}Q(z) & -\sqrt{\frac{a}{1-a}}H^*B(z) \\
O & \sqrt{\frac{1-a}{1-a}}z^kB(-z)^*
\end{bmatrix}.
\]
Then under the condition of Theorem 5, orthogonal multiwavelets \( \Psi^\#(x) \) corresponding to \( \Phi^\#(x) \) is given, in terms of Fourier transform, by
\[
\hat{\Psi}^\#(w) = Q^\#(e^{-iw/2})\hat{\Phi}^\#(\frac{w}{2}).
\]
Proof: It is easy to verify that
\[
\begin{bmatrix}
P^\#(z) & P^\#(-z) \\
Q^\#(z) & Q^\#(-z)
\end{bmatrix}
\]
is a unitary matrix. Hence, according to wavelet construction theorem, we are able to define the orthogonal multiwavelets \( \Psi^\#(x) \) by (16).

4 Examples

We will illustrate by some examples how to construct higher multiplicity orthogonal multiwavelets in terms of any given orthogonal uniwavelet or multiwavelets based on our method.

Example 1 (Case 1: \( r = 2, s = 1 \)) Let \( \Phi(x) = (\phi_1, \phi_2)^T \) be an orthogonal multiscaling functions, satisfying the following equation:

\[
\Phi(x) = P_0 \Phi(2x) + P_1 \Phi(2x - 1) + P_2 \Phi(2x - 2),
\]

where
\[
P_0 = \begin{bmatrix}
0 & \frac{2+\sqrt{7}}{4} \\
0 & \frac{2-\sqrt{7}}{4}
\end{bmatrix}, \quad P_1 = \begin{bmatrix}
\frac{3}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{3}{4}
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
\frac{2-\sqrt{7}}{4} & 0 \\
\frac{2+\sqrt{7}}{4} & 0
\end{bmatrix}.
\]

The corresponding orthogonal multiwavelets \( \Psi(x) \) satisfies the following equation

\[
\Psi(x) = Q_0 \Phi(2x) + Q_1 \Phi(2x - 1) + Q_2 \Phi(2x - 2),
\]

where
\[
Q_0 = \begin{bmatrix}
0 & \frac{3}{4} \\
0 & \frac{1}{4}
\end{bmatrix}, \quad Q_1 = \begin{bmatrix}
-\frac{2+\sqrt{7}}{4} & -\frac{2-\sqrt{7}}{4} \\
-\frac{2-\sqrt{7}}{4} & -\frac{2+\sqrt{7}}{4}
\end{bmatrix}, \quad Q_2 = \begin{bmatrix}
\frac{3}{4} & 0 \\
\frac{3}{4} & 0
\end{bmatrix}.
\]

Hence,
\[
P(z) = \frac{1}{2} \begin{bmatrix}
\frac{3}{4}z + \frac{2+\sqrt{7}}{4}z^2 & \frac{2+\sqrt{7}}{4} + \frac{1}{4}z \\
\frac{1}{4}z + \frac{2+\sqrt{7}}{4}z^2 & \frac{2-\sqrt{7}}{4} + \frac{3}{4}z
\end{bmatrix}, \quad Q(z) = \frac{1}{2} \begin{bmatrix}
\frac{2+\sqrt{7}}{4}z + \frac{1}{4}z^2 & \frac{3}{4} & -\frac{2-\sqrt{7}}{4}z \\
-\frac{2-\sqrt{7}}{4}z + \frac{3}{4}z^2 & \frac{3}{4} & 2+\sqrt{7}z
\end{bmatrix}.
\]

