On stop-loss strategies for stock investments

Shih-yu Shen
Associate Professor
Department of Mathematics
National Cheng-Kung University

Andrew Minglong Wang
Associate Professor
Department of Accounting
National Cheng-Kung University

Please send all correspondence to Shih-yu Shen,
Department of Mathematics, Cheng-Kung University, Tainan, Taiwan 70101, ROC
Abstract—
This paper studies the expected return of a stock investment with a stop-loss strategy. The probability density function for the investment value is formulated as the solution for a boundary value problem of a partial differential equation. Then, the expected value is manipulated as a function of the stop-loss probability. Two examples are solved by an analytic method. Finally, we design a boundary element method to solve the boundary value problem for a general stop-loss criterion.

Keywords— Stop-loss strategy, Boundary value problem, Convection-diffusion equation, Boundary element methods.
On stop-loss strategies for stock investments

§1. Introduction

One of the most frequently used investment strategies to control risk is to use stop-loss order. A stop-loss strategy means setting a criterion at the beginning of the investment and selling the stock when the stock price meets the criterion. This paper examines the impact of a stop-loss order upon the expected return of a stock investment, and evaluate the possibility of igniting it.

The stock price follows a stochastic process is a widely accepted assumption in finance. Bachelier [1] used an arithmetic Brownian motion process to analyze stock investment. Black and Scholes [2], and Merton [3] drived the first explicit general equilibrium solution to the option pricing under the assumption of the stock price following a geometric Brownian motion. Since the publication of the Black-Scholes paper, the geometric Brownian motion has become the standard model for stock price behavior.

A Monte Carlo method may be applied to evaluate the risk and the expected return of such an investment; but it takes a huge computing time. Another method is using the first hitting time of a Winer process to model the stop-loss probability. In some cases, the distribution of the first hitting time is available [4][5]. Nevertheless, the method in this paper is more realistic for a practical problem.

In this study, the partial differential equation (PDE) methodology has been applied to evaluate probability density function (p.d.f.) of a stock investment. With the geometric Brownian motion model, the implicit assumption is that the probability density function for a stock price is a log normal distribution. Therefore, we define a transformed price $x$ as logarithm of stock price. Then, the probability density for $x$ is a normal density function with a volatility $\sigma$ and drift $\eta$. When the present stock price is $s_0$, the density $u^s(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0-\eta t)^2}{2\sigma^2 t}}$, where $x_0 = \ln s_0$. When a stop-loss strategy is used, the calculation of p.d.f. for $x$ becomes complicated. We consider the continuous model as the limit of a discrete model. Therefore, the density function can be formulated as the solution of a boundary value problem of a partial differential equation.
The impact of a stop-loss strategy upon the expected return of an investment is an important practical issue. If we define $u(x, t)$ as the p.d.f. of $x$ at time $t$ when a pre-specified stop-loss criterion not been met, then, the expected value of an investment adopting such a strategy can be evaluated with $u(x, t)$. So does the time varying instantaneous expected return and probability of executing the stop-loss order.

In this paper, a boundary element method (BEM) is designed to solve the boundary value problem of the PDE. The PDE for $u(x, t)$ is a convection-diffusion equation which is a linear homogeneous equation. Boundary value problems of linear PDE appear in many fields such as heat conduction and wave propagation [6]. Numerical methods for boundary value problems of PDE have been extremely intensively studied in recent decades. Finite element methods [7] and finite difference methods [8] are among the most important approaches for solving PDE. However, BEMs are more efficient and accurate for solving linear PDE than the above methods [9]. PDE appearing in many financial derivative pricing models [10] are in the form of the convection-diffusion equation, BEM could play a pivotal role in the pricing of financial assets.

The outline of the paper is as follows: In section 2, the p.d.f. for continuous time is modeled as the limit of the p.d.f. for a discrete model. Then, a boundary value problem for the p.d.f. is established. Section 3 derive the formulations for instantaneous annualized expected returns and expected values. Two examples with linear criteria are solved analytically in section 4. In section 5, we design a boundary element method to solve the probability density for general criteria. The BEM is applied to an example to verify the accuracy of the method. The last section is discussion and conclusion.

§2. The PDE and the boundary condition

In this paper, the stock price behavior is modeled as geometric Brownian motion. We define a quantity $x = \ln s$ as the transformed price where $s$ is the stock price-per-share so that the transformed price is in Brownian motion with a drift.
Therefore, as the present stock price is $s_0$, the probability density for the transformed price $x$ at time $t$, $u^*(x, t)$, is

$$u^*(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0-\eta t)^2}{2\sigma^2 t}},$$

where $x_0 = \ln s_0$, and $\sigma$ and $\eta$ are the stock price volatility and the drift respectively. We assume a function of time $s^*(t)$ be preset at the initial time such that the investor will sell the stock if the stock price reaches $s^*(t)$. In other words, the stock will be sold if the transformed price $x$ is $l(t)$, where $l(t)$ denotes $\ln(s^*(t))$. Here, $l(t)$ is called the criterion of the stop-loss strategy. In this study, one of the goals is to formulate the mathematical model for the probability density function, $u(x, t)$ for the investor holding the stock at transformed price $x$ and time $t$ under a specific stop-loss criterion.

