OPTIMAL BURN-IN TIME TO MINIMIZE THE COST
FOR REPAIRABLE ASSEMBLY PRODUCTS
UNDER WARRANTY

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Abstract: In this paper, we present an optimal burn-in model for repairable assembly products (RAP) sold under warranty. The general repairable products, sold under warranty and optimal burn-in model by Sheu (2004) are further extended to the case in which the probability of type II failure is independent for a parent item. This paper investigates a parent item (product or system) made up of two items (sub-system), which has a bathtub-shaped failure rate. There are two types of failure possible for the parent item. One is the type/item I failure, which can be removed by a major repair, and the other is a type/item II failure, which can be removed by a minimal repair. The expected total cost per unit sold is derived for each of the two warranty policies (failure-free policy and rebate policy) as well as the conditions required for burn-in to be beneficial under failure-free policy.

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1. Introduction

This paper addresses a repairable assembly product (RAP), sold under warranty
with optimal burn-in time to minimize the cost. There are two types of failure of the RAP combined by two sub-systems (items) which has a bathtub-shaped failure rate, as shown in Figure 1. One is type/item I failure, which can be removed by a complete repair, replace with a successful item I, and the other is type II failure, item II failure, which can be removed by a minimal repair, replace one as bad as old one.

The early literature considers repairable and non-repairable products separately during objective model construction [7], [8], [14], [21], [13], [24]. Sheu [19] presented an optimal burn-in and warranty model which generalizes the models of Cha [4], [5]. In the models of Cha [4], [5] and Sheu [19], they addressed a repairable good as one unique item (system) when that good fails. A type I failure occurs with probability 1-p, and type II failure occurs with probability p, (0≤p≤1), these two types of failure are dependent upon each other. Sheu [19] further extended the model of Cha [4], for the two cases considered by Sheu [19] and Cha [4], in which a type II failure probability is dependent upon a type I failure (p with 1-p or p(t) with 1-p(t).). For most of the assemble-to-order goods such as a personal computer, a type II failure is a minor failure (minor cost), and thus can be corrected by a minimal repair (replace fuse, resistance, capacitor or adjust variable resistance ....); whereas a type I failure is a catastrophic failure (a major item failure such as a central process unit, liquid crystal display or hard disc) which can only be removed by replacing the CPU, LCD or hard disc (which are major items of product). The type II failure probability is independent of the type I failure. However, from a practical perspective, a RAP with two types of sub-systems (items), with one being repairable and the other one non-repairable, and with a type II failure probability being independent from the type I, seems to be more realistic.
A warranty is a contractual agreement between a manufacturer (seller) and a consumer (buyer) which requires the manufacturer to rectify all failures occurring within the warranty period. The terms of the contract may take different forms [1], [2]. One is a failure-free warranty that obligates the manufacturer to maintain the product free of charge during the warranty period. Another is a prorated rebate warranty that obligates the manufacturer to refund a fraction of the purchase price if the product fails within the warranty period.

When a RAP is sold with a failure-free warranty, the manufacturer has the option of rectifying a failure by either repairing the failed item II (adjusting and reset the failed item or replace a minor item), or replacing it with a new item I. The decision to repair or to replace depends on the failure type of the RAP.

Providing warranties is a trend among manufacturers. The warranty cost usually relates to the failure rate of the RAP, and to the repair cost in the field. Since electric goods have a high failure rate during the initial period (infant mortality), reducing warranty cost and improving reliability by means of burn-in is a good idea. Many burn-in models have been proposed for estimating the reliability measures and determining the optimum burn-in policies [12], [15], [8], [14], [21], [13], [20], [24], [9].

Burn-in is designed to expose latent defects, thereby preventing sub-standard units from being delivered to the customer. Thus, a burn-in program eliminates infant mortality, and ensures good finished product quality, for a product that operates in the region of constant near-failure rate \( r(t) = \lambda \) following its purchase. However, burn-in programs increase manufacturing costs. Therefore, manufacturers attempt to determine the duration of the burn-in procedure. The optimum time for stopping the burn-in according to a given set of criteria is termed the optimal burn-in time [7], [3], [12], [15], [19], [20].