Take \( A(z) = \frac{1}{2}Q^2(z) = \frac{1}{4}[\frac{2-\sqrt{7}}{4}z + \frac{3}{4}z^2, \frac{1}{4} - \frac{2+\sqrt{7}}{4}z], \quad B(z) = \frac{\sqrt{3}}{3}[\frac{1+\sqrt{7}}{4}z + \frac{3+\sqrt{3}}{4}z + \frac{3-\sqrt{3}}{4}z^2 + \frac{1-\sqrt{3}}{4}z^3]. \) It is easy to verify that \( A(z)A(z)^* + A(-z)A(-z)^* = \frac{1}{4}, \quad B(z)B(z)^* + B(-z)B(-z)^* = \)
Thus, by (9), and taking
\[
P^{\text{new}}(z) = \begin{bmatrix}
\frac{3}{8} z + \frac{2-\sqrt{7}}{8} z^2 & \frac{2+\sqrt{7}}{8} + \frac{1}{8} z \\
\frac{1}{8} z + \frac{2+\sqrt{7}}{8} z^2 & \frac{2-\sqrt{7}}{8} + \frac{3}{8} z \\
\frac{1}{16} - \frac{2-\sqrt{7}}{16} z + \frac{3}{16} z^2 & \frac{3+\sqrt{3}}{12} + \frac{3\sqrt{3}}{12} z^2 + \frac{\sqrt{3}-3}{12} z^3
\end{bmatrix}.
\] (17)

Again applying Theorem 3, we obtain a new orthogonal multiscaling functions \(\Phi^{\text{new}}(x) = [\phi_1(x), \phi_2(x), \phi_3(x)]^T\), with two scale matrix symbol \(P^{\text{new}}(z)\) given by (17).

Let \(X = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix}, Y = [-\frac{\sqrt{3}}{2}, 0]^T\). It is easy to verify that matrices \(X, Y\) satisfy (10).

Thus, by (9), and taking \(k = 3\), we construct the matrix \(Q^{\text{new}}(z)\) by
\[
Q^{\text{new}}(z) = \begin{bmatrix}
-\frac{2+\sqrt{7}}{16} z + \frac{3\sqrt{3}}{16} z^2 & \frac{\sqrt{3}}{16} - \frac{2\sqrt{7}+\sqrt{21}}{16} z & -\frac{1+\sqrt{3}}{12} - \frac{3+\sqrt{3}}{12} z^2 - \frac{3-\sqrt{7}}{12} z^2 - \frac{1-\sqrt{3}}{12} z^3 \\
-\frac{2-\sqrt{7}}{8} z + \frac{3}{8} z^2 & \frac{1}{8} - \frac{2-\sqrt{7}}{8} z & 0 \\
0 & 0 & \frac{1+\sqrt{3}}{6} z^3 - \frac{3+\sqrt{7}}{6} z^2 + \frac{3-\sqrt{7}}{6} z - \frac{1-\sqrt{3}}{6}
\end{bmatrix}.
\] (18)

Hence, applying Theorem 4, an orthogonal multiwavelets \(\Psi^{\text{new}}(x) = [\psi_1(x), \psi_2(x), \psi_3(x)]^T\) corresponding to \(\Phi^{\text{new}}(x)\) can be constructed by two scale matrix symbol \(Q^{\text{new}}(z)\) defined in (18).

**Example 2** (Case 2: \(r = s = 2\)) Let us consider the orthogonal multiscaling functions and multiwavelets \(\Phi(x)\) and \(\Psi(x)\) in [8], the corresponding two scale matrix symbol \(P(z)\), and \(Q(z)\) are given by, respectively,
\[
P(z) = \begin{bmatrix}
\frac{1}{4} + \frac{1}{2} z + \frac{1}{4} z^2 & \frac{1}{4} - \frac{1}{4} z^2 \\
-\frac{\sqrt{7}}{8} + \frac{\sqrt{7}}{8} z^2 & -\frac{\sqrt{7}}{8} + \frac{1}{2} z - \frac{\sqrt{7}}{8} z^2
\end{bmatrix},
Q(z) = \begin{bmatrix}
-\frac{1}{4} + \frac{1}{2} z - \frac{1}{4} z^2 & -\frac{1}{4} + \frac{1}{4} z^2 \\
\frac{1}{8} - \frac{1}{8} z^2 & \frac{1}{8} + \frac{\sqrt{7}}{4} z + \frac{1}{8} z^2
\end{bmatrix}.
\]

For simplicity, we take \(B(z) = \text{diag}[\frac{1}{2} p(z), \frac{1}{2} p(z)]\), where \(p(z)\) is two scale symbol associated with the \(N\) order Daubechies orthogonal scaling function. Hence, \(B(z)B(z)^* + B(-z)B(-z)^* = (1 - \frac{3}{4})I_{2 \times 2}\). Thus, take \(a = \frac{3}{4}\). Suppose \(H\) is any unitary matrix. By (14), we can construct a new two scale matrix symbol \(P^\#(z)\), which can generated a new orthogonal multiwavelets functions \(\Phi^\#(x)\) with multiplicity 4.

Similarly, by (15), and letting \(k\) be any odd number, we can construct \(Q^\#(z)\), which can generated the orthogonal multiswavelets \(\Psi^\#(x)\) corresponding to \(\Phi^\#(x)\).
Example 3  (Case 3: $r = s = 1$) Let $\phi(x)$ be the 2 order Daubechies orthogonal scaling function, $\psi(x)$ be orthogonal wavelet associated with $\phi(x)$, with two scale symbol $p(z), q(z)$, respectively, where

\[
\begin{align*}
p(z) &= \frac{1}{2} \left( \frac{1 + \sqrt{3}}{4} + \frac{3 + \sqrt{3}}{4} z + \frac{3 - \sqrt{3}}{4} z^2 + \frac{1 - \sqrt{3}}{4} z^3 \right), \\
q(z) &= \frac{1}{2} \left( \frac{\sqrt{3} - 1}{4} + \frac{3 - \sqrt{3}}{4} z + \frac{3 + \sqrt{3}}{4} z^2 + \frac{1 + \sqrt{3}}{4} z^3 \right)
\end{align*}
\]

Take $B(z) = \frac{1}{2}, H = 1$. Then $a = \frac{1}{2}$. By (14) in Theorem 5, we obtain

\[
P^\#(z) = \frac{1}{2} \begin{bmatrix} 1 + \frac{\sqrt{3}}{2} + \frac{3 + \sqrt{3}}{2} z + \frac{3 - \sqrt{3}}{2} z^2 + \frac{1 - \sqrt{3}}{2} z^3 & 0 \\
\frac{\sqrt{3} - 1}{4} + \frac{3 - \sqrt{3}}{4} z + \frac{3 + \sqrt{3}}{4} z^2 + \frac{1 + \sqrt{3}}{4} z^3 & 1 \end{bmatrix}.
\]

Applying Theorem 5, an orthogonal multiscaling functions $\Phi^\#(x) = [\phi_1(x), \phi_2(x)]^T$ is constructed by

\[
\begin{align*}
\phi_1(x) &= \frac{1 + \sqrt{3}}{4} \phi_1(2x) + \frac{3 + \sqrt{3}}{4} \phi_1(2x - 1) + \frac{3 - \sqrt{3}}{4} \phi_1(2x - 2) + \frac{1 - \sqrt{3}}{4} \phi_1(2x - 3), \\
\phi_2(x) &= \frac{\sqrt{3} - 1}{4} \phi_1(2x) + \frac{3 - \sqrt{3}}{4} \phi_1(2x - 1) - \frac{3 + \sqrt{3}}{4} \phi_1(2x - 2) + \frac{1 + \sqrt{3}}{4} \phi_1(2x - 3) + \phi_2(2x).
\end{align*}
\]

Similarly, applying Theorem 6, an orthogonal multiwavelets $\Psi^\#(x) = [\psi_1(x), \psi_2(x)]^T$ corresponding to $\Phi^\#(x)$ can be constructed.

REFERENCE