We consider a discrete time model first. In a discrete model, an investment period $t$ is divided into $n$ steps. The treading can be made only at the end of the steps. At the end of $i$th step, the probability density for the investor owning the stock at the transformed price $x$ is denoted by $u_i(x)$. The probability density at the end of $(i+1)$th step, $u_{i+1}(x)$, can be calculated from the density at $i$th step by a convolution. Accordingly,

$$u_{i+1}(x) = \int_{l_i}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 \Delta t}} e^{-\frac{(x-\tau-\eta \Delta t)^2}{2\sigma^2 \Delta t}} u_i(\tau) d\tau,$$

where $l_i = l(t_i)$ and $\Delta t = t/n$. The continuous time model is taken as the limit of the discrete model; i.e.

$$u(x, t) = \lim_{n \to \infty} u_n(x).$$

The density function $u(x, t)$ for the continuous model will be shown to fulfill a partial differential equation.

Let $p_i(x)$ denote the probability of that the transformed price larger than $x$ at the end of $i$th time step. Thus

$$p_i(x) = \int_x^{\infty} u_i(\tau) d\tau \quad \text{for} \quad x > l_i.$$
Substituting equation (1) into equation (2), we have

\[ p_{i+1}(x) = \int_x^\infty \int_{l_i}^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-(\bar{x} - \tau - \eta \Delta t)^2/2\sigma^2 \Delta t} u_i(\tau) d\tau d\bar{x}, \]

for \( x > l_{i+1} \). The probability difference between two steps is

\[ p_{i+1}(x) - p_i(x) = \int_x^\infty \int_{l_i}^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-(\bar{x} - \tau - \eta \Delta t)^2/2\sigma^2 \Delta t} u_i(\tau) d\tau d\bar{x} \]

\[- \int_x^\infty u_i(\bar{x}) \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-(\bar{x} - \tau - \eta \Delta t)^2/2\sigma^2 \Delta t} d\tau d\bar{x}. \]

Using a substitution, \( v = \bar{x} - \tau - \eta \Delta t \), we have

\[ p_{i+1}(x) - p_i(x) = \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-(v - x - \eta \Delta t)^2/2\sigma^2 \Delta t} (u_i(\bar{x}) - u_i(\bar{x} - v - \eta \Delta t)) d\bar{x} dv \]

\[- \int_{-\infty}^\infty \int_x^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-(v - x - \eta \Delta t)^2/2\sigma^2 \Delta t} u_i(\bar{x}) d\tau d\bar{x} \]

where

\[ a(v) = \begin{cases} 
  x & \text{if } v \leq x - l_i - \eta \Delta t \\
  v + l_i + \eta \Delta t & \text{if } v > x - l_i - \eta \Delta t 
\end{cases} \]

Consequently,

\[ p_{i+1}(x) - p_i(x) = \int_{-\infty}^\infty -\frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-v^2/2\sigma^2 \Delta t} \int_{a(v)}^\infty u_i(\bar{x}) - u_i(\bar{x} - v) + u_i(\bar{x} - v) - u_i(\bar{x} - v - \eta \Delta t) d\bar{x} dv \]

\[- \int_{-\infty}^\infty \int_x^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-v^2/2\sigma^2 \Delta t} u_i(\bar{x}) d\bar{x} dv, \]

and then

\[ \frac{p_{i+1}(x) - p_i(x)}{\Delta t} = \int_{-\infty}^\infty -\frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-v^2/2\sigma^2 \Delta t} \int_{a(v)}^\infty u_i(\bar{x} - v) - u_i(\bar{x} - v - \eta \Delta t) d\bar{x} dv \]

\[ + \int_{-\infty}^\infty \frac{-v}{\Delta t \sqrt{\pi \sigma^2 \Delta t}} e^{-v^2/2\sigma^2 \Delta t} \int_{a(v)}^\infty u_i(\bar{x} - v) - u_i(\bar{x}) d\bar{x} dv \]

\[- \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-v^2/2\sigma^2 \Delta t} \frac{1}{\Delta t} \int_{x}^{a(v)} u_i(\bar{x}) d\bar{x} dv. \quad (3) \]

Note that, (ref. page 9 of [11])

\[ \lim_{t \to 0^+} \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{v^2}{4t}} f(v) dv = f(0), \]
and
\[
\lim_{t \to 0^+} \int_{-\infty}^{\infty} \frac{v}{4t^{3/2}} e^{-\frac{v^2}{8t}} f(v) dv = f'(0).
\]