In this investigation, we consider a RAP sold under warranty, and determines the burn-in time required before the RAP is put for sale. The burn-in time is optimized to minimize the expected total cost (i.e., manufacturing cost plus warranty costs) under two warranty policies (failure-free policy and rebate policy). The remainder of the paper is organized as follows: in Section 3 the repair modes are formulated, in Section 4 the manufacturing cost models are formulated, in Section 5 the modified warranty policy is proposed, and in Section 6 the warranty cost modes are formulated. Finally, in Section 7 the optimization models are addressed.

It can is evident that the present models and results found in this study are a generalization of those reported by Nguyen and Murthy [1], Sheu and Chein [19], [20].
2. Notation

$X_i$: random variable, the lifetime of item $i$;

$F_i(t)$: Cdf of $X_i$, $= \Pr(X_i \leq t)$;

$F_i(t)$: survival function of $X_i = \exp\{-\int_0^t r_i(u)du\}$;

$r_i(t)$: failure rate of item $i = \frac{f_i(t)}{F_i(t)}$;

$r_0(t)$: failure rate of a RAP;

$Y_1$: random variable, the failure time of item I passed burn-in time $\tau$;

$G_1(t)$: Cdf of $Y_1$, $= \Pr(Y_1 \leq t)$;

$G_1(t)$: survival function of $Y_1 = \exp\{-\int_0^t r_1(\tau + u)du\}$

For the assembly of a complex electronic product or system, we assume it is comprised of two sub-system (items) that have a bathtub-shaped failure rate, as shown in Figure 2. The first change point of $r_i(t)$ is $t_{i1}$, the second change point is $t_{i2}$, meaning that $r_i(t)$ decreases for $t < t_{i1}$; $r_i(t)$ increases for $t > t_{i2}$; $r_i(t) = \lambda_i$ for $t_{i1} \leq t \leq t_{i2}$. The failure rate is to depend on the item characteristic. However, most literature considered a product (system) a unit that has two different type failures, with the two types being dependent upon one another [6], [4], [5], [19], [20].

3. Modified General Repair Model

If a general repairable product is an assembly of two items with two different failure models, then the type I failure is independent from the type II failure model. The model will be distinct from the general failure model described by Sheu [20], who describes a general repairable product as: "when the product fails, the type II failure occurs with probability of $1- p$, and type I failure occurs
with probability $p$, $(0 \leq p \leq 1)$. In practice, the probability $p$ for a RAP is not constant, as it is a function of $r_i(t)$.

The RAP fails randomly when an item breaks down. Let $X$ be the failure time of a new RAP without burn-in, and the failure characteristics of the RAP are described as follows: One is the type I failure (major item I failure), which can be removed by a complete repair, and the other is a type II failure (minor item II failure), which can be removed by a minimal repair (minor cost).

Therefore, let the random variable $X_1$ denote the time to the first type I failure of the product without burn-in. If $F_1(t)$ is defined as the cumulative distribution function of $X_1$ and the survival function is given by

$$
\overline{F}_1(t) = Pr(X_1 \geq t) = \exp\{-\int_0^t r_1(u)du\} = \exp\{-\wedge_1(t)\}.
$$

**Remark 1.** The relationship of the failure rate between item with parent product as follows:

The failure rate of parent product: $r_0(t) = r_1(t) + r_2(t)$ and the probability of type I failing at time $t$: $p(t) = \frac{r_1(t)}{r_1(t) + r_2(t)}$

In general, the probability $p(t)$ for an assembly product is not constant, hence equation (1) in Sheu [20] $Pr\{Y \geq t\} = \exp\{-\int_0^t ph(u)du\}$ is a special case.

After the burn-in time $\tau$, let random variable $Y_1$ denote the type I failure time of product passed burn-in time $\tau$. If $G_1(t)$ is defined as the cumulative distribution function of $Y_1$ and the survival function is given by

$$
\overline{G}_1(t) = Pr\{Y_1 \geq t\} = \exp\{-\int_0^t r_1(\tau + u)du\} = \frac{\overline{F}_1(\tau + t)}{\overline{F}_1(\tau)}.
$$

