Pricing and replenishment policies in dual-channel supply chain under continuous unit cost decrease

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A R T I C L E   I N F O

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Unit cost decrease
Replenishment policy
Coordination

A B S T R A C T

This paper explores pricing and replenishment policies for a high-tech product in a dual-channel supply chain that consists of a brick-and-mortar channel and an internet channel. The unit cost of the product decreases over its short life cycle. Assuming the manufacturer as the Stackelberg leader, the optimal pricing and replenishment policy is analysed mathematically. It is found that there is a severe price competition between the retail and online channel, and product compatibility has a significant impact on the pricing policy. In particular (i) customers' higher retail channel preference above a threshold leads to non-coexistence of dual-channel, (ii) the dual-channel is non-profitable for product compatibility outside an interval and (iii) higher or lower retail price in comparison to online price is dependent on product compatibility. Also, the retailer's higher setup cost may lead to non-existence of online channel. Finally, a profit sharing mechanism through wholesale price adjustment resolves channel conflict. A numerical example is illustrated to justify our proposed model.

1. Introduction

The rapid growth of internet based electronic commerce has attracted the manufacturers of several companies such as, IBM, HP, Sony, Kodak, Panasonic, Cisco, etc to introduce direct online channels to their existing brick-and-mortar retail networks. Reduced cost for searching, increasing contact with the customers and detail specification and information of the products through the internet enable the manufacturer to enhance it’s market coverage. The growth of US online marketing is forecasted at 8% in 2010 and is set to reach 14% by 2012. Two third (2/3) of the marketers believe that online business must be complemented by traditional marketing activities [1]. As a result, manufacturers redesign their traditional channel structures by engaging in direct sales to reach different customer’s segments that cannot be reached by the traditional retail channel. This channel structure births to the dual channel. In fact, manufacturers who sell only through retailers are now considering the option of selling directly to end customers. Since, in dual-channel of same/substitutable product is sold through retail store as well as online channel. Consequently, the customers have alternatives to choose the channel that is better suited for their needs [2].

Decreasing property of price component and diminishing of demand over time due to introduction of upgraded versions of components are now important characteristics of high-tech industrial market. In high-tech industries such as communication
and computers, some component cost is decreasing around one percent per week [3]. Thus, production and sale in one week earlier or later leads to about 1% loss or gain. As a matter of fact, in decreasing unit cost environment, the decision maker always remains in searching the appropriate selling price because it has a considerable impact on demand as well as on optimal ordering policy. Thus, optimal pricing strategy [4–12] is a major issue to attract the customers in any business organization in a given economy.

Various aspects of dual-channel supply chain, such as advantages and disadvantages of online channel in addition to brick-and-mortar channel, when to open an online channel, pricing policies, replenishment policies, price competition, retail services, sales effort in retail channel, return policies, etc have been explored extensively in supply chain literature. Interestingly, there is no research till date that has discussed pricing and replenishment policies for the hi-tech products, whose unit costs decrease continuously in their short life span. Hi-tech products have high online compatibility and tech savvy customers that generally considers the specifications of the products through online channels and compare the retail prices with the products in online manufacturers’ suggested retail prices. In such situation there is a need for the manufacturer to identify online price and replenishment/production policy of a product that reduces total channel cost effectively that increases channel profit.

In inventory literature, there are a few models concerning continuous cost changing. Both Buzacott [13] and Erel [14] have proposed two inventory models where the unit cost of the product increases under inflationary situation. Erel [14] has developed the model under the assumption that the unit cost of the product increases in compound nature. Whereas, Buzacott [13] has assumed compound increments of both unit cost and setup cost. Goyal et al. [15] have developed inventory models under decreasing feature of unit cost. Khouja and Goyal [16] have suggested that the model of Buzacott [13] can be used for continuous unit cost change, if the rate of change of unit cost is same as the ordering cost. Erel's [14] model is also applicable in this purpose. But, if the rate of inflation is less than 10%, then the models provide wrong approximation. Teng and Yang [17] and Teng et al. [18] have developed inventory models under partial backlogging where demand and cost fluctuate over time. In both the models, optimal replenishment policy and optimal purchasing policy have been determined to minimize system running cost. They have claimed that this policy fits for todays high-tech market. Khouja and Goyal's [16] model may be considered as a special case of Teng and Yang [17] and Teng et al. [18] with constant demand and unit cost dependent holding cost. In unit cost decrementation inventory literature, interested readers may consult the paper of Khouja and Park [19] which provides interesting review of the literature associating it with the existing industrial scenario. Khouja and Park [19] have developed an inventory model to determine optimal operating policy in which the unit cost of the product decreases continuously by a constant percentage. Under the restriction of equal cycle lengths for finite time horizon, they have derived an approximate close form value for the optimal cycle length to minimize system operating cost. Panda [20] has determined the optimal pricing and replenishment policy in a decreasing demand with time and price sensitive market where the unit cost of the product decreases linearly with time. Cardenas-Barron et al. [21] have suggested a heuristic algorithm to solve the vendor management inventory system with multi-product and multi-constraint based EOE model with backorders, considering two classical backorders costs: linear and fixed. Sarkar and Majumder [22] have investigated an integrated vendor-buyer supply chain model to reduce total cost of the channel by considering the setup cost reduction of the vendor.

As indicated above, in addition to traditional brick-and-mortar channel, a new channel provided to the customers directly through internet is prevailing in practice because of it’s intuitive advantages. As a result, dual-channel supply chain has got enormous attention and become in main stream. Extensive researches have been done addressing variety of problems in dual-channel supply chain. For example, Levary and Mathieu [23] have examined the profits of retail store, online store and dual-channel, and have concluded that the dual-channel provides maximum profit. Ahn et al. [24] have discussed about the pricing decisions of a dual-channel supply chain, where the retail channel and the online channel operate in spatially separated markets. Huang et al. [25] have determined the optimal pricing and replenishment policy in a decreasing demand with time and price sensitive market where the unit cost of the product decreases linearly with time. Cardenas-Barron et al. [21] have suggested a heuristic algorithm to solve the vendor management inventory system with multi-product and multi-constraint based EOE model with backorders, considering two classical backorders costs: linear and fixed. Sarkar and Majumder [22] have investigated an integrated vendor-buyer supply chain model to reduce total cost of the channel by considering the setup cost reduction of the vendor.

Coordination among the channel members has potentiality to realize the benefits of the members of the chain. To coordinate the members of a supply chain, contracts are designed effectively among the decentralized decision makers such that the difference between outcome of a centralized decision and decentralized decisions can be neutralized. The basic objective behind designing a coordination contract is to incentivize decentralized channel members to act coherently with one another. Variety of side-payment contracts like as quantity discount [30,33], quantity flexibility [31,34], two-part tariff [35], revenue sharing [36,37], sales rebate [38], buy back [39], credit option [40], mail-in-rebate [41], disposal cost sharing [42,43] etc., have been used in supply chains as the ways of cutting out of channel conflict. These contracts differ by contractual clauses among channel members and are primarily concerned with quantity, time, quality and price. For detail discussion on channel coordination, the survey papers of Cachon [44] and Sarmah et al. [45] are referred to the readers.

Although supply chain literature has rich content on two-channel supply chain coordination, there are few papers which are dealt with resolving channel conflict in a dual-channel supply chain. Cai [46] has showed that hybrid revenue
sharing and linear price relationship between the retail channel price and direct channel price coordinate a dual channel supply chain. Boyaci [47] has proposed that revenue sharing, wholesale price, buy back contracts can't resolve channel conflict though a penalty contract coordinates the channel. Cai et al. [46] have proposed that it is possible to achieve win–win profits in a dual channel supply chain applying price discounting, though they have not discussed channel coordination issues. Yao and Liu [48] have compared the profit gains under Bertrand and Stackelberg equilibrium pricing strategies but they have not discussed about double marginalization of the channel. In a manufacturer-Stackelberg dual-channel supply chain, Chen et al. [27] have investigated that a contract with a wholesale price and a direct channel price offered by the manufacturer can resolve the channel conflict. They have also suggested that complementary agreements such as two-part-tariff, negotiated profit sharing in some specific ranges coordinate the channel and the channel members' profits are win–win.

The purpose of the paper is to determine optimal pricing and replenishment policy in a supply chain while the manufacturer operates an online channel besides retail channel. The manufacturer sales the product through retail and online channels simultaneously. As the product has a short life time and it’s cost decreases continuously with respect to time, demand of the products in both the channels is sensitive with price that results in different selling prices per unit. In conventional inventory models with pricing strategy, the number of price changes is pre-determined, i.e., the times of price changes are known earlier over a finite time horizon. Relaxing this assumption, in this paper, we assume that several replenishments of equal cycle lengths may be done over a finite time horizon and the decision maker has the opportunity to adjust the unit selling prices in each of the replenishment cycle to maximize the profit. Generally, in decentralized decision making, the retailer decides the number of replenishment cycle that maximizes it’s profit, whereas, in centralized channel, the manufacturer proposes the replenishment number. Moreover, we apply negotiation of profit sharing mechanism to determine the optimal wholesale price of the manufacturer that resolves channel conflict. The research reported in this paper differs from the prior works is the following aspects. First, we consider a product whose unit cost decreases continuously with time and becomes obsolete after it’s finite life time. These features have not considered in earlier studies. Second, we consider the system over the finite lifetime of the product instead of a single replenishment/production cycle. As a result, in addition to selling prices in the channels, the optimal number of replenishment cycles in decentralized and centralized channel are different. We use profit sharing mechanism to align the replenishment cycles of both the channel structure. Third, the paper addresses the effect of product compatibility on the optimal selling prices and optimal order quantity of the channel.