Thus, when \( \Delta t \) approaches 0, the first term of the right-hand side of equation (3) is \( \eta u_i(x) \), the second term is \( -\frac{\sigma^2}{2} \frac{d}{dx} u_i(x) \) and the last term is 0. Let \( \Delta t \) approach 0. The temple derivative of the probability function \( p(x, t) \) for continuous time is obtained;

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\sigma^2}{2} u_x(x, t) + \eta u(x, t) \tag{4}
\]

Since \( \frac{\partial}{\partial x} p(x, t) = -u(x, t) \), differentiating both sides of equation (4) with respect to \( x \), we have

\[
u_t(x, t) = \frac{\sigma^2}{2} u_{xx}(x, t) - \eta u_x(x, t). \tag{5}
\]

Partial differential equation (5) is a convection-diffusion equation.

The criterion \( l(t) \) is assumed to be differentiable, and, therefore, there exists a positive number \( \kappa \) not depending on \( \Delta t \) such that \( |l_{i+1} - l_i| < \kappa \Delta t \). Consider the probability density near the boundary \( l_{i+1} \)

\[
u_{i+1}(l_i + k \Delta t)) = \int_{l_i}^{l_i + \Delta t} \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-\frac{(l_i - \tau - (n-k)\Delta t)^2}{2\sigma^2 \Delta t}} u_i(\tau) d\tau
\]

\[
= \int_{l_i}^{l_i + \Delta t} \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-\frac{(l_i - \tau)^2}{2\sigma^2 \Delta t}} \left( e^{-\frac{2(n-k)(l_i - \tau)}{2\sigma^2 \Delta t}} e^{\frac{(n-k)^2\Delta t}{2\sigma^2 \Delta t}} u_i(\tau) d\tau, \right. \tag{6}
\]

where \( |k| < \kappa \). Note that

\[
\lim_{\Delta t \to 0} \int_{l_i}^{l_i + \Delta t} \frac{1}{\sqrt{2\pi \sigma^2 \Delta t}} e^{-\frac{(l_i - \tau)^2}{2\sigma^2 \Delta t}} f(\tau) d\tau = \frac{1}{2} f(l_i)
\]

Therefore, as \( k = \frac{l_{i+1} - l_i}{\Delta t} \),

\[
\lim_{\Delta t \to 0} u_i(l_{i+1}) = \frac{1}{2} u_i(l_i)
\]

Consequently,

\[
\lim_{\Delta t \to 0} u_{i+j}(l_{i+j}) = \left( \frac{1}{2} \right)^j u_i(l_i) \tag{7}
\]

For the continuous model,

\[
u(l(t), t) = 0, \tag{8}
\]
if \( l(t) \) is smooth. Equation (8) is the boundary condition for the p.d.f. \( u(x, t) \). Equation (5), boundary condition (8) and the initial condition

\[
u(x, 0) = \delta(x - x_0),
\]  

(9)

where \( \delta(x) \) is the Dirac delta function, compose a well-posed initial-boundary value problem, so that we may calculate the expected value of the investment and the stop-loss probability for a specific criterion. Let \( p_l(t) \) denote the stop-loss probability.

\[
p_l(t) = 1 - \int_{l(t)}^{\infty} u(x, t) \, dx
\]  

(10)

The change rate of stop-loss probability per unit time

\[
\frac{d}{dt} p_l(t) = - \int_{l(t)}^{\infty} u_t(x, t) \, dx + u(l(t), t)
\]

\[
= - \int_{l(t)}^{\infty} \sigma^2 \frac{x}{2} u_{xx}(x, t) - \eta u_x(x, t) \, dx = \frac{\sigma^2}{2} u_x(l(t), t).
\]  

(11)

Consequently,

\[
p_l(t) = \int_0^t \frac{\sigma^2}{2} u_x(l(\tau), \tau) \, d\tau.
\]  

(12)

§3. The expected value

In this section, we derive formulations for the expected value of an investment \( E(t) \) and the stop-loss probability \( p_l(t) \). Before we treat the general case, consider a simple special case first. The simple case is no strategy; i.e. \( l(t) = -\infty \). The expected value \( E(t) \) at time \( t \) is the expected value for the stock price \( E^s(t) \). The stock expected value at time \( t \)

\[
E^s(t) = \int_{-\infty}^{\infty} u^s(x, t) s(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} u^s(x, t) e^x \, dx.
\]  

(13)

The change rate of the expected value is that

\[
\frac{d}{dt} E^s(t) = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u^s(x, t) s(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{\sigma^2}{2} u^s_{xx}(x, t) - \eta u^s_x(x, t) \right) e^x \, dx.
\]  

(14)
Using integration by parts, we have

\[
\frac{d}{dt} E^s(t) = \left( \frac{\sigma^2}{2} u_x(x, t) - \left( \frac{\sigma^2}{2} + \eta \right) u^s(x, t) \right) e^x + \left( \frac{\sigma^2}{2} + \eta \right) \int_{-\infty}^{\infty} u^s(x, t) s(x) dx. \]

Note that \( \lim_{x \to \infty} u^s(x, t) e^x = 0 \) and \( \lim_{x \to \infty} u_x^s(x, t) e^x = 0 \) for any \( t \).

