Affine TS-Model-Based Optimal Fuzzy Regulating Control

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Abstract—Affine TS fuzzy system is much preferred than linear type in providing one more adjustable parameter for computation-intelligent (neural-fuzzy-evolution) modelling of model-free physical system or highly nonlinear and complex model-based system. However, few researches are devoted in intrinsic analysis of affine-type fuzzy system and in developing controllers to regulate affine TS-based nonlinear systems. In this paper, the affine-type global optimal fuzzy control design scheme is theoretically derived. The generated closed-loop fuzzy systems are demonstrated to be stable. The performance of the proposed fuzzy controller is demonstrated via two nonlinear systems. Simulation results show the proposed controller can stabilize the affine fuzzy system in very short time.

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I. INTRODUCTION

The research in fuzzy modelling and fuzzy control have come of age. There are two model-based approaches to theoretically construct T-S fuzzy system of a nonlinear system: One is from local linear approximation, which generates linear consequent part with a constant term included in each rule, called affine T-S fuzzy system; the other is via sector nonlinearity concept [1], which results in constant-free linear consequence for each rule, called linear T-S fuzzy system [2]. Both are demonstrated to be universal approximation to any smooth nonlinear systems. It is noticed that the consequent parts in both models are represented by linear state equations, but there exists a constant singleton in the fuzzy rule consequence for each rule, called constant-free linear T-S fuzzy system [2].

II. AFFINE-TYPE FUZZY CONTROLLER

We consider the following affine T-S fuzzy system to describe a nonlinear physical system:

\[
R^i : \quad \text{if } x_1 = T_{1i}, \ldots, x_n = T_{ni}, \text{ then } Y(t) = CX(t), \\
X(t) = A_1X(t) + B_1u(t) + D_1, \quad i = 1, \ldots, r, \quad (1)
\]

where \( R^i \) denotes the \( i \)th rule of the fuzzy model; \( x_1, \ldots, x_n \) are system states; \( T_{1i}, \ldots, T_{ni} \) are the input fuzzy terms in the \( i \)th rule; \( X(t) = [x_1, \ldots, x_n]^{T} \in \mathbb{R}^n \) is the state vector, \( Y(t) = [y_1, \ldots, y_n]^{T} \in \mathbb{R}^m \) is the system output vector, and \( u(t) \in \mathbb{R}^m \) is the system input; and \( A_i, B_i, C_i, D_i \) are, respectively, \( n \times n, n \times m, n' \times n \) and \( n \times m \) matrices. We shall design the rule-based fuzzy controllers, \( (i = 1, \ldots, \delta) \)

\[
R^i : \quad \text{if } y_1 = S_{1i}, \ldots, y_n = S_{ni}, \text{ then } u(t) = r_i(t) \quad (2)
\]

to minimize the quadratic cost functional,

\[
J(u(\cdot)) = \int_{t_0}^{\infty} [X^T(t)LX(t) + u^T(t)Su(t)]dt, \quad (3)
\]

However, it is impractical to theoretically convert the mathematical model into T-S fuzzy model if the nonlinear system is too complex to describe. More and more researchers attempt to learn fuzzy model from input-output data [11]. For the learning-based modelling, the affine T-S fuzzy model is much preferred than linear type in providing one more adjustable parameter for computation-intelligent (neural-fuzzy-evolution) learning [12]. However, there so far no corresponding servo and few regulating controller are proposed for affine T-S fuzzy systems. E. Kim and coauthors synthesize the affine-type fuzzy controller via convex optimization technique and recast it into LMI problem [13], [14]. They further specialize in affine T-S fuzzy system with constant input matrix and transform the regulating problem into bilinear matrix inequality [15]. P. Bergsten and coauthors try to derive affine-type observer; but the constant-term consequence in affine T-S fuzzy system is just a trivial term for observer derivation and the simulation is in fact a typical linear-type formulation [16].