2. Notation

Following notations are used to develop the proposed model.

- \( L \) the time horizon under consideration
- \( T \) the cycle time during the planning horizon
- \( n \) the total number of replenishments over \([0, L]\) (a decision variable)
- \( D^r \) the demand rate of the product in retail channel of ith replenishment
- \( D^i \) the demand rate of the product in direct channel of ith replenishment
- \( h_r \) the inventory holding cost per unit per unit time of the retailer
- \( h_m \) the inventory holding cost per unit per unit time of the manufacturer
- \( s_r \) the ordering or/and set-up cost of the retailer
- \( s_m \) the ordering or/and set-up cost of the manufacture

For ith replenishment \((i = 1, 2, \ldots, n)\):

- \( c(t) \) unit production cost of the manufacturer
- \( w_i \) wholesale price of the manufacturer to the retailer
- \( p^r_i \) unit selling price of the retailer
- \( p^d_i \) unit selling price of the manufacturer in direct channel.

3. Model formulation

Assume that a manufacturer sales a hi-tech product through an online channel in parallel with the retail channel. The channel is operated over the finite time horizon \( L \) in which the retailer replenishes \( n \) times after every time interval \( T \) such that \( nT = L \). As in Khouja [16], Khouja and Goyal [13], Panda [17], the unit cost of the product decreases continuously with respect to time at a rate \( c(t) = \alpha - \beta t, \ t \in (0, \alpha/\beta) \). The \( \alpha \) is the introductory unit cost of the product and \( \beta \) is the unit cost’s time sensitive parameter. It is quite reasonable to assume that \( \alpha/\beta > 1 \), i.e., the unit cost of the product is positive over the planning horizon \( L \). In the ith \((i = 1, 2, \ldots, n)\) replenishment cycle the demands in the retail and direct channels are linear in unit selling prices (Yan [26], Yue and Liu [49]) and are of the forms

\[
D^r = \theta a - b_r p^r_i + r_1 (p^d_i - p^r_i), \quad i = 2, \ldots, n
\]
and 
\[ D^d = (1 - \theta)a - b_2p_i^d + r_2(p_i^r - p_i^d), \quad i = 1, 2, \ldots, n, \]

where, \( a > 0 \) is the market potential. The parameter \( \theta, \) \( 0 \leq \theta \leq 1 \) is the compatibility of the product with the retail channel. Product compatibility is how the consumer perceives the product into the customer's lifestyle choices. When the product closely matches the individual's needs, wants, beliefs, values, and consumptions patterns of the customers, it can be considered highly compatible with the consumers' choice. The percentage of the primary demand \( a \) that goes to the retail channel is \( \theta \) and when the value of \( \theta \) is greater, the product's compatibility with the retail channel is larger and more consumers would purchase the product from the retail channel. Computer-related products, books, information, magazines, and digital products have more compatibility with the direct channel than the products like water, rice, gasoline, and milk. Here, \( b_1 > 0 \) and \( b_2 > 0 \) are the price sensitivity factors in retail channel and online channel respectively. The \( r_1 > 0 \) and \( r_2 > 0 \) are the cross-price effects which reflect the degree of price competition between the channels. According to Yan [26], we assume that the price sensitivity factors and cross-price effect in the retail channel and in the online channel are equal, i.e., \( b_1 = b_2 = b \) and \( r_1 = r_2 = r \). It is quite reasonable that \( b > r \), i.e., the effect of price sensitivity of a channel is greater than cross-price effect. Thus, in the \( i \)th \((i = 1, 2, \ldots, n)\) replenishment cycle, the demands of the product in the retail channel and in the online channel are respectively as follows.

\[ D^r = \theta a - (b + r)p_i^r + rp_i^d, \quad i = 1, 2, \ldots, n \]

and

\[ D^d = (1 - \theta)a - (b + r)p_i^d + rp_i^d, \quad i = 1, 2, \ldots, n. \] (1)

Quite often in the \( i \)th \((i = 1, 2, \ldots, n)\) replenishment cycle, the selling price in the online channel is higher than the manufacturer's wholesale price, i.e., \( p_i^r > w_i \), \( i = 1, 2, \ldots, n \). Otherwise, the retailer purchases the product through the online channel rather than from the manufacturer. Also, for profitability of the retailer, in the \( i \)th \((i = 1, 2, \ldots, n)\) replenishment cycle, the selling price of the retailer is higher than the manufacturer's wholesale price, i.e., \( p_i^r > w_i \), \( i = 1, 2, \ldots, n \). Under this model setting, our objective is to find out optimal decisions in decentralized as well as centralized systems.

### 3.1. Decentralized dual-channel-supply chain

In decentralized decision making, the manufacturer and the retailer are interested to achieve maximum individual profit. Interactions between the manufacturer and the retailer are considered as a Stackelberg game. The manufacturer acts as the Stackelberg leader of the channel and the retailer is its follower. In Stackelberg game, leader makes first move and follower then reacts by consistent playing the best move with available information. The objective of the leader is to design own move in such a way that his own profit is maximum, after considering all rational moves follower can devise [50]. In this way, the manufacturer first announces the wholesale price and selling price of the product in the online channel. Based on the manufacturer’s decision, the retailer determines the retail price and orders \( n' \) replenishments to the manufacturer each of length \( T' \) over \( L \) so that \( n'T' = L \). In the \( i \)th \((i = 1, 2, \ldots, n')\) replenishment cycle, profit function of the retailer is

\[ \pi_i^{r/ds}(p_i^{r/ds}) = D_i^r T' p_i^{r/ds} - D_i^r T' w_i - s_i - D_i^d T' \frac{h_m T'}{2}, \quad (i = 1, 2, \ldots, n'). \] (3)

In Eq. (3), 1st, 2nd, 3rd and 4th terms represent sales revenue, purchase cost, set up cost and holding cost of the product respectively. The profit function of the retailer over the planning horizon is

\[ \pi^{r/ds}(n', p_i^{r/ds}) = \sum_{i=1}^{n'} \pi_i^{r/ds}(p_i^{r/ds}) = \frac{L}{n'} \sum_{i=1}^{n'} \left[ D_i^r \left( p_i^{r/ds} - w_i - \frac{h_m L}{2n'} \right) \right] - n's_i. \] (4)

In the decentralized decision making, the profit function of the manufacturer is the sum of two profit functions. One is for the quantities which is sold through the retail channel and the other is for the online channel. The profit function of the manufacturer, in the \( i \)th replenishment cycle, is

\[ \pi_i^{m/ds}(w_i, p_i^{d/ds}) = D_i^r T' w_i - D_i^r T' c[(i - 1)T'] + D_i^d T' p_i^{d/ds} - D_i^d T' \frac{h_m T'}{2} - D_i^d T' c[(i - 1)T'] - s_m, \quad (i = 1, 2, \ldots, n). \] (5)

In Eq. (5), first and second terms represent the manufacturer’s net profit from the retail channel and the remaining terms represent the manufacturer’s profit from the online channel. Total profit of the manufacturer over the planning horizon is

\[ \pi^{m/ds}(w_i, p_i^{d/ds}) = \sum_{i=1}^{n'} \pi_i^{m/ds}(w_i, p_i^{d/ds}) = \frac{L}{n'} \sum_{i=1}^{n'} \left[ D_i^r \left( w_i - c \left( i - 1 \frac{L}{n'} \right) \right) + D_i^d \left( p_i^{d/ds} - c \left( i - 1 \frac{L}{n'} \right) - \frac{h_m L}{2n'} \right) \right] - n's_m. \] (6)

Total profit of the retailer in the planning horizon is a function of \( p_i^{d/ds}, \) \((i = 1, 2, \ldots, n')\) and \( n' \), where \( n' \) is a discrete variable. On the other hand, the profit function of the manufacturer is a function of \( w_i \) and \( p_i^{d/ds}, \) \((i = 1, 2, \ldots, n)\). Since the manufacturer is the Stackelberg leader of the channel, it first determines the wholesale price and selling price of the product.
through the online channel where it assumes that the retailer will consider multiple replenishment cycles. As a follower, the retailer then sets the retail price and number of replenishment cycles in the time horizon. Now, for the concavities of the profit functions we have the following proposition.