Thus,

\[
\frac{d}{dt} E^s(t) = \left( \frac{\sigma^2}{2} + \eta \right) E^s(t) \tag{15}
\]

Equation (15) is a first order linear ordinary differential equation (ode). With the initial condition, \( E^s(0) = s_0 \), equation (15) can be solved uniquely. The solution for equation (15) is

\[
E^s(t) = s_0 e^{\left( \frac{\sigma^2}{2} + \eta \right) t}.
\]

The annualized stock return is a constant \( \frac{\sigma^2}{2} + \eta \) which will be denoted by \( R \) in the rest of this paper.

For a general criterion \( l(x) \) with \( l(0) = l_0 < \ln s_0 \), the expected value \( E(t) \) of the stop-loss investment is that

\[
E(t) = \int_{l(t)}^{\infty} u(x, t) s(x) dx + \int_{0}^{t} \frac{\partial}{\partial t} p_1(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau, \tag{16}
\]

where \( r \) is the interest rate. The second term of the right-hand side in equation (16) is the expected value under the condition that the stop-loss be ignited. Substituting formula (11) into equation (16), we have

\[
E(t) = \int_{l(t)}^{\infty} u(x, t) e^x dx + \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau. \tag{17}
\]

Differentiating both sides of equation (17) and using PDE (5), we have

\[
\frac{d}{dt} E(t) = \int_{l(t)}^{\infty} u_t(x, t) e^x dx + u(l(t), t) e^{l(t)} e^{r(t)(t-\tau)} + \frac{\sigma^2}{2} u_x(l(t), t) e^{l(t)}
\]

\[
+ r \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau
\]

\[
= \int_{l(t)}^{\infty} \left( \frac{\sigma^2}{2} u_{xx}(x, t) - \eta u_x(x, t) \right) e^x dx + \frac{\sigma^2}{2} u_x(l(t), t) e^{l(t)}
\]

\[
+ r \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau. \tag{18}
\]
Using an integration by parts on the first term of the right-hand side of equation (18) and applying boundary condition (8), we have

\[
\frac{d}{dt} E(t) = (\frac{\sigma^2}{2} u_x(x, t) - (\frac{\sigma^2}{2} - \eta) u(x, t)) e^{\xi t} |_{l(t)} + (\frac{\sigma^2}{2} + \eta) \int_{l(t)}^{\infty} u(x, t) e^{\xi t} dx
\]

\[
+ \frac{\sigma^2}{2} u_x(l(t), t) e^{t(l(t))} + \frac{\sigma^2}{2} r \int_{0}^{t} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau
\]

\[
- \frac{\sigma^2}{2} u_x(l(0), 0) e^{l(0)} + (\frac{\sigma^2}{2} + \eta) \int_{l(0)}^{\infty} u(x, 0) e^{\xi t} dx + \frac{\sigma^2}{2} u_x(l(0), 0) e^{l(0)}
\]

\[
+ r \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau
\]

\[
= R \int_{l(0)}^{\infty} u(x, 0) e^{\xi t} dx + r \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau.
\]

Therefore,

\[
\frac{d}{dt} E(t) = RE(t) - (R - r) \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau, \quad (19a)
\]

or

\[
\frac{d}{dt} E(t) = rE(t) + (R - r) \int_{l(t)}^{\infty} u(x, t) e^{\xi t} dx, \quad (19b)
\]

From equation (19),

\[
e(t) = R - \frac{(R - r)}{E(t)} \int_{0}^{t} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(t-\tau)} d\tau \quad (20)
\]

The instantaneous annualized expected return \(e(t)\) varies with time and is always between \(R\) and \(r\), because \(u_x(l(t), t) \geq 0\) for \(t \geq 0\). Furthermore, the solution of the differential equation (19) can be expressed explicitly as

\[
E(t) = s_0 e^{Rt} - (R - r) \int_{0}^{t} e^{-R(\tau - t)} \int_{0}^{\tau} \frac{\sigma^2}{2} u_x(l(\tau), \tau) e^{l(\tau)} e^{r(\tau - \tau)} d\tau d\xi \quad (21)
\]

In expression (21), \(u_x(l(t), t)\) has to be solved. For some stop-loss criteria, close-forms for \(u_x(l(t), t)\) are available. In general, \(u_x(l(t), t)\) has to be solved by a numerical method.