**Remark 2.** Equation (2) in Sheu [20] $\overline{G}_\tau(t) = (\overline{F}_\tau(t))^p$ can be simplified as follows:

$$
\overline{G}_\tau(t) = (\overline{F}_\tau(t))^p = (\overline{F}(\tau + t)/\overline{F}(\tau))^p = (\overline{F}(\tau + t))^p/(\overline{F}(\tau))^p
$$

$$
= \exp\{-\int_0^{\tau+t} ph(u)du\}/\exp\{-\int_0^\tau ph(u)du\}
$$

$$
= \exp\{-\int_0^{\tau+t} r_1(u)du\}/\exp\{-\int_0^\tau r_1(u)du\} = \frac{\overline{F}_1(\tau + t)}{\overline{F}_1(\tau)}.
$$

For the RAP that does not undergo burn-in, the sequence of type I failures (item I failure) followed by replacement constitutes a renewal process, and the
expected number of replacements in $[0, T]$, $V(t)$ is given by the following renewal equation:

$$V(T) = F_1(t) + \int_0^T V(T - t) dF_1(t).$$  \hspace{1cm} (1)$$

Similarly, for a RAP with burn-in time $\tau$, the expected number of replacements for item I in $[0, T]$ is given by

$$V_1(T) = G_1(t) + \int_0^T V_1(T - t) dG_1(t).$$  \hspace{1cm} (2)$$

4. Manufacturing Cost Model

The manufacturing cost model contains five costs:
- $c_{01}$: the manufacturing cost per item I without burn-in;
- $c_{02}$: the manufacturing cost per item II without burn-in;
- $c_1$: the fixed set-up cost of burn-in per item;
- $c_2$: the cost of burn-in per unit of time per item;
- $c_3$: the minimal repair cost per item II during burn-in.

Let $v(\tau)$ be the total manufacturing cost per unit for the RAP with burn-in time $\tau$. We formulate the manufacturing cost per unit product

$$v(\tau) = \text{manufacturing cost} + \text{set up cost} + \text{burn-in cost} + \text{repair cost}$$

$$= c_m + c_s + c_b + c_r.$$

As in Sheu [20], we define $\eta_1$ -1 to be the random variable which is the number of shop major repairs until the first item I surviving the burn-in is obtained. It is evident that the random variable $\eta_1$ has a geometric distribution given by

$$P_r\{\eta_1 = k\} = F_1(\tau) : F_1^{k-1}(\tau), \quad \forall k \geq 1.$$  \hspace{1cm} (3)$$

Hence

$$c_m = \sum_{i=1}^{\eta_1} c_{01} + c_{02}, \quad c_s = \sum_{i=1}^{\eta_1} c_1 + c_l,$$

$$c_b = \sum_{i=1}^{\eta_1-1} c_2 X_{1,i} + 2c_2 \tau, \quad c_r = c_3 \int_0^{\tau} r_2(t) dt,$$
and
\[ v(\tau) = \sum_{i=1}^{\eta-1} [c_{01} + c_1 + c_2 X_{1,i}] + c_{01} + c_{02} + 2(c_2 \cdot \tau + c_1) + c_3 \int_0^\tau r_2(t) dt . \]

The s-expect cost of \( v(\tau) \) is given by
\[ v(\tau) = E[v(\tau)] = E[\eta - 1] \{ c_{01} + c_1 + c_2 \cdot E[X_1 | X_1 < \tau] \} + c_{01} + 2(c_2 \cdot \tau + c_1) + c_{02} + c_3 \int_0^\tau r_2(t) dt \]
\[ = \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^\tau F_1(t) dt] + c_{02} + c_1 + c_2 \cdot \tau + c_3 \int_0^\tau r_2(t) dt . \] (4)

The manufacturing cost, \( v(\tau) \), increases with the burn-in time. (the function \( \frac{1}{F_1(\tau)} \int_0^\tau F_1(t) dt \) and \( \int_0^\tau r_2(t) dt \) increase with \( \tau \).
\[ \frac{\partial}{\partial \tau} E[v(\tau)] = r_1(\tau) \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^\tau F_1(t) dt] + 2c_2 + c_3 r_2(\tau) > 0 . \] (5)

**Remark 3.** The following three special case for the manufacturing cost model with burn-in time \( \tau \) are consider:

1. \( r_1(t)=0 \) indicates the case of repairable products consider by Nguyen and Murthy [13]. In this case, it is easy to see that
\[ E[v(\tau)] = c_{01} + c_{02} + c_2 \cdot \tau + c_3 \int_0^\tau r_2(t) dt \] (6)
and
\[ \frac{d}{d\tau} E[v(\tau)] = c_2 + c_3 \cdot r_2(t) > 0 \]
which are the same as (8) and (9) in [13].