Proposition 1.

(i) For given \( n' \), the manufacturer’s profit function over \( L \) is a concave function of \((w_i, p_i^{d/ds})\), \((i = 1, 2, \ldots, n')\) and optimal wholesale price and retail price in the online channel are

\[
W_i^{d} = \frac{a[r + \theta b]}{2[(b + r)^2 - r^2]} - \frac{Lh_i}{4n'} + \frac{1}{2} c [c - \frac{1}{n'}(i - 1)]L, \quad (i = 1, 2, \ldots, n'),
\]

\[
P_i^{d/lds} = \frac{a[r + (1 - \theta)b]}{2[(b + r)^2 - r^2]} + \frac{Lh_m}{4n'} + \frac{1}{2} c [c - \frac{1}{n'}(i - 1)]L, \quad (i = 1, 2, \ldots, n').
\]

(ii) For given \( n' \) and the manufacturer’s optimal \((w_i, p_i^{d/lds})\), \((i = 1, 2, \ldots, n')\) pair, the retailer’s profit function is concave over \( L \) and it’s optimal retail price is

\[
P_i^{r/lds}(W_i, p_i^{d/lds}) = \frac{w_i}{2} + \frac{r}{2(b + r)}p_i^{d/lds} + \frac{a\theta}{2(b + r)} + \frac{Lh_i}{4n'}, \quad (i = 1, 2, \ldots, n')
\]

or

\[
p_i^{d/lds} = \frac{a}{2(b + 2r)} \left[ \frac{r}{b} + \left( \frac{3b + 4r\theta}{2(b + r)} \right) \right] + \frac{L}{8n'} \left[ h_i + \frac{r}{b + r}h_m \right] + \left( \frac{b + 2r}{4(b + r)} \right) c \left( \frac{1}{n'} - \frac{1}{n'} \right), \quad (i = 1, 2, \ldots, n')
\]

Proof. See Appendix A.

Using the optimal selling prices of the retail channel and online channel, optimal order quantities, optimal profits in the ith replenishment cycle over the entire time horizon can be found which are presented in Table 1.

From (9), we have \( \partial p_i^{r/lds}(W_i, p_i^{d/lds})/\partial p_i^{d/lds} = r/[2(b + r)] > 0 \), i.e., in the ith replenishment cycle, the optimal selling price of the retailer decreases when the selling price of the online channel decreases. Further, \( \partial p_i^{r/lds}(W_i, p_i^{d/lds})/\partial w_i = 1/2 > 0 \), i.e., the optimal retail price is also decreasing with decreasing optimal wholesale optimal of the manufacturer. As the selling price of the retailer decreases with decreasing values of \( p_i^{d/lds} \) and \( W_i \), in the ith replenishment cycle, the manufacturer has the option to control the retailer’s selling price by introducing an online channel, where he/she sets the wholesale price \( w_i \) and unit selling price \( p_i^{d/lds} \). This result is quite similar to Chiang et al. [51]. It is also observed that

\[
\frac{\partial \pi_i^{r/lds}(p_i^{d/lds})}{\partial W_i} = \frac{L}{n'} \sum_{i=1}^{n'} \left( D_i^{r/lds} - \frac{r}{2(b + r)} \right) - \frac{L}{2} \left( \frac{b + r}{2n'} \right) > 0.
\]

That means, the optimal total profit of the retailer over \( L \) increases with increasing optimal online selling price. The intuitive reason is straightforward. When the online selling price increases, it forces some demands of the online channel to switch to the traditional retail channel. As a result, the retailer’s profit increases. On the other hand,

\[
\frac{\partial \pi_i^{r/lds}(p_i^{d/lds})}{\partial w_i} = - \frac{L}{n'} \sum_{i=1}^{n'} \left( D_i^{r/lds} - \frac{r}{2(b + r)} \right) - \frac{L}{2} \left( \frac{b + r}{2n'} \right) < 0.
\]

Here, the optimal total profit of the retailer decreases with increasing wholesale price of the manufacturer that is quite obvious. However, the optimal selling prices and wholesale prices satisfy the following proposition.

Proposition 2. In the planning horizon \( L \), for given \( n' \), (i) \( p_i^{r/lds} > p_2^{r/lds} > \ldots > p_{n'}^{r/lds} \), (ii) \( p_1^{d/lds} > p_2^{d/lds} > \ldots > p_{n'}^{d/lds} \) and (iii) \( w_1 > w_2 > \ldots > w_n' \) hold.

Proof. See Appendix A.

Proposition 2 indicates that the optimal selling prices in the channels and optimal wholesale price of the manufacturer decrease with the increasing replenishment number. That means, the optimal prices in the ith cycle are higher than those of (\( i + 1 \))th, \((i = 1, 2, \ldots, n' - 1)\) cycle. The reasonable explanation is as follow. As the product has limited lifetime, it’s unit cost
### Table 1
Optimal solutions in decentralized and centralized decision making (where $A = [r_{hn} - (b + r)h]$ and $B = \frac{2b + 3r + 7h_{hn} + 2b + r}{b_{hr}}$).

#### Decentralized scenario

<table>
<thead>
<tr>
<th>Replenishment $i = 1, 2, \ldots, n^t$</th>
<th>Manufacturer</th>
<th>Retailer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wholesale price</td>
<td>$\frac{\alpha_i(r - b)}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Selling price</td>
<td>$\frac{\alpha_i(r - b)}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Quantity sold</td>
<td>$\frac{a_{i}L}{2h_{r}}\left[1 - \frac{0(2b + r)}{2(r + b)} + \frac{L^2}{8h_{r}}\right] - \frac{2b + 3r}{8b + r} + \frac{c[(i-1)\frac{1}{2}]}{(r + b)} - \frac{bl(2b + 3r)}{4n(r + b)} + \frac{c[(i-1)\frac{1}{2}]}{8r}$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Manufacturing quantity</td>
<td>$\frac{a_{i}L}{2h_{r}}\left[1 - \frac{0(2b + r)}{2(r + b)} + \frac{L^2}{8h_{r}}\right] - \frac{2b + 3r}{8b + r} + \frac{c[(i-1)\frac{1}{2}]}{(r + b)} - \frac{bl(2b + 3r)}{4n(r + b)} + \frac{c[(i-1)\frac{1}{2}]}{8r}$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Profit</td>
<td>$\frac{a_{i}L}{8b_{r}(b + 2r)}\left[2(b + r) - 2(0) - \frac{b}{b + r} + \frac{b^2a_{i}^2}{b + r}\right] + \frac{b(3b + 4r)}{8b + r} + \frac{c[(i-1)\frac{1}{2}]}{(r + b)} - \frac{bl(2b + 3r)}{4n(r + b)} + \frac{c[(i-1)\frac{1}{2}]}{8r}$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
</tbody>
</table>

#### Centralized scenario

<table>
<thead>
<tr>
<th>Replenishment $i = 1, 2, \ldots, n^t$</th>
<th>Direct channel</th>
<th>Retail channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selling price</td>
<td>$\frac{a_{i}(1 - \bar{b})}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Demand</td>
<td>$\frac{\alpha_i(r - b)}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Channel profit</td>
<td>$\frac{L}{2r_{hr}}\left[r + b - 2b(1 - \frac{1}{r}) + \frac{a_{i}(1 - \bar{b})}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]\right] - \frac{bl(2b + 3r)}{4n(r + b)} + \frac{c[(i-1)\frac{1}{2}]}{8r}$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Order quantity</td>
<td>$\frac{L}{2r_{hr}}\left[r + b - 2b(1 - \frac{1}{r}) + \frac{a_{i}(1 - \bar{b})}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]\right] - \frac{bl(2b + 3r)}{4n(r + b)} + \frac{c[(i-1)\frac{1}{2}]}{8r}$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
<tr>
<td>Channel profit</td>
<td>$\frac{L}{2r_{hr}}\left[r + b - 2b(1 - \frac{1}{r}) + \frac{a_{i}(1 - \bar{b})}{2h_{hn}(r + \frac{1}{2})} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]\right] - \frac{bl(2b + 3r)}{4n(r + b)} + \frac{c[(i-1)\frac{1}{2}]}{8r}$</td>
<td>$\frac{h}{7b_{r} + 2r} + \frac{1}{8r} + \frac{1}{16}c[(i-1)\frac{1}{2}]$</td>
</tr>
</tbody>
</table>
decreases continuously with time. Now, the wholesale price of the product decreases because it depends on the unit cost of the product. As a result the selling prices in the channels also decrease because the channel members are willing to sell more product by lowering the selling prices before the products obsolesce.