§4. Examples
When the stop-loss criterion is a linear function of time, the exact solution for \( u(x, t) \) can be obtained. A portfolio insurance and a leveraged investment strategies are used as numerical examples. The instantaneous annualized expected return and stop-loss probability for both investment strategies are calculated to demonstrate the practical application.

**Example 1: portfolio insurance**

In this example, we assume that the investor adopts a portfolio insurance strategy, the goal is to guarantee the investing fund at least equal to the initial investment multiplied by a positive constant \( z \) at end of the investment period \( T \). The portfolio insurance strategy will be a feasible strategy when \( z < e^{r_f T} \) where \( r_f \) is the risk-free lending rate. Under this scenario, when the stock price hits \( z_0 e^{-r_f (T-t)} \), the stock position will be liquidated completely and reallocated to the Treasury-bill position. This strategy implies that the stop-loss criterion will increase with time at a rate of \( r_f \). The stop-loss criterion will be

\[
    l(t) = \ln(s_0 e^{-r_f T}) + r_f t
\]

Then, the p.d.f. \( u(x, t) \) is subject to

\[
    u_t(x, t) - \frac{\sigma^2}{2} u_{xx}(x, t) + \eta u_x(x, t) = 0 \tag{5}
\]

\[
    u(l((t), t) = 0 \tag{8}
\]

and

\[
    u(x, 0) = \delta(x - x_0). \tag{9}
\]

There is an exact solution for \( u(x, t) \).

\[
    u(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0-\eta t)^2}{2\sigma^2 t}} - \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-(x_0-2b)-\eta t)^2}{2\sigma^2 t}} + p
\]

where \( b = r_f T - \ln z \) and \( p = 2b(r_f - \eta) / \sigma^2 \). On the boundary, \( x = l(t) \),

\[
    u_x(l(t), t) = \frac{4b}{2\sigma^2 t \sqrt{2\pi\sigma^2 t}} e^{-\frac{(r_f - \eta) t - b}{2\sigma^2 t}}
\]
By using equation (10), we have

\[
p_l(t) = 1 - \frac{1}{2} \left( \text{erfc} \left( \frac{r_f t - \eta t - b}{\sqrt{2\sigma^2 t}} \right) - e^p \text{erfc} \left( \frac{r_f t - \eta t + b}{\sqrt{2\sigma^2 t}} \right) \right)
\]

Then, the o.d.e for the expected value is

\[
\frac{d}{dt} E(t) = RE(t) - (R - r_f)e^{l(t)}p_l(t).
\]  \hspace{1cm} (23)

The solution for the o.d.e (23) with the initial condition is

\[
E(t) = s_0 e^{Rt} \left( 1 - \int_0^t (R - r_f)e^{l(\tau)}p_l(\tau)e^{-R\tau} d\tau \right).
\]

As \( z = 0.8, T = 2, R = 0.12, r_f = 0.07 \) and \( \sigma = 0.3 \), the p.d.f. \( u(x, t) \) and the stop-loss probability \( p_l(t) \) may be obtained, and then the expected value \( E(t) \) can be calculated by a composite Simpson rule. When \( t = 2 \), the p.d.f., \( u(x, 2) \), is shown in figure 1. Figure 2 shows the p.d.f. with respect to the stock price \( s \). The p.d.f. with respect to \( s \) is \( u(\ln s, 2) \frac{1}{s} \). Figure 3 shows the probability \( p_l(t) \) with respect to time. Figure 4 and figure 5 show the expected value \( E(t)/s_0 \) and the instantaneous annualized expected return \( e(t) \) respectively. The dashed lines in figure 4 and figure 5 indicate the values without any stop-loss strategy.

Example 2: A leveraged investment

In this example, we consider a leveraged investment strategy to buy a share of stock with \( B \) fraction of borrowing fund. To avoid the possibility of default, the stop-loss criterion has to be preset. If \( s(t) \) hits \( Bs_0 e^{r_b t} \), where \( r_b \) is the borrowing rate, the stock position should be liquidated immediately and the value of investment will be zero. Thus, the criterion should be

\[
l(t) = \ln(Bs_0) + r_b t
\]  \hspace{1cm} (24)
The expected value for the investment is

\[ E(t) = \int_{l(t)}^{\infty} u(x,t)(e^x - B s_0 e^{r_b t}) dx \] (25)

Differentiating equation (25) and using a method similar to the method in section 3, we have

\[ \frac{d}{dt} E(t) = \left( \frac{\sigma^2}{2} + \eta \right) \int_{l(t)}^{\infty} u(x,t)e^x dx - re^{l(t)} \int_{l(t)}^{\infty} u(x,t)dx \]

or

\[ \frac{d}{dt} E(t) = RE(t) + (R - r_b)e^{l(t)}(1 - p_l(t)) \] (26)

Then, the instantaneous annualized expected return

\[ e(t) = R + (R - r_b)(1 - p_l(t)) \frac{e^{l(t)}}{E(t)}. \]