2. \( r_2(t)=0 \) indicates the case of non-repairable products consider by Nguyen and Murthy [13]. In this case,
\[ [E[v(\tau)] = \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^\tau F_1(t) dt]] \] (7)
and
\[ \frac{d}{d\tau} E[v(\tau)] = c_2 + v(\tau) \cdot r_2(t) > 0 \]
which are the same as (10) and (11) in [13].
(3) \( p(t) = \frac{r_1(t)}{r_1(t) + r_2(t)} = p \) indicates the general repairable products consider by Sheu [20]. In this case, the product is mentioned as one item. \( E[v(\tau)] \) is modified as (5) in Sheu [20].

\[
E[v(\tau)] = \frac{1}{G(\tau)} [C_0 + C_1 + C_2 \int_0^\tau G(t)dt + C_3 (\frac{1}{p} - 1) G(\tau)].
\] (8)

5. Modified Warranty Policy

There is a variety of warranty policies among different manufacturers. In this paper we address the failure-free policy and the rebate policy. The failure-free policy can be further divided into two categories, renewing and non-renewing. In this paper, we have focused on different warranty periods of items with a renewing policy.

(1) non-renewing policy – replacement of a failed item I or the repair item II does not alter the original warranty of the RAP. Under the non-renewing policy, the warranty period is fixed and identical for the repairable item II and the non-repairable item I.

(2) renewing policy – if a RAP fails through item I within the warranty time, it is replaced by a new item I with a new warranty for item I. In effect, the warranty of item I begins a new with each replacement, but item II does not alter the original warranty when it failed and is repaired. In other words, with renewing policy, the warranty period of item I is a variable interval, but the warranty period of item II remains fixed.

6. Warranty Cost Model

6.1. Failure-Free Policy

6.1.1. Non-Renewing

Let \( Z_n \) be the waiting time until the occurrence of the \( n \)th replacement for the burned-in item I during \([0, T]\), hence \( Z_n = \sum_{i=1}^{n} Y_i \) and \( Y_{1,1}, Y_{1,2}, Y_{1,3} \ldots \) are i.i.d. random variable with a survival function \( G_1 \). The renewal function will be produced as

\[
V_1(t) = \sum_{n=1}^{\infty} P(Z_n \leq t) = \sum_{n=1}^{\infty} G_1^{(n)}(t),
\] (9)
where $G_1^{(n)}(t)$ denotes the $n$-fold convolution of the distribution $G_1(t)$ itself. Let $C_i$ be the cost of $i$-th replacement item I during $[0, T]$, then from equation (7) and extra cost $(c_4)$ in the field, $C_i$ is obtained as follows:

$$C_i = \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^\tau F_1(t) dt] + c_4. \quad (10)$$

Let the warranty cost of item I be equal to $\sum_{i=1}^N C_i$. $N(t)$ is the numbers of replacement for item I during warrant interval $[0, T]$. The counting process $N(t)$ is to be a renewal process [16]. Hence, we formulate the warranty cost $K_1(T, \tau)$ of item I as follows

$$K_1(T, \tau) = \sum_{i=1}^N C_i$$

and s-expectation $K_1(T, \tau)$ is introduced by condition $Y_{1,1}$ (the first failure time of item I)

$$E[K_1(T, \tau)] = E[K_1(T, \tau)|Y_{1,1} < T] + E[K_1(T, \tau)|Y_{1,1} > T]$$

$$= C_i P(Y_{1,1} < T) + \int_0^T E[K_1(T-y, \tau)] dG_1(y) + 0. \quad (11)$$

The warranty cost of item II is independent from item I, and the s-expectation total warranty cost of a RAP can be expressed as follows:

$$E[K(T, \tau)] = E[K_1(T, \tau)] + E[K_2(T, \tau)]$$

$$= C_i G_1(T) + \int_0^T E[K_1(T-y, \tau)] dG_1(y) + (c_3 + c_4) \int_0^T r_2(\tau + u) du. \quad (12)$$

Let $E[K_1(T, \tau)] = k_1(T, \tau)$,

$$k_1(T, \tau) = w(T, \tau) + \int_0^T k_1(T-y, \tau) dG_1(y). \quad (13)$$

Thus $k_1(T, \tau)$ satisfies the