Now, the optimal pricing strategy provided in Table 1 is acceptable to the channel members only when \( p^{d_{i}} > w_{i} \) and \( p^{d_{i}} > w_{i} \) for \( i = 1, 2, \ldots, n' \). Therefore, \( p^{d_{i}} > w_{i} \) holds if

\[
\theta > \frac{b}{a} \left[ (i - 1) \frac{L}{n'} - \frac{L(rh_{m} + 3(b + r)h_{r})}{2an'} \right]
\]

(11)

Also, \( p^{d_{i}} > w_{i} \) holds if

\[
\theta < 0.5 + \frac{L(b + 2r)(h_{r} + h_{m})}{4an'} = \theta^{\text{max}}, \quad \text{(say)}
\]

(12)

The right hand side of (11) depends on \( n' \) and \( i \), whereas right hand side of (12) is dependent on \( n' \) but independent of \( i \). Since, the system operates over the planning horizon \( L \), where \( n' \) replenishment cycles are made, the maximum value of the right hand side of (11) is at \( i = 1 \) is

\[
\frac{bx}{a} - \frac{L(rh_{m} + 3(b + r)h_{r})}{2an'} = \theta^{\text{min}}, \quad \text{(say)}
\]

and we have the following Lemma. □

**Lemma 1.** For given \( n' \), the manufacturer will participate in the dual-channel for a product whose unit cost decreases continuously over \( L \) if \( \theta \in (\theta^{\text{min}}, \theta^{\text{max}}) \).

**Lemma 1** indicates that the customer’s channel preference is one of the determining factors for operating an online channel besides the traditional retail channel. When \( \theta < \theta^{\text{min}} \), the retailer can’t do business because it’s selling price is less than the manufacturer’s wholesale price. On the other hand, for \( \theta > \theta^{\text{max}} \), the manufacturer can’t set the optimal selling price as the online price is there. Note that, we consider the maximum of lower threshold of product’s compatibility with the retail channel. As there are multiple replenishment cycles over \( L \), the retailer’s optimal selling prices are profitable in these cycles only when the lower limit of \( \theta \) is maximum among all of these values. As online selling price is higher than the wholesale price, it does not ensure that the retailer will participate in the profit making retail-e-tail channel for a product. This experiences continuous unit cost decrease over \( L \) because of its setup cost. In the \( i \)th \( (i = 1, 2, \ldots, n') \) replenishment cycle, the retailer will participate in the dual-channel only when it’s profit is positive, i.e., \( \pi^{d_{i}} > 0 \), i.e., if

\[
\theta > \frac{1}{a} \left[ 4 \sqrt{\frac{n'(b + r)s_{r}}{L}} + \frac{L[(b + r)h_{r} - rh_{m}]}{2n'} + bc \left( i - 1 \right) \frac{L}{n'} \right] = \theta^{i}_{\pi}, \quad (i = 1, 2, \ldots, n') \quad \text{holds}.
\]

(13)

But, the value of \( \theta^{i}_{\pi} \) depends on \( n' \) and \( i \) and attains its maximum value when \( i = 1 \). That is

\[
\theta^{i}_{\pi} = \frac{1}{a} \left[ 4 \sqrt{\frac{n'(b + r)s_{r}}{L}} + \frac{L[(b + r)h_{r} - rh_{m}]}{2n'} + bx \right] .
\]

On the other hand, the manufacturer will operate the online channel until the demand in the online channel is positive, i.e., \( D^{d_{i}} > 0 \), i.e., if

\[
\theta < \frac{2(b + r)}{2b + r} \left[ \frac{(2(b + r)^{2} - r^{2})h_{m} - r(b + r)h_{r}}{2an'(2b + r)} \right] L - \frac{b(2b + 3r)}{(2b + r)a} \left( i - 1 \right) \frac{L}{n'} = \theta^{m}_{\theta}, \quad (i = 1, 2, \ldots, n'),
\]

where \( \theta^{m}_{\theta} \) also depends on \( n' \) and \( i \) and attains its minimum value when \( i = 1 \), i.e.,

\[
\theta^{m}_{\theta} = \frac{2(b + r)}{2b + r} - \frac{(2(b + r)^{2} - r^{2})h_{m} - r(b + r)h_{r}}{2an'(2b + r)} \frac{b(2b + 3r)}{(2b + r)a} .
\]

(14)

Unlike the retailer’s case, for positive profit of the manufacturer we consider here the positive online demand because the setup cost for operating the online channel is included in the system from where the manufacturer supplies the product. It is assumed that the setup cost of the manufacturer consists of setup cost for manufacturing the product and setup cost for operating the online channel because we concentrate only on overall profitability (profit from the retail channel and profit from the online channel) of the manufacturer.

When the customers’ retail channel preference lies in between \( \hat{\theta} \) and \( \hat{\theta} \), Eqs. (13) and (14) suggest that the manufacturer can successfully operate a profitable dual-channel. Now,

\[
\hat{\theta} - \theta^{\text{min}} = \frac{1}{a} \left[ 4 \sqrt{\frac{n'(b + r)s_{r}}{L}} + \frac{2L(b + r)h_{r}}{n'} \right] > 0
\]
and
\[ \hat{\theta} - \hat{\theta}_{\text{max}} = \frac{(2b + 3r)(a - 2bx)}{2a(b + r)} + \frac{(2b^2 + 7br + 4r^2)h_r - (2b^2 + 3br + r^2)h_m}{4an'(2b + r)} > 0 \]
hold. This implies that \( \max(\hat{\theta}, \hat{\theta}_{\text{max}}) = \hat{\theta} \) and \( \min(\hat{\theta}, \hat{\theta}_{\text{max}}) = \hat{\theta}_{\text{max}} \), i.e., \((\hat{\theta}, \hat{\theta}_{\text{max}})\) are nested in \((\hat{\theta}_{\text{min}}, \hat{\theta})\). This result is quite obvious because the manufacturer will operate the online channel only when both the channels are profitable. The retail channel of the manufacturer will be profitable not only for the retail price is equitable to the wholesale price but also reasonably higher because it must over compensate the costs related entire retail channel running cost. It is possible only when the customers’ retail channel preference is higher than \( \hat{\theta} \) but lower than \( \hat{\theta}_{\text{max}} \). Similarly, the retailer will participate in the dual-channel and will make profit if \( \hat{\theta} > \hat{\theta} \) holds. Thus, the proposition is as follows:

**Proposition 3.** Over the planning horizon \( L \), for given \( n' \), the manufacturer can operate a profitable retail-online channel when the customers’ retail channel preference \( \theta \in (\hat{\theta}, \hat{\theta}_{\text{max}}) \).

However, as far as the competition between the retail channel and online channel is concerned, we have the following proposition.

**Proposition 4.** For given \( n' \), in the \( i \)-th \( (i = 1, 2, \ldots, n') \) replenishment cycle the optimal retail price is higher than online selling price if \( \theta \in (\theta_{\text{dc}}', \theta_{\text{max}}') \), while reverse may be noted for \( \theta \in (\hat{\theta}, \theta_{\text{dc}}') \).

**Proof.** See Appendix A.

Proposition 4 suggests that the customer’s channel preference intensifies the channel competition. To attract more customers, the retailer sets the retail price less than online selling price when the customers’ preference for the retail channel is below a threshold of the products channel compatibility. On the other hand, the retailer sets the retail price greater than the online selling price to earn higher profit margin when \( \theta \) is above the threshold \( \theta_{\text{dc}}' \). Thus, before making any pricing decision, the channel members must consider the customers’ channel preference as a decision factor. This result can be further justified as follows.

\[ \frac{d\pi_{t_{ds}}^r}{d\theta} = -\frac{ab}{2[(b + r)^2 - r^2]} < 0 \]

\[ \frac{d\pi_{t_{ds}}^d}{d\theta} = \frac{a(3b + 4r)}{4(b + r)(b + 2r)} > 0, \]

i.e., selling price in the retail channel increases and online selling price decreases when the customers’ preference for the retail channel increases. As more profit gains are the objectives of the channel members, the channel members decide about the selling prices based on the product compatibility. Also, note that

\[ \frac{d\omega_i}{d\theta} = \frac{ab}{2[(b + r)^2 - r^2]} > 0, \]

i.e., the wholesale price of the manufacturer increases with the increment of the customers’ retail channel preference. This is quite reasonable. Since the retailer sets higher selling price because of customers’ higher preference for the retail channel, the manufacturer increases also it’s wholesale price in order to acquire some margins from the retailer’s profit. However, the channel members can apply such strategy and counter strategy until the product compatibility with the retail channel lies in \((\hat{\theta}, \theta_{\text{max}}')\). It is very interesting to note that \( d\pi_{t_{ds}}^r/ds_r = (2/a)\sqrt{(b + r)n'}/Ls_r > 0 \), i.e., the lower threshold of profitable product compatibility with the retail channel is directly proportional to the retailer’s set up cost.