With the initial condition,

\[ E(0) = (1 - B)s_0, \]

o.d.e. (26) has an unique solution,

\[ E(t) = (1 - B)s_0 e^{R t} + (R - r_b)e^{R t} \int_0^t e^{l(\tau) - R \tau}(1 - p_l(\tau))d\tau. \]

We assume \( R > r_b \). \( e(t) \) is always greater than \( r_b \). The instantaneous annualized expected return equals \( R + (R - r_b)B \) at \( t = 0 \); but, decreasing with time. As \( t \) approaches infinity, \( e(t) \) approaches \( R \). When \( B, R, r_b \) and \( \sigma \) are 0.8, 0.12, 0.07 and 0.3 respectively, the instantaneous annualized expected return \( e(t) \) is plotted in figure 6. Figure 7 shows the expected value compared with the expected value without any leverage.

[Insert Figure 6 about here]

[Insert Figure 7 about here]

§5. Boundary element method
In this section, we derive a boundary element method for a general criterion. A boundary integral equation will be derived for the function \( u_x(l(t), t) \). Let \( g(x, t; \bar{x}, \bar{t}) \) be a fundamental solution of the dual equation of equation (5); i.e.

\[
-\frac{\partial}{\partial t} g(x, t; \bar{x}, \bar{t}) - \eta \frac{\partial}{\partial x} g(x, t; \bar{x}, \bar{t}) - \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} g(x, t; \bar{x}, \bar{t}) = \delta(x - \bar{x}, t - \bar{t})
\]  

(27)

where \( \delta(x, t) \) is the 2-D Dirac delta function. The fundamental solution

\[
g(x, t; \bar{x}, \bar{t}) = \frac{1}{\sqrt{2\pi\sigma^2(t-t')}} e^{-\frac{(x-\bar{x}+u(t-t'))^2}{2\sigma^2(t-t')}} H(t-t')
\]

where \( H(t) \) is the Heviside step function. Since \( u(x, t) \) fulfill equation (5),

\[
\int_0^T \int_{l(t)}^{\infty} \left( \frac{\sigma^2}{2} u_{xx}(x, t) - \eta u_x(x, t) - u_t(x, t) \right) g(x, t; \bar{x}, \bar{t}) dx dt = 0
\]

Using integration by parts, we have

\[
\int_0^T \left( -\frac{\sigma^2}{2} u_x(l(t), t) g(l(t), t; \bar{x}, \bar{t}) + \frac{\sigma^2}{2} u(l(t), t) g_x(l(t), t; \bar{x}, \bar{t}) + (\eta + l'(t)) u(l(t), t) g(l(t), t; \bar{x}, \bar{t}) dt
\]

\[
+ \int_{l(0)}^{\infty} \left( \frac{\sigma^2}{2} g_{xx}(x, t; \bar{x}, \bar{t}) + \eta g_x(x, t; \bar{x}, \bar{t}) + g_t(x, t; \bar{x}, \bar{t}) \right) u(x, t) dt dx = 0
\]

(28)

Substituting the boundary condition, \( u(l, t) = 0 \) and equation (27) into equation (28), we have the integral equation,

\[
u(\bar{x}, \bar{t}) = -\frac{\sigma^2}{2} \int_0^T u_x(l(t), t) g(l(t), t; \bar{x}, \bar{t}) dt + \int_{l(0)}^{\infty} u(x, 0) g(x, 0; \bar{x}, \bar{t}) dx
\]

(29)

Substitute initial condition (9) into equation (29) yield

\[
u(\bar{x}, \bar{t}) = g(x_0, 0; \bar{x}, \bar{t}) - \frac{\sigma^2}{2} \int_0^T u_x(l(t), t) g(l(t), t; \bar{x}, \bar{t}) dt
\]

(30)

For convenience, we use \( f(t) \) to denote \( u_x(l(t), t) \). When \( (\bar{x}, \bar{t}) \) approaches \( (l(\bar{t}), \bar{t}) \), equation (30) becomes

\[
\int_0^T f(t) g(l(t), t; l(\bar{t}), \bar{t}) dt = \frac{2}{\sigma^2} g(x_0, 0; l(\bar{t}), \bar{t})
\]

(31)

12
Equation (31) is the boundary integral equation for the function $u_x(l(t), t)$. Equation (31) may be solved by a numerical method.