In this paper, we shall develop the relative controlling techniques for affine TS-model-based nonlinear systems. Technical contributions of this paper can be described as follows. The global optimal regulating control for affine TS-based nonlinear systems is theoretically derived. The proposed closed-loop fuzzy regulating systems are demonstrated to be stable. Several nonlinear systems are concerned to examine the performance of the regulating controllers.
where \( X^i(t)LX(t) \) is state-trajectory penalties with \( L \) belonging to symmetric positive semi-definite \( n \times n \) matrices, and \( u^i(t)Su(t) \) denotes energy consumption; \( y_1, \ldots, y_o \) are the elements of output vector \( Y(t) \), \( S_{1i}, \ldots, S_{ni} \) are the input fuzzy terms in the \( i \)-th control rule, and the plant input (i.e., control output) vector \( u(t) \) or \( r_i(t) \) is in \( \mathbb{R}^n \) space.

From the essence of the dynamic programming formalism, the operation of minimizing \( J(u(\cdot)) \) in Eq. (3) can be decomposed as follows:

\[
\min_{u \in [t, \infty)} J(u(\cdot)) = \min_{u \in [t, \infty)} \left\{ \int_t^\infty (X^i(t)LX(t) + u^i(t)Su(t)) dt \right\} + \min_{u \in [t, \infty)} \left\{ \int_t^\infty (X^i(t)LX(t) + u^i(t)Su(t)) dt \right\},
\]

where we use lower index to denote time-dependence for notation simplification, i.e., \( X_t \) for \( X(t) \). Hence, the quadratic optimization problem is, in fact, a successively on-going dynamic problem with regarding to the state resulting from the previous decision, i.e., the initial state (at time \( t \)) \( X_{0t} = X^*_t \). Moreover, according to the signal flow of a fuzzy inference system [12], we know, at any time \( t \), the overall behavior of the fuzzy system can be captured by fuzzily blending all the fuzzy subsystems; in other words, the entire T-S type fuzzy system in Eq. (1) can be represented as

\[
\dot{X}_t = \sum_{i=1}^r h_i(X_t)(A_iX_t + B_iu_t + D_i), \quad t \in [t, \infty), \tag{5}
\]

with \( u_t = \sum_{i=1}^\delta w_i(Y_t)r_{it} \) and \( X_{0t} = X^*_t \in \mathbb{R}^n; h_i(X_t) \) and \( w_i(Y_t) \) denote, respectively, the normalized firing-strength of the \( i \)-th rule of the fuzzy model and of the \( i \)-th fuzzy control rule; i.e., \( h_i(X_t) = \alpha_i / \sum_{i=1}^\delta \alpha_i \) with \( \alpha_i = \Pi_{j=1}^\delta \mu_j(X_t) \), where \( \mu_j(X_t) \) is the membership function of fuzzy term \( T_{ji} \), and \( w_i(Y_t) = \beta_i / \sum_{i=1}^\delta \beta_i \) with \( \beta_i = \Pi_{j=1}^\delta \mu_{ji}(Y_t) \), where \( \mu_{ji}(Y_t) \) is the membership function of fuzzy term \( S_{ji} \).

Therefore, the continuous optimization dynamic issue is on successively finding the continuous optimal global decision (global optimal fuzzy controller) \( u^*_t \) for minimizing the continuous cost functional,

\[
J_t(u_t) = \int_t^\infty (X^i(t)LX(t) + u^i(t)Su(t)) dt, \quad t \in [t, \infty), \tag{6}
\]

and estimating \( X^*_t \) with regarding to the initial state \( X^*_t \), where \( t^+ \) denotes the time instant slightly later time \( t \); and then, with the new initial state, \( X^*_t \), resolving \( u^*_t \) to minimize \( J_t(u_t) \). In other words, the quadratic optimal continuous fuzzy control problem can be restated as the following dynamic problem:

**PROBLEM 1.1**: Given the continuous fuzzy system in Eq. (5) with \( u_t = \sum_{i=1}^\delta w_i(Y_t)r_{it} \), successively find the continuous optimal global decision, \( u^*_t \), for minimizing the continuous quadratic cost functional \( J_t(u_t) \) in Eq. (6), where the initial state is the optimal state resulting from the previous decision, i.e., \( X_{00} = X_0 \) and \( X_{0t} = X^*_t \), \( t \in [t, \infty) \).