As such in the \( i \)-th replenishment cycle, \( \theta_{\text{dc}}' < \hat{\theta} \) holds if

\[ s_r > \frac{L}{n'(b + r)} \left( \frac{a\theta_{\text{dc}}'}{4} - \frac{L[(b + r)h_r - rh_m]}{8n'} - \frac{bx}{8} \right)^2. \]  

(15)

In such case, \( \theta_{\text{dc}}' < \hat{\theta} < \theta_{\text{max}}' \) holds, i.e., the lower limit of customers retail channel preference is higher than the upper limit of product compatibility in between which the manufacturer can set higher online price in comparison to the retail price. Obviously in this case the manufacturer can not run the online channel because it cannot set the profitable online selling price. Thus the manufacturer can operate the profitable online only when the retailer’s set up cost is reasonably low. Then, we have the following proposition.

**Proposition 5.** The manufacturer cannot operate the online channel if

\[ s_r > \frac{L}{n'(b + r)} \left( \frac{a\theta_{\text{dc}}'}{4} - \frac{L[(b + r)h_r - rh_m]}{8n'} - \frac{bx}{8} \right)^2 \]
holds.
Notice that \( \frac{d}{dn} \left( \sum_{i=1}^{n'} p_i^{dr} \right) / d\theta = La/4 > 0 \) and \( \frac{d}{dn} \left( \sum_{i=1}^{n'} p_i^{dr} \right) / d\theta = -La(2b + r)/[4(b + r)] < 0 \), i.e., over the planning horizon \( L \), the optimal order quantity of the retail channel increases, whereas the optimal order quantity of the online channel decreases with the increment of the customer’s retail channel preference. This result is quite obvious because more customers will buy the product from the retail channel. As a result, the retailer’s order quantity will increase. Interestingly, \( dQ^{dr}/d\theta = -Lab/[4(b + r)] < 0 \), i.e., the total order quantity over \( L \) decreases with increasing product compatibility with the retail channel. Total order quantity is the sum of product sold through the retail channel and online channel. The decrement of sold product through the online channel with increment of the customer’s retail channel preference is higher than the product selling through the retail channel. As a result, the volume of total order quantity decreases with the increment of the customers retail channel preference.

Note that
\[
\frac{d}{dn} \left( \frac{1}{n'} \sum_{i=1}^{n'} w_i \right) = \frac{L}{4n'^2} (h_r - \beta) < 0 \text{ if } h_r < \beta.
\]

That is, average wholesale price of the manufacturer over the planning horizon \( L \) decreases as the retailer’s number of replenishments increases. This is quite reasonable because the unit cost of the product decreases continuously over \( L \). As \( n' \) increases, the manufacturer supplies the product to the retailer for shorter replenishment cycles. Since the wholesale of the manufacturer is dependent on the product’s unit cost, multiple replenishments lead to decrement of average wholesale price. Now,
\[
\frac{d}{dn} \left( \frac{1}{n'} \sum_{i=1}^{n'} p_i^{dr} \right) = -\frac{L}{4n'^2} (h_m + \beta) < 0
\]
and
\[
\frac{d}{dn} \left( \frac{1}{n'} \sum_{i=1}^{n'} p_i^{rs} \right) = -\frac{L}{8n'^2} \left[ h_r + \frac{r}{b + r} h_m + \beta \frac{b + 2r}{b + r} \right] < 0.
\]

Therefore, average retail price and online price over \( L \) decrease as the number of replenishment increases. The selling prices in the channels are dependent on the wholesale price and unit cost of the product. Unit cost of the product decreases with increasing \( n' \). As a result, the selling prices also decrease. On the other hand,
\[
\frac{dQ^{dr}}{dn} = \frac{L^2 b}{8n'^2} \left[ h_r + \frac{(2b + 3r)}{b + r} h_m + \beta \frac{3b + 4r}{b + r} \right] > 0
\]
holds. This indicates that an optimal order quantity of the channel increases as the number of replenishment increases. As the selling prices in the channels decrease for increasing \( n' \), the customers buy more and hence \( Q^{dr} \) increases over \( L \). As far as the optimal number of replenishment over the time horizon \( L \) is concerned, we have the following proposition.

**Proposition 6.** Over the selling season \( L \) the retailer’s profit is maximum for \( n_0' \) number of replenishments, where \( n_0' \) is given by
\[
n_0' = \begin{cases} [n_0] & \text{if } \pi^{rs}(\lfloor n_0 \rfloor) < \pi^{dr}(\lfloor n_0 \rfloor) + 1 \\ [n_0] + 1 & \text{otherwise} \end{cases}
\]
and
\[
n_0 = (-d + \sqrt{d^2 + b^2})^{\frac{1}{2}} + (-d - \sqrt{d^2 + b^2})^{\frac{1}{2}},
\]
where \( b = L^2 (A - \beta \bar{p})[2a\bar{w} - b(2\bar{x} - \beta L)/96(b + r)s_r] \) and \( d = L^2 [3A(A - \beta \bar{p}) + 2b^2 \bar{p}^2]/192(b + r)s_r \).

**Proof.** See Appendix A.

**Proposition 6** suggests that, based on the manufacturer’s wholesale price and online selling price, the retailer chooses \( n_0' \) number of replenishment over \( L \) that maximizes its profit. As the manufacturer is dependent on the retail channel, he/she will follow the retailer’s replenishment policy and, based on it, the manufacturer will apply the online pricing schedule.

### 3.2. Centralized policy

The traditional centralized policy views the system as single entity where there is one central planner who makes all decisions so as to maximize the profit of the whole chain. The centralized policy determines suitable selling prices of the product for both retail and direct channel as well as production cycle so as to maximize the total system profit. The relevant costs
considered for the retailer and the manufacturer in this policy are similar to those in the decentralized replenishment policy. Profit function of the integrated channel in the ith replenishment cycle is

\[ \pi_i^c = D_i r c_i - D_i r c_i^* s - \frac{D_i r c_i c_s}{2} [i - 1] r c_i - D_i r c_i^* c_s [i - 1] r c_i] - s - s_m. \]  

(17)

The optimal solution of (17) is presented in the following proposition.

**Proposition 7.** (i) For given \( n^c \), the profit function of the integrated channel over \( L \) is a concave function of \( \theta \). (ii) \( \theta \) depends on \( L > \theta \)

Proposition 8. In the centralized channel over the planning horizon \( L \), for given \( n^c \), (i) \( p_i^c > p_i^d > \ldots > p_i^c \), (ii) \( p_i^c > p_i^d > \ldots > p_i^c \).

**Proof.** See Appendix A.

Using the optimal values of \( p_i^c \) and \( p_i^d \), we obtain the demand of the product in retail and direct channel in the ith replenishment cycle and multiplying these with cycle length, we get the amount of quantity sold through retail and online channel respectively which are displayed in Table 1. Also, the order quantity which are produced by the manufacturer is equal to the product of total demand and cycle length, the profit of the integrated channel in ith (\( i = 1, 2, \ldots, n^c \)) replenishment cycle as well as over the planning horizon are displayed in Table 1. Now, the optimal values of the centralized decision satisfy the following properties. \( \square \)

**Proposition 8.** In the centralized channel over the planning horizon \( L \), for given \( n^c \), (i) \( p_i^c > p_i^d > \ldots > p_i^c \), (ii) \( p_i^c > p_i^d > \ldots > p_i^c \).

**Proof.** See Appendix A.

This result is quite similar to the decentralized decision making process. As the unit cost of the product decreases continuously, higher number of replenishment always leads to decrement of selling prices in both the channels. Now, the manufacturer would be interested on opening the online channel only when online channel demand in the ith replenishment cycle is positive, i.e., \( D_i r c_i^* > 0 \), which, after simplification, yields

\[ \theta < 1 - \frac{(b + r) \ h_m - \ h_r \ L}{2 \ a n c} - \frac{b \ c}{a} [i - 1] \frac{L}{n^d} = \theta_{max}. \]

Here, \( \theta_{max} \) depends on \( n^c \) and \( i \) and it attends minimum value for \( i = 1 \), i.e.,

\[ \theta < 1 - \frac{(b + r) \ h_m - \ h_r \ L}{2 \ a n c} = \theta_{min}. \]

(20)

Although, in centralized channel, the manufacturer and the retailer co-operate and take decision jointly, the product compatibility has an impact on the manufacturer’s decision for opening the online channel. In the ith replenishment cycle, if the customers’ retail channel preference is higher than threshold \( \bar{\theta} \), then the manufacturer’s decision for opening an online channel is not profitable because its online demand is negative in such cases. On the other hand, there must be a competition between the retail channel and the online channel in the centralized process though the channel members co-operate. The channel members take decision jointly but the market potential remains same. When the manufacturer operates an online channel, some customers switches to the online channel. As a result the retailer’s demand decreases and it earns less profit. Besides the selling prices of the retail and online channel, the customers’ channel preference determines the divisions of potential market demand. So, like the decentralized decision making process, here also the selling prices of the retail channel may be higher than the online channel, i.e., \( p_i^c > p_i^d \). (i = 1, 2, …, \( n^c \)), i.e., if

\[ \theta > 0.5 - \frac{(b + 2 r) \ h_r - h_m \ L}{4 a n c} = \theta_{min}. \]

(21)

Also, note that for any \( i = 1, 2, \ldots, n^c \),

\[ \bar{\theta} - \theta_{min} = \frac{1}{2} \frac{b a}{a} + \frac{(b + 4 r) \ h_r - (3 b + 4 r) h_m}{4 a n c} > 0. \]

Thus from (25), (26), (27), we have the following proposition. \( \square \)
Proposition 9. For given \( n' \), the manufacturer will operate an online channel if \( \theta \in (0, \theta') \). The online selling price is higher than retail price for any \( \theta \in (0, \theta') \) and the retail price is higher than the online price for any \( \theta \in (\theta', \theta^\circ) \).