Here, we design a boundary element method to approximate the function $f(t)$ in equation (31), so that the rate of the stop-loss probability, $\frac{d}{dt} p_l(t) = \frac{\sigma^2}{2} f(t)$, may be evaluated in the time interval $[0, T]$. Let $t_i = i\Delta t$ be the nodes where $\Delta t = \frac{T}{n}$ and $n$ is the number of nodes. The approximation for $f(t)$ is chosen to be piecewise constant. Thus, the approximation

$$f^*(t) = \sum_{i=1}^{n} a_i \phi_i(t), \quad (32)$$

where

$$\phi_i(t) = \begin{cases} 
1 & \text{if } t_{i-1} \leq t < t_i \\
0 & \text{if } t < t_{i-1} \text{ or } t \geq t_i 
\end{cases}$$

Therefore,

$$\int_{0}^{T} f^*(t)g(l(t), t; l(\bar{t}), \bar{t})dt = \sum_{i=1}^{n} a_i \int_{0}^{T} \phi_i(t)g(l(t), t; l(\bar{t}), \bar{t})dt \quad (33)$$

Let $\psi_i(\bar{t})$ denote $\int_{0}^{T} \phi_i(t)g(l(t), t; l(\bar{t}), \bar{t})dt$. Thus,

$$\psi_i(\bar{t}) = \int_{t_{i-1}}^{t_i} g(l(t), t; l(\bar{t}), \bar{t})dt \quad (34)$$

Substituting equation (33) into equation (31) and using a collocation method, we have the equations

$$\sum_{i=1}^{n} a_i \psi_i(t_k^*) = \frac{2}{\sigma^2} g(x_0, 0, l(t_k^*), t_k^*), \quad (35)$$

where $t_k^*$ are the collocation points. Note that $\psi_i(t_k^*) = 0$, when $t_k^* < t_i$. In this method, the collocation points are chosen to be the midpoints of the elements; i.e. $t_k^* = (k - \frac{1}{2})\Delta t$. Therefore $\psi_i(t_k^*) = 0$ if $k < i$. Thus the equations (35) can be rewritten as

$$a_k \psi_k(t_k^*) = \frac{2}{\sigma^2} g(x_0, 0, l(t_k^*), t_k^*) - \sum_{i=1}^{k-1} a_i \psi_i(t_k^*), \quad (36)$$
Because $\psi_k(t^*_k) \neq 0$, $a_k$ may be solved step by step. $\psi_i(t^*_k)$ are evaluated by the three-point Gaussian quadrature rule when $k > i$.

$$
\psi_i(t^*_k) = \frac{\Delta t}{18\sqrt{2\pi \sigma^2}} \left( \frac{5}{\sqrt{t_l}} e^{-\frac{(t_i^*_k - t_{l+1}^*_k - \eta t_l)^2}{2\sigma^2 t_l}}
\right.
\left. + \frac{8}{\sqrt{t_c}} e^{-\frac{(t_i^*_k - t_{c+1}^*_k - \eta t_c)^2}{2\sigma^2 t_c}}
\right.
\left. + \frac{5}{\sqrt{t_r}} e^{-\frac{(t_i^*_k - t_{r+1}^*_k - \eta t_r)^2}{2\sigma^2 t_r}} \right),
$$

(37)

where $t_l = (k - i + \frac{1}{2}\sqrt{\frac{3}{5}})\Delta t$, $t_r = (k - i - \frac{1}{2}\sqrt{\frac{3}{5}})\Delta t$ and $t_c = (k - i)\Delta t$. When $i = k$, $\psi_k(t_k)$ needs to be evaluated more accurately. Let $d$ denote the derivative $\frac{d}{dt}l(t_k)$. While $\Delta t$ is very small, the $k$’th element may be approximated by a straight segment, because $l(t)$ is smooth. Therefore,

$$
\psi_k(t_k) \approx \int_0^{\frac{\Delta t}{2}} \frac{1}{\sqrt{2\pi \sigma^2 t}} e^{-\frac{(d-\eta)^2 t}{2\sigma^2}} dt
\approx \sqrt{\frac{\pi}{4\sigma^2}} \int_0^{\frac{(d-\eta)^2 \Delta t}{4\sigma^2}} \frac{1}{\sqrt{\tau}} e^{-\tau} d\tau
= \frac{1}{|d-\eta|} \text{erf} \left( \frac{|d-\eta|}{2\sigma} \sqrt{\Delta t} \right)
$$

(38)

Using the expansion, we have

$$
\psi_k(t^*_k) \approx \frac{\sqrt{\Delta t}}{\sqrt{\pi \sigma^2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)j!} t^*^j
$$

(39)

where $t^* = \frac{(d-\eta)^2 \Delta t}{2\sigma^2}$.

**Example 3**

In order to verify the accuracy of the numerical method, we recalculate the stop-loss density $f(t)$ in example 2 by the boundary element method. Because the stop-loss criterion (24) is linear with time, therefore,

$$
\psi_i(t^*_k) = \psi_1(t^*_{k-i+1}).
$$
Let $d_i$ denote $\psi_1(t^*_i+1)$, the equation (36) becomes

$$a_k = \frac{2}{d_0 \sigma^2} g(x_0, 0, l(t_k^*), t_k^*) - \frac{1}{d_0} \sum_{i=1}^{k-1} a_i d_{k-i}. \tag{40}$$

When $k \neq 0$, $d_k$ may be calculated by formula (37); but $d_0$ has to be evaluated more accurately. Taking 6 terms in equation (39), we have

$$d_0 = \sqrt{\Delta t} \sqrt{\frac{\pi \sigma^2}{1 - t^*_3 + t^*_2 - t^*_1 + t^*_4 - t^*_5}}$$

where $t^* = \frac{(r_f - \eta)^2 \Delta t}{4 \sigma^2}$. Here, the drift $\eta$, which is $R - \frac{\sigma^2}{2}$, is 0.075. $a_k$ can be calculated by using formulation (40) step by step.