At any time-instant \( t \), the optimal local decision (local optimal fuzzy control law) is from minimizing \( J_t(u_t) \) in Eq. (6) with regard to the fuzzy subsystem,

\[
\dot{X}_t = A_iX_t + B_iu_t + D_i, \quad t \in [t, \infty), \quad i = 1, \ldots, r; \tag{7}
\]

the optimal global decision is from minimizing \( J_t(u_t) \) with regard to the entire fuzzy system in Eq. (5). For clarity, since \( u^*_t \) is only a variable to be solved no matter for the aforementioned local optimization problem or for the global optimization issue in Problem 1.1, we can use \( r^*_t \) to denote the optimal local decision of the \( i \)-th fuzzy subsystem.

Now, let \( \zeta_t(X_t, u_t) \) and \( \zeta_t(X_t, r_{it}) \), \( i = 1, \ldots, r \), denote, respectively, the entire energy and local energy at any time-instant \( t \), \( t \in [t, \infty) \). Then, \( J_t(u_t) = \int_t^\infty \zeta_t(X_t, u_t) dt \) and \( J_t(r_{it}) = \int_t^\infty \zeta_t(X_t, r_{it}) dt \). At any time-instant, the energy of the entire fuzzy system is some kinds of (nonlinear) summation of the energy of fuzzy subsystems. This nonlinear summation is not necessary to be the same summation as fuzzily blending subsystems into entire system and is only state dependent; it is nothing to do with input since the physical properties such as stability will not affect by different inputs. Therefore, we use \( h^i(X(t)) \) to denote that (a) nonlinear summation is only state dependent; (b) the energy-relationship between entire system and subsystems could be fully different from the behavior-relationship, denoted by normalized membership function, \( h(X(t)) \). We then write \( \zeta_t(X_t, u_t) = \sum_{i=1}^r h_i^i(X_t) \zeta_t(X_t, r_{it}) \), where \( \zeta_t(X_t, u_t) \) and \( \zeta_t(X_t, r_{it}) \), \( i = 1, \ldots, r \), denote, respectively, the entire energy and local energy at any time-instant \( t \), \( t \in [t, \infty) \). At the time-instant \( t \) with initial condition \( X^*_t \), let \( r^*_t \) denote the optimal local decision to minimize \( J_t(r_{it}) \) for all \( i = 1, \ldots, r \), i.e.,

\[
\frac{\partial J_t(r_{it})}{\partial r_{it}} |_{r^*_{it}} > 0, \tag{7}
\]

Then, their corresponding global decision \( \dot{u}_t = \sum_{i=1}^r h_i(X_t) \zeta_t(X_t, r_{it}) \) can satisfy

\[
\frac{\partial J_t(u_t)}{\partial u_t} |_{\dot{u}_t} = \sum_{i=1}^r h_i(X_t) \frac{\partial^2 \zeta_t(X_t, r_{it})}{\partial r_{it}^2} |_{\dot{u}_t} = 0, \tag{7}
\]

where \( \dot{u}_t = u^*_t \). So, at any time instant \( t \), if we can find \( r^*_t \) to minimize \( J_t(r_{it}) \) then their composed global decision \( u^*_t \) can be the global minimizer of the total cost \( J_t(u_t) \).

**Theorem 1**: For affine T-S fuzzy system in Eq. (1) and fuzzy controller in Eq. (2), if \( A_i \) is nonsingular, \( (A_i, B_i, C) \) is c.b., \( (A_i, C) \) is c.o. and \( \tilde{\pi}_i^{-1}(L + A_i^t\tilde{\pi}_i) > 0, \forall i = 1, \ldots, r \), then

(1) the local optimal fuzzy regulating law is

\[
r^*_i(t) = -S^{-1}B_i^t\pi_iX^*_i(t) + \dot{r}^*_i, \quad i = 1, \ldots, r, \tag{8}
\]
where $\bar{r}^i_\pi = -S^{-1}B^i_1(\bar{p}_i\bar{X}^i_t + b^i_t)$ and $\bar{X}^i_t = A^{-1}_iD_i$; their “blending” global optimal fuzzy controller, $u^*_i(t) = \sum_{i=1}^{r} h_i(X^*(t))r^i_t(t)$, minimizes $J(u(\cdot))$ in (3), where

$$\bar{b}^i_t = -\int_0^\infty e^{l(A_i - B_iS^{-1}B^i_1\pi_i)|r \cdot d\tau \cdot L\bar{X}^i_t},$$