Proposition 9 demonstrates that, when the channel members cooperate and take decision jointly, the manufacturer’s decision is to open an online channel that is profitable only when the customers’ retail channel preference is below the threshold number \( \theta' \). Interestingly below this threshold number of the product compatibility, there exists a price competition between the retail channel and the online channel. If the customers’ retail channel preference is within\((0, \theta')\), then the online price will be higher than the retail price and the reverse is set for \( \theta \in (\theta', \theta^\circ) \). Thus, for a profitable centralized retail-online channel, the channel will set the selling prices according to the customers’ channel preference.

Observe that \( \frac{dQ_c}{d\theta} = \frac{L}{2} > 0 \) and \( \frac{dQ_{dc}}{d\theta} = -\frac{L}{2} < 0 \), i.e., in centralized channel over the time horizon \( L \), optimal order quantity in the retail channel increases, whereas it decreases in the online channel when the customers’ retail channel preference increases. This is quite natural. But, \( \frac{dQ_c}{d\theta} = \frac{dQ_{rc}}{d\theta} + \frac{dQ_{dc}}{d\theta} = 0 \), i.e., customers’ channel preference has no overall impact on the order quantity when the channel members cooperatively operate a profitable retail-online channel. The other characteristics of the selling prices and order quantity with respect to the number of replenishment in the centralized channel are same as the decentralized decision making process. According to the optimal number of replenishment over the planning horizon \( L \), we have the following proposition.

Proposition 10. Over the selling season \( L \), the number of replenishments for which system’s profit is maximum is given by

\[
\eta_0^* = \begin{cases} 
[n_0'] & \text{if } \pi^{r/d_i}([n_0']) > \pi^{r/d_i}([n_0'] + 1) \\
[n_0'] + 1 & \text{otherwise}
\end{cases}
\]

and

\[
\eta_0 = -d_c + \left( d_c^2 + b_c^2 \right)^{\frac{1}{2}} + \left( -d_c - \sqrt{d_c^2 + b_c^2} \right)^{\frac{1}{2}}
\]

where \( d_c = \left[ \frac{2at^2(\beta + \theta h_t) + bL^2(2\alpha - \beta L)(2\beta + h_t + h_m)}{24(s_r + s_m)} \right], \)

\( d_c = L^3 \left[ \frac{h_t^2 + h_m^2 - 2rh_t h_m + 4b\beta^2 + 3b(\beta h_t + h_m)}{48(s_r + s_m)} \right] \) and \([n_0'] \) denotes the largest integer which is not greater than \( n_0^* \).

Proof. Same as Proposition 6.

Now, \( \pi^i > \pi^{r/d} + \pi^{m/d} \) holds, i.e., the channel is not coordinated. This is quite obvious as indicated in supply chain literature that cooperative integrated decision is always more profitable than decentralized system. In the next section, we demonstrate a profit sharing mechanism assuming that the manufacturer and the retailer jointly take the centralized decision, which is the channel best decision and share the total channel profit in a portion that ensures win–win profit.

3.3. Profit sharing for channel coordination

As the manufacturer and the retailer are separate and independent economic entities, a key issue is to develop mechanisms that can align their objectives and coordinate their activities so as to optimize system performance. To obtain centralized channel profit, there is a need to devise coordination mechanisms that are not only able to coordinate the activities but also able to align the objectives of independent supply chain members. The difficulties for coordination in this model are due to different optimal cycle length and hence different pricing policy for centralized and decentralized scenarios. For accepting the centralized cycle length, which is less than the decentralized cycle length, the retailer’s cost will increase and there is no reason that the retailer will adopt centralized policy unless proper incentive. As an incentive, manufacturer can offer the retailer to share the surplus profit if the retailer adopt centralized decisions. Under profit sharing mechanisms, the system performance is first optimized and the resultant benefit is then shared between the manufacturer and the retailer. This solution can be considered as a cooperative solution. Its implementation, however, depends on the development of a profit sharing scheme that is acceptable to both parties.

The manufacturer provides incentive to the retailer for accepting centralized selling price, \( p_i^c \) and cycle length, \( L/\eta_0^* \) by offering him/her to the surplus profit proportionally according to their decentralized profit. To obtain centralized channel profit manufacturer has also sell the product at a price \( p_i^c \) through the online channel. Surplus profit for accepting centralized policy over the planning horizon \( L \) is \( \pi_{sp} = \pi^c - (\pi^{r/d} + \pi^{m/d}) \). The manufacturer and the retailer will get additional profits \( [\pi^{m/d}/(\pi^{r/d} + \pi^{m/d})]\pi_{sp} \) and \( [\pi^{r/d}/(\pi^{r/d} + \pi^{m/d})]\pi_{sp} \) respectively over the planning horizon \( L \). Now, the question is how they implement the profit sharing policy in different cycle. For this purpose, we propose that the surplus profit can be shared between them by just adjusting wholesale price properly. Thus, for a particular cycle, the manufacturer and the retailer get additional profit \( [\pi^{m/d}/\eta_0^*](\pi^{r/d} + \pi^{m/d})]\pi_{sp} \) and \( [\pi^{r/d}/\eta_0^*](\pi^{r/d} + \pi^{m/d})]\pi_{sp} \) respectively.

In the \( i \)th \((i = 1, 2, \ldots, n_0^*) \) replenishment cycle, profit of the retailer is given by
\[ \frac{LD_i^c}{n_0^c} \left( \frac{w_{ps}^r - L_i^c - \frac{h_{r}}{2n_0^c}}{n_0^c} \right) - s_r = \pi^{r/\theta}_{i} + \left( \frac{\pi^{r/\theta}_{i}}{n_0^c(\pi^{r/\theta}_{i} + \pi^{m/\theta}_{i})} \right) \pi^m_{sp}. \] (24)

After simplification, the wholesale price of the product in the ith \((i = 1, 2, \ldots, n_0^c)\) replenishment cycle can be found as

\[ w_{ps}^r = p_{r}^c - \frac{L_i^c}{2n_0^c} - \frac{n_0^c}{LD_i^c} \left[ \pi^{r/\theta}_{i} + \left( \frac{\pi^{r/\theta}_{i}}{n_0^c(\pi^{r/\theta}_{i} + \pi^{m/\theta}_{i})} \right) \pi^m_{sp} + s_r \right]. \] (25)

Thus, through proper choice of wholesale price, profit sharing mechanism can be implemented and the decentralized channel can achieve profit equal to centralized profit which also assure win–win outcomes for all the channel members.

4. Numerical illustration

Assume that a manufacturer sales a hi-tech product and decides the sales season \( L \) as 120 days. At the beginning of the sales season the unit cost of the product is \( \alpha = $50 \) and the cost decreases at a rate \( \beta = $0.25 \) per day. Other parameter values are \( a = 100, b = 0.4, r = 0.1, h_{r} = $0.15, h_{m} = $0.12 \) per unit per day, \( s_m = $5000, s_r = $1000 \) per replenishment cycle. Also, assume that the customer’s retail channel preference is \( \theta = 0.5 \). The optimal values are presented in Table 2.

In decentralized setting the retailer replenishes thrice over the planning horizon whereas in centralized channel total number of replenishment is 4. As indicated, the wholesale price and the selling prices in retail and online channel decrease in every next replenishment cycle. From Table 2, one can easily observe that the manufacturer’s and the retailer’s profit increase 12.8% and 15.4% respectively from the decentralized channel when they adopt centralized policy through profit sharing mechanism. Both in centralized and decentralized channel, optimal order quantities over the planning horizon increase 12.8% and 15.4% respectively from the decentralized channel when they adopt centralized policy through profit sharing mechanism. Both in centralized and decentralized channel, optimal order quantities over the planning horizon increase in each and every next replenishment. Optimal online channel price and retail price are higher than the manufacturer’s wholesale price for \( \theta \in (0.1526, 0.5162) \). Also, \( \theta = 0.8564 \) and \( \theta = 0.354 \). Thus for profitable retail-online channel customers’ retail channel preference must be \((0.354, 0.5126)\). Interestingly as indicated, customers’ retail channel preference intensifies price competition between the channel. In the present model, selling in decentralized setting for \( \theta > 0.4295, \theta = 0.4203, \theta = 0.4111 \) in the 1st, 2nd and 3rd replenishment cycles, the retail prices are higher than the online selling prices (see Fig. 1). In centralized setting there is also price competition that is presented in Fig. 2. However, as in the decentralized case, the manufacturer can operate a profitable integrated channel when the customers’ retail channel preference \( \theta \in (0.0791) \). Also, the retail price is higher than the online price for customers’ retail channel preference in \((0.4982, 0.791)\). Notice that the product compatibility has lesser impact on centralized channel when compared with decentralized channel.