After $a_k$ are calculated, $f(t)$ may be approximated by equation (32). When $T = 2$ and $n = 200$, the result of $f(t)$ is shown in figure 8 compared with the exact solution. The resulting curve coincides with the exact solution. In figure 8, the dashed line shows the magnified difference between the numerical result and the exact solution. Table 1 lists the relative $L_2$ error for different number of elements where the relative $L_2$ error is $(\int_0^T \left( \sum_{i=1}^{n} a_i \phi_i(t) - f(t) \right)^2 dt) / \int_0^T f^2(t) dt \frac{1}{2}$. From table 1, we find that the convergence rate of the method is better than first order.

[Insert Figure 8 about here]

§6. Discussion and conclusion

In this study, the stock price behavior is modeled as geometric Brownian motion. This assumption is widely accepted for option pricing theories [12]. Intuitively, the assumption is equivalent to that the probability density for the transformed price $x$ ($x = \ln s$), $u(x, t)$, fulfills a convection-diffusion equation; i.e. $u_{xx} + \frac{2\eta}{\sigma^2} u_x = \frac{\sigma^2}{2} u_t$, because the density distribution for a Brownian motion is a fundamental solution for a convection-diffusion equation. In section 2, it is shown that the density function locally fulfills the convection-diffusion equation. The boundary condition of the probability density function for a stop-loss strategy is established as well. Thus, the equation for the density function is mathematically well-posed. After the p.d.f. is solved, the expected value of the investment can be evaluated. Moreover, the
expected value may be expressed as a solution of an o.d.e. When the stop-loss criterion is a linear function of time, the exact form for the p.d.f. is available. Results for two examples with exact p.d.f. are demonstrated. The boundary element methods (BEM), which are the most efficient systematical numerical method for linear partial differential equations, are introduced to solve the boundary value problem. This is the first time that a BEM is used to solve problems arisen from finance applications. The BEM may accurately and efficiently produce an approximation. Because of the existence of boundary integral presentation (29), we may have whole field function of the probability density $u(x,t)$, when the boundary value $u_x(l(t), t)$ is solved by the BEM. Therefore, for any stop-loss strategy $l(t)$, the expected return of the investment can be evaluated.

For derivative securities, the expected values depend on the probability density function of the underlying instruments. Thus, this technique can also be applied to calculate the expected returns of a preset stopping strategy for investing in derivative securities. Furthermore, American option pricing problem is equivalent to free boundary problem of a convection-diffusion equation. BEM would be a simple, accurate and efficient technique for solving it.
<table>
<thead>
<tr>
<th>( N_e )</th>
<th>( L_2 ) error</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.00617</td>
</tr>
<tr>
<td>200</td>
<td>0.001911</td>
</tr>
<tr>
<td>400</td>
<td>0.0006284</td>
</tr>
<tr>
<td>800</td>
<td>0.000212</td>
</tr>
<tr>
<td>1600</td>
<td>0.0000725</td>
</tr>
<tr>
<td>3200</td>
<td>0.00002518</td>
</tr>
</tbody>
</table>

Table 1
References


Figure 1. The probability density function with respect to the transformed price $x$, when $t = 2$ years. The dashed line shows the density without any stop-loss strategy.

Figure 2. The probability density function with respect to the stock price $s$, when $t = 2$ years. The dashed line shows the density without any stop-loss strategy.

Figure 3. The stop-loss probability $p(t)$ in example 1.

Figure 4. The expected value $E(t)/s_0$ with respect to time. The dashed line indicates the expected value without any stop-loss strategy.

Figure 5. The expected annalized return $e(t)/s_0$ with respect to time. The dashed line indicates the expected return without any stop-loss strategy.

Figure 6. The expected instataneous annalized return $e(t)$ with respect to time. The borrowing ratio $B$, the annalized stock return $R$, interest rate $r_b$, and stock volitivity $\sigma$ are 0.8, 0.12, 0.07 and 0.3 respectively. The dashed line shows the annalized stock return.

Figure 7. The expected value for the investment in example 2. The dashed line shows the expected value without any leverage.

Figure 8. The numerical result of the boundary derivative $f(t)$. The dotted line shows the exact values. The dashed line shows the magnified difference between the numerical result and the exact solution.