(9)

and $\bar{\pi}_i$ is the unique symmetric positive semi-definite solution of the Riccatti equation

$$K_iA_i + A^T_iK_i - K_iB_iS^{-1}B^i_1K_i + L = 0;$$

(10)

the following entire optimal feedback fuzzy system is stable.

$$\bar{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))[A_i - B_iS^{-1}B^i_1\pi_i]X^*(t) + B_i\bar{r}^i_t + D_i.$$  

(11)

**Proof.** (1) Based on the aforementioned local viewpoint of the global optimal fuzzy control, we know that solving the quadratic optimal control problem in Problem 1.1 is to find only one corresponding optimal solution of the fuzzy controller for each rule of the fuzzy model. We further assume $A_i$, $i = 1, \ldots, r$, is nonsingular and let $\bar{X}(t) = X(t) + \bar{X}^i_t$, where $\bar{X}^i_t = A^{-1}_iD_i$. Therefore, the optimal global decisions $u^*_i \in \mathbb{R}^{\infty}$ can be regarded as a series of optimal global decision $u^*_i$ based on the following successively on-going local quadratic optimal constant-target tracking issue with the initial state resulting from the previous decision. In other words, the affine-type quadratic problem is reformulated into successively on-going augmented linear-type quadratic tracking problem: Find the optimal local decision at time instant $t$, $r^i_t$, for minimizing the cost functional,

$$J_i(r^i_t) = \int_t^\infty ((\bar{X}^i_t - \bar{X}^i_t)L(\bar{X}^i_t - \bar{X}^i_t) + r^i_tS^i_t)dl$$

with regarding to the fuzzy subsystem, $\dot{\bar{X}}^i_t = A_i\bar{X}^i_t + B_i\bar{r}^i_t$, $l \in [t, \infty)$; and then, the derived local decisions are fuzzily blended to generate the optimal global decision at time instant $t$, $u^*_i$, for minimizing the entire cost functional $J_i(u^*_i)$ in (6) , i.e., $u^*_i = \sum_{i=1}^{r} h_i(X^*(t))r^i_t$. Since the local fuzzy system (i.e., fuzzy subsystem) is linear, its quadratic optimization problem is the same as the general linear quadratic tracking issue. Therefore, it is realizable that solving the optimal control problem for fuzzy subsystem can be achieved by simply generalizing the classical quadratic tracking theorem from the deterministic case to the fuzzy case: regulating law $r^i_t(t) = -S^{-1}B^i_1[\bar{\pi}_i(X^*(t) + \bar{X}^i_t) + b^i_t(t)]$, fuzzy subsystem $\dot{X}^*(t) = (A_i - B_iS^{-1}B^i_1\pi_i)X^*(t) + \bar{X}^i_t) - B_iS^{-1}B^i_1b^i_t(t)$ with $b^i_t(t)$ satisfies

$$b^i_t(\infty) = 0_{n \times 1}.$$  

(12)

We further obtain $b^i_t(t) = \bar{b}^i_t$ in (9). Hence, the optimal fuzzy control design scheme for an affine T-S fuzzy system in Theorem 1 can then be derived.

(2) For stability analysis, we regard $\bar{X}^i_t$ as an artificial target. Let $\bar{U}^i_\pi(t) = B_i\bar{r}^i_t + D_i$, an artificial-target associated constant input. Then, we know the stability of the resultant feedback fuzzy system in Eq. (11) concurs with the zero-input system,

$$\bar{X}^*(t) = \sum_{i=1}^{r} h_i(X^*(t))[A_i - B_iS^{-1}B^i_1\pi_i]X^*(t).$$  

(13)