### Table 2
Optimal values in decentralized channel, centralized channel and for profit sharing mechanism. (Bold faces denote the optimal solution).

<table>
<thead>
<tr>
<th>System</th>
<th>( n )</th>
<th>( w_i )</th>
<th>Retail price</th>
<th>Online price</th>
<th>Order quantity</th>
<th>( \pi^{r/\theta}_{i} )</th>
<th>( \pi^{m/\theta}_{i} )</th>
<th>Total profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decentralized</td>
<td>1</td>
<td>83</td>
<td>105.1</td>
<td>91.1</td>
<td>2582</td>
<td>9312</td>
<td>81819</td>
<td>91131</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>85.2, 77.7</td>
<td>103.8, 99.3</td>
<td>89.3, 81.8</td>
<td>3019</td>
<td>12652</td>
<td>109758</td>
<td>122410</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>86, 81, 76</td>
<td>103.4, 100.4, 97.4</td>
<td>88.7, 83.7, 78.7</td>
<td>3165</td>
<td>13239</td>
<td>116685</td>
<td>129924</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>86.4, 82.6, 78.9, 75.1</td>
<td>103.2, 100.9, 98.6, 96.4</td>
<td>88.4, 84.6, 80.9, 77.2</td>
<td>3237</td>
<td>13058</td>
<td>117826</td>
<td>130884</td>
</tr>
<tr>
<td>Centralized</td>
<td>1</td>
<td>–</td>
<td>92</td>
<td>91.1</td>
<td>3211</td>
<td>–</td>
<td>–</td>
<td>101444</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>–</td>
<td>89.8, 82.2</td>
<td>89.3, 81.8</td>
<td>3765</td>
<td>–</td>
<td>–</td>
<td>137663</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>–</td>
<td>89, 84.79</td>
<td>88.7, 83.7, 78.7</td>
<td>3950</td>
<td>–</td>
<td>–</td>
<td>146162</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>–</td>
<td>88.6, 84.9, 81.1, 77.4</td>
<td>88.4, 84.6, 80.9, 77.2</td>
<td>4043</td>
<td>–</td>
<td>–</td>
<td>147942</td>
</tr>
<tr>
<td>Profit sharing</td>
<td>4</td>
<td>75.8, 71.2, 66.5, 61.8</td>
<td>88.8, 84.9, 81.1, 77.4</td>
<td>88.4, 84.6, 80.9, 77.2</td>
<td>4043</td>
<td>15075</td>
<td>132867</td>
<td>147942</td>
</tr>
</tbody>
</table>

Fig. 1. Price competition with respect to product compatibility in decentralized system.
The setup cost of the retailer has an impact on the manufacturer’s decision for operating the retail-online channel. As the lower threshold of product compatibility is directly proportional to the retailer’s setup cost, $\theta$ may be higher than $\theta_{dc}^c$. In such cases, the manufacturer can not set the online selling price and hence can not operate the retail-online channel. In the present model setting, if $s_r > 2361$ holds then the manufacturer can not operate the online channel (see Fig. 3). Thus, for a profitable retail-online channel, apart from the customers’ retail channel preference, the setup cost of the retailer should be reasonably low. From Table 2, it is noticed that profit sharing mechanism coordinates the channel and divides the surplus profit between the channel members. Also the average wholesale price of the manufacturer in profit sharing mechanism is 68.8 and it is lower than the average wholesale price, 83 of the decentralized decision making. This is quite obvious.

5. Summary and concluding remarks

In this paper we consider a retail–etail channel supply chain for a product that experiences continuous unit cost decrease over the planning horizon. Existing literature in this direction discusses the replenishment and pricing policy over the planning horizon for a single business entity like retailer or manufacturer. But as indicated before, in the current business scenario, the overall channel performance optimization is prevailing in practice and coexistence of retail and online channels are quite common. In such scenario, the present model proposes pricing and replenishment policy and analyzes how the customers’ channel preference affects the individual and integrated decisions of the channel members. The paper also discusses, in decentralized decision making, how the channel performance can be maximized through a transfer pricing policy. The model proposes the following managerial insights.

First, the retail price is directly proportional to online price and wholesale price. The manufacturer has the control, to some extent, on the retail price because it sets the wholesale price and online selling price. Second, the optimal selling prices and wholesale price of the product decrease continuously over the planning horizon because the products’ unit cost decreases continuously. In this scenario, the manufacturer must determine a planning horizon shorter than the lifetime of the product. Third, product compatibility has significant impact on the successful operation of profitable retail-online channel. As indicated earlier, if the customers retail channel preference lies in an interval then the manufacturer operates an online channel besides the traditional brick-and-mortar channel for a product that experiences continuous cost decrease. It is interesting that there is a price competition between the retail and online channel and customers’ channel preference intensifies.
the price competition. When the customers’ retail channel preference is below a certain threshold, the manufacturer sets online price higher than the retail price otherwise the retail price is higher. In integrated channel, there is also a price competition though the channel members cooperate. Thus, in centralized channel, there is an interval of customers’ retail channel preference for profit making retail-online channel. Fourth, the setup cost of the retailer influences the manufacturer’s decision for opening the online channel. The manufacturer cannot operate the online channel when setup cost of the retailer is above a threshold number because the manufacturer cannot set the online price in that case. Fifth, the profit sharing mechanism coordinates the channel and divides the surplus profit.

The model presented in this paper has some limitations. First, we have considered linear price dependent demand function. Although price dependent demand functions are used extensively in economics and marketing, still it is necessary to examine whether the managerial implications found in this paper can be generalized to other demand function or not. Second, we have assumed that the channel members know all information though it is not common in practice. Thus, model may be developed by considering that the channel members have private information. Third, we have used simple profit sharing mechanism to coordinate the channel. Instead of this mechanism, some other well established coordination contracts may be used to study the proposed model elaborately. Another interesting direction of future research is to be considered for multiple retailers while there is retail price competition. Also, the manufacturer wants to sell the product through an online channel. Although members of different replenishment cycles and price competition among the retailers make the model more complex in comparison to the present one, still it will be more dynamic and will be close to real business, in practice.

Appendix A

Proof of Proposition 1. For given \( n' \)

\[
\frac{\partial^2 n_1^{i/ds}(p_1^{d/i})}{\partial p_1^{d/i}} = \frac{L}{n'} \left[ a 0 - (b + r) p_1^{i/ds} + r p_1^{d/i} - (b + r) \left( p_1^{i/ds} - w_i - \frac{Lh_i}{2n'} \right) \right]
\]

(A.1)

\[
\frac{\partial^2 n_1^{i/ds}(p_1^{d/i})}{\partial p_1^{d/i}^2} = -\frac{2L}{n'} (b + r) < 0
\]

From (A.1), equating right hand side to zero and simplifying the stationary point, we have

\[
p_1^{i/ds}(w_i, p_1^{d/i}) = \frac{w_i}{2} + \frac{r}{2(b + r)} p_1^{d/i} + \frac{a 0}{2(b + r)} + \frac{Lh_i}{4n'}, (i = 1, 2, \ldots, n').
\]

Again,

\[
\frac{\partial^2 n_1^{m/ds}(p_1^{d/i})}{\partial p_1^{d/i}} = \frac{L}{n'} \left[ (1 - \theta) a + r \frac{a 0 + (b + r) \frac{Lh_i}{2(b + r)}}{2(b + r)} - \frac{r}{2} \left( (i - 1) \frac{L}{n'} \right) - \frac{Lh_i}{2n'} \right] - \left( \frac{r^2}{2(b + r)} - (b + r) \right) + r w_i
\]

\[
+ \frac{r^2}{2(b + r)} (b + r) p_1^{d/i}, \quad i = 1, 2, \ldots, n'
\]

(A.2)

\[
\frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial w_i} = \frac{L}{n'} \left[ \theta a + (b + r) \frac{a 0 + (b + r) \frac{Lh_i}{2(b + r)}}{2(b + r)} - \frac{r}{2} \left( (i - 1) \frac{L}{n'} \right) - \frac{Lh_i}{2n'} \right] + \frac{1}{2} (b + r) c \left( (i - 1) \frac{L}{n'} \right) - (b + r) w_i + r p_1^{d/i},
\]