Defining Lyapunov function $V(X) = X^TPX$, where $P$ is a symmetric positive matrix. Via Eq. (10), we have $A_i - B_iS^{-1}B^i_1\pi_i = -\pi^{-1}_i(L + A^T_i\pi_i)$, and hence,

$$\dot{V}(X) = X^TP\bar{X}^* + \bar{X}^*PX$$

$$\leq -\sum_{i=1}^{r} h_i(X^*)\{X^*[(L + A^T_i\pi_i)]^TPX$$

$$+X^TP\sum_{i=1}^{r} h_i(X^*)[A_i - B_iS^{-1}B^i_1\pi_i]X]\}

$$= -\sum_{i=1}^{r} h_i(X^*)\{X^*[\pi^{-1}_i(L + A^T_i\pi_i)]X$$

$$-2\sum_{i=1}^{r} h_i(X^*)[X^*\pi^{-1}_i(L + A^T_i\pi_i)]X < 0$$

for choosing $P = I > 0$ and $\pi^{-1}_i(L + A^T_i\pi_i) > 0$, since $h_i(X^*)$ is a positive number always.

**III. Numerical Simulation**

In this section, a four-poles active magnetic bearing system and a sinusoidal nonlinear system are adopted to examine the performance of the proposed affine-type fuzzy control scheme. We first consider a horizontal four-poles differential-driving magnetic bearings driven by sum of bias current and control current [17]. On the assumption of symmetric structure and rigid floating mass, we have the dynamic equation of rotor motion,

$$m\ddot{x}(t) = \lambda(-i_b + i(t))^2 - \lambda(-i_b - i(t))^2,$$

(14)

where $i(t)$ is control current, rotor mass $m = 0.0126$ (lb · sec$^2$/in), nominal air gap $G = 0.02$ in, force constants $\lambda = 0.00186$ (lb · in$^2$/Amp$^2$), sensitivity of air gap to shaft displacement $\beta = 0.974$ and the bias current $i_b = 0.3$ Amp [17]. From Figs. 1(a) and 1(b), we found, for operating $x(t) = 0.01$ in, the magnetic-force evolution is parabolic function with an extreme point at around $i(t) = -0.3$ Amp, and, for $x(t) = 0.005$ in, the magnetic-force evolution is parabolic function with an extreme point at around $i(t) = -0.6$ Amp. Taylor’s expansion technique is used to linearize the magnetic bearing system with respect to operating point $(x(t), i(t)) = (0.0, (0.005, 0.01), (0.005, -0.3)$ and $(0.01, -0.3)$. And, the following affine T-S fuzzy system is obtained [17]

$$R^1: \text{If } x \text{ is ZE, then } \bar{X}(t) = A_1X(t) + B_1i(t) + D_1,$$

$$R^2: \text{If } x \text{ is PM, } i \text{ is ZE, then } \bar{X}(t) = A_2X(t) + B_2i(t) + D_2,$$
conditions rotor and the proposed control current with initial regulators could propose more flexible con-

We now consider another nonlinear system [15],

\[
\begin{align*}
A & = \begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix} \\
B & = \begin{bmatrix}
1 \\
2
\end{bmatrix}
\end{align*}
\]

The affine T-S fuzzy system is (i = 1, 2, 3) [15]

\[
R^i : \text{If } x(t) \in \mathcal{R}^i, \text{ then } \dot{X}(t) = A_iX(t) + B_iu(t) + D_i,
\]

where \(X(t) = [x(t) \ \dot{x}(t)]^T\) and the membership functions of the fuzzy terms are shown in Figs. 1(c) and 1(d); \(A_1 = \begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix} \) and \(A_2 = \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix} \) and \(A_3 = \begin{bmatrix}
3 & 4 \\
4 & 5
\end{bmatrix} \) for each rule.