(i = 1, 2, \ldots, n'),

(A.3)

\[
\frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial p_1^{d/i}^2} = 2 \left( \frac{r^2}{2(b + r)} - (b + r) \right)
\]

\[
\frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial w_i \partial p_1^{d/i}} = - (b + r)
\]

\[
\frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial p_1^{d/i} \partial w_i} = r = \frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial w_i \partial p_1^{d/i}}
\]

\[
\frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial p_1^{d/i}^2} \times \frac{\partial^2 n_1^{d/i}(w_i, p_1^{d/i})}{\partial w_i^2} \left( \frac{\partial^2 n_1^{m/ds}(w_i, p_1^{d/i})}{\partial w_i \partial p_1^{d/i}} \right)^2 = 2(b + r)^2 - r^2 > 0.
\]
Equating (A.2) and (A.3) to zero and solving for \( p_{i}^{d/ds} \) and \( w_{i} \), the result can be realized. Substituting the optimal values of \( p_{i}^{d/ds} \) and \( w_{i} \) in Eq. (9), we get the retailer’s optimal selling price in \( i = 1, 2, \ldots, n' \) replenishment cycle displayed in Eq. (10). Substituting the optimal values of \( p_{i}^{d/ds} \) and \( p_{i}^{d/ds} \) in Eqs. (1) and (2), we get the demand of the product in retail and direct channel per unit time for \( i = 1, 2, \ldots, n' \) replenishment cycle and multiplying these with cycle length, we get the amount of quantity sold through retail and online channel respectively which are displayed in Table 1. Also, the order quantity that the manufacturer faces is equal to the product of total demand and cycle length, the profit functions of the manufacturer and the retailer in \( i = 1, 2, \ldots, n' \) replenishment cycle are displayed in Table 1.

**Proof of Proposition 2.** For given \( n' \) we have, \( c(t) = \alpha - \beta t \). Thus for \( i = 1 \),

\[
p_{i}^{d/ds} - p_{i+1}^{d/ds} = \frac{1}{2} [c(0) - c(L/n')] = \frac{\beta L}{2n'} > 0
\]

that is, \( p_{i}^{d/ds} > p_{i+1}^{d/ds} \).

For \( i = 2 \),

\[
p_{i}^{d/ds} - p_{i+1}^{d/ds} = \frac{1}{2} [c(L/n') - c(2L/n')] = \frac{\beta L}{2n'} > 0
\]

that is, \( p_{2}^{d/ds} > p_{3}^{d/ds} \).

For \( i = m \),

\[
p_{i}^{d/ds} - p_{i+1}^{d/ds} = \frac{1}{2} [c((m - 1)L/n') - c((m)L/n')] = \frac{\beta L}{2n'} > 0
\]

that is, \( p_{m}^{d/ds} > p_{m+1}^{d/ds} \).

Hence, we can say \( p_{i}^{d/ds} > p_{i+1}^{d/ds} \) for all \( i = 1, 2, \ldots, n' \) i.e., \( p_{1}^{d/ds} > p_{2}^{d/ds} > \ldots > p_{n'}^{d/ds} \).

Other results can be obtained in similar way and hence omitted.

**Proof of Proposition 4.** Comparing the selling prices of the product in retail channel and direct channel in \( i = 1, 2, \ldots, n' \) replenishment cycle, we get \( p_{i}^{d/ds} \geq p_{i}^{c/dc} \) if

\[
\theta \geq \frac{2(b + r)}{5b + 6r} + \frac{(2b + r)(b + 2r)L(h_m - h_r)}{2(5b + 6r)n'a} + \frac{b(b + 2r)}{(5b + 6r)a} c \left[ \left( i - 1 \right) \frac{L}{m} \right] = \theta_{ic}\tag{A.2}
\]

Again, comparing \( \theta_{ic} \) with \( \theta_{im} \), we get

\[
\theta_{im} - \theta_{ic} = \frac{(b + 2r)(a - 2bc)}{2(5b + 6r)a} + \frac{(b + 2r)(7b + 8r)h_r + (b + 4r)h_m L}{4anr(5b + 6r)} > 0
\]

and comparing \( \theta_{ic} \) with \( \theta_{imin} \), we get

\[
\theta_{imin} - \theta_{ic} = \frac{(b + r)(a - 2bc)}{2(5b + 6r)a} + \frac{(b + r)(7b + 8r)h_r + (b + 4r)h_m L}{anr(5b + 6r)} > 0
\]

Hence, \( \theta_{ic} \in (\theta_{imin}, \theta_{im}) \). Thus one can easily realize that \( p_{i}^{r/ds} > p_{i}^{d/ds} > w_{i} \) if \( \theta \in (\theta_{imin}, \theta_{im}) \) and \( p_{i}^{d/ds} > p_{i}^{r/ds} > w_{i} \) if \( \theta \in (\theta_{imin}, \theta_{im}) \).

**Proof of Proposition 6.** \( \frac{d\pi^{c/ds}}{dn} = 0 \) gives

\[
16(b + r)s_{r}n^{3} + \left[ \left( a\theta - b \left( \frac{\alpha - \frac{\beta L}{2}}{2} \right) \right) (A - b\beta) \right] n^{2} + \frac{L}{6} \left[ 3A(A - 2b\beta) + 2b^{2}\beta^{2} \right] = 0
\]

Using Cardano’s method for solving the cubic equation, we get

\[
n_{0}^{r} = \left( -d + \sqrt{d^{2} + b^{2}} \right)^{1/3} + \left( -d - \sqrt{d^{2} + b^{2}} \right)^{1/3},
\]

where \( b = L^{2} [A - b\beta] / 96(b + r)s_{r} \) and \( d = L^{3} [3A(A - 2b\beta) + 2b^{2}\beta^{2}] / 192(b + r)s_{r} \).

The analytical solution for finding the number of replenishment over the selling season \( L \) gives the optimal profit of the retailer which can be an integer or can not be an integer. But the number of replenishment must be integer. It is very simple to find the integer solution of replenishment for the retailer. Suppose \( n_{0}^{r} \) denotes the largest integer not greater than \( n_{0}^{r} \).

Then the retailer will accept \( n_{0}^{r} \) if \( \pi^{r/ds}(n_{0}^{r}) > \pi^{d/ds}(n_{0}^{r} + 1) \) otherwise \( n_{0}^{r} + 1 \) is the better solution for the retailer. Hence optimal number of replenishment for the retailer is given by
\[ n_0^* = \begin{cases} \lceil n_0 \rceil & \text{if } \pi^{c,d}(\lceil n_0 \rceil) > \pi^{c,d}(\lceil n_0 \rceil + 1) \\ n_0 + 1 & \text{otherwise} \end{cases} \]

**Proof of Proposition 7.** From Eq. (16) \( \frac{\partial \pi^i}{\partial \mathbf{p}^i} = 0 \) gives

\[
2rp_i^c - 2(b + r)p_i^c + \frac{(b + r)h_i T_c}{2} - \frac{rh_m T_c}{2} + bc[(i - 1)T_c] = 0
\]

and \( \frac{\partial \pi^i}{\partial \mathbf{p}^c} = 0 \) gives

\[
-2(b + r)p_i^c + 2rp_i^c + (1 - \theta)a - \frac{rh_i T_c}{2} + \frac{(b + r)h_m T_c}{2} + bc[(i - 1)T_c] = 0
\]

Solving we have the selling price of the product for retail and direct channel in \( i \)th, \( i = 1, 2, \ldots, n^c \) replenishment cycle. Again,

\[
\frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc}} = -2(b + r)
\]

\[
\frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc}} = -2(b + r)
\]

\[
\frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc}} = 2r = \frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc} \partial p_i^{dc}}
\]

\[
\frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc}^2} \times \frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc}^2} - \left( \frac{\partial^2 \pi^i_c(p_i^c, p_i^{dc})}{\partial p_i^{dc} \partial p_i^{dc}} \right)^2 = 4[(b + r)^2 - r^2] > 0
\]

That is, \( \pi^i_c \) is a concave function of \( p_i^c \) and \( p_i^{dc} \).

**Proof of Proposition 8.** We have \( c(t) = \alpha - \beta t \). Thus for \( i = 1 \),

\[
p_i^c - p_i^{c+1} = p_i^c - p_i^{c+1} = \frac{1}{2}c(0) - c(L/n^c)] = \frac{\beta L}{2n^c} > 0
\]

that is, \( p_i^c > p_i^{c+1} \).

For \( i = 2 \),

\[
p_i^c - p_i^{c+1} = \frac{1}{2}c(L/n^c) - c(2L/n^c)] = \frac{\beta L}{2n^c} > 0
\]

that is, \( p_i^c > p_i^{c+1} \).

For \( i = m \),

\[
p_i^c - p_i^{c+1} = \frac{1}{2}c((m - 1)L/n^c) - c((m)L/n^c)] = \frac{\beta L}{2n} > 0
\]

that is, \( p_i^c > p_i^{c+1} \).

Hence, we can say \( p_i^c > p_i^{c+1} \) for all \( i = 1, 2, \ldots, n^c \) i.e., \( p_1^c > p_2^c > \ldots > p_{n^c}^c \).

Other results can be obtained in similar way and hence omitted.

**References**