Each fuzzy subsystem is c.c. and c.o. since \(\text{rank}[B_i] = 2\) and \(\text{rank}[C_i^T A_i^T C_i]^T = 2\) for

\[
\begin{align*}
A_1 & = \begin{bmatrix}
0 & 1 \\
1 & 2
\end{bmatrix} \\
A_2 & = \begin{bmatrix}
1 & 2 \\
2 & 3
\end{bmatrix} \\
A_3 & = \begin{bmatrix}
3 & 4 \\
4 & 5
\end{bmatrix}
\end{align*}
\]

and \(\dot{\pi}_i - 1(L + A_i^T \dot{\pi}_i) > 0\) for each rule. Figure 2 shows the evolution of the position and velocity of the shaft rotor and the proposed control current with initial conditions \(x(0) = [0.0082 \ 0]^T\) and \(i(0) = -0.15 \ Amp; x(0) = [0.006 \ 0]^T\) and \(i(0) = -0.33 \ Amp; x(0) = [0.004 \ 0]^T\) and \(i(0) = -0.53 \ Amp; x(0) = [0.002 \ 0]^T\) and \(i(0) = -0.73 \ Amp. \) Simulations results show that the dynamic system can be stabilized in around just 0.4 seconds. We now consider another nonlinear system [15],

\[
\begin{align*}
\dot{x}'_1(t) & = -x'_2(t) + \epsilon x(t) cos x'_2(t) + (1 + \epsilon cos x'_2(t)) u(t), \\
\dot{x}'_2(t) & = x'_1(t) + u(t), \\
\dot{x}'_3(t) & = x'_1(t) + 3x'_2(t) + sin x'_2(t)
\end{align*}
\]

with \(\epsilon = 0.5.\) A diffeomorphic transform, \[
\begin{bmatrix}
x'_1(t) \\
x'_2(t) \\
x'_3(t)
\end{bmatrix} = \begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
\]

is used to get a constant input function. We then have

\[
\begin{align*}
\dot{x}_1(t) & = x_2(t) + 4x_3(t) + (\epsilon + 1) sin x_3(t), \\
\dot{x}_2(t) & = -x_1(t) - x_2(t) - x_3(t) - \epsilon sin x_3(t), \\
\dot{x}_3(t) & = x_2(t) + x_3(t) + \epsilon sin x_3(t) + u(t).
\end{align*}
\]

The affine T-S fuzzy system is (i = 1, 2, 3) [15]

\[
R^i : \text{If } x_3(t) \in \mathcal{M}_i, \text{ then } \dot{X}(t) = A_iX(t) + B_iu(t) + D_i,
\]

where \(\pi = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\]

\[
\begin{align*}
D_1 & = \begin{bmatrix}
0 \\
-1 \\
-1
\end{bmatrix} \\
D_2 & = \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix} \\
D_3 & = \begin{bmatrix}
0 \\
1 \\
-2
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\pi_1 & = 1.1050 \\
\pi_2 & = 0.0443 \\
\pi_3 & = 0.9547
\end{align*}
\]

\[
\begin{align*}
\pi_1 & = 1.1050 \\
\pi_2 & = 0.0443 \\
\pi_3 & = 0.9547
\end{align*}
\]

\[
\begin{align*}
\pi_1 & = 1.1050 \\
\pi_2 & = 0.0443 \\
\pi_3 & = 0.9547
\end{align*}
\]

\[
\begin{align*}
\pi_1 & = 1.1050 \\
\pi_2 & = 0.0443 \\
\pi_3 & = 0.9547
\end{align*}
\]

IV. CONCLUSIONS

Though linear T-S fuzzy system is the most popular fuzzy model and has been successfully applied in various fields, affine T-S is still much preferred in neural-fuzzy modelling of mode-free physical system or even highly nonlinear and complex model-based system. We here propose the affine-type optimal fuzzy controller to stabilize an nonlinear system. Global optimum effect is guaranteed through theoretically derivation in Section 2. And, the generated closed-loop control system is demonstrated to be stable. Two nonlinear systems are considered to show the proposed regulating controller can achieve stabilizing effect in very short time.

REFERENCES


Fig. 1. (a) Magnetic-force evolution with respect to control current $i(t)$ and rotor displacement $x(t)$; (b) Magnetic-force evolution with control current $i(t)$ at $x(t) = 0$, 0.005, 0.01 in; (c) and (d) membership functions for $x(t)$ and $u(t)$.

Fig. 2. The rotor position $x^*(t)$, velocity $\dot{x}^*(t)$ and optimal control current $i^*(t)$ of 4-poles AMB system.

Fig. 3. State response $X^*(t)$ in Eq. (16) and the proposed optimal controller $u^*(t)$. 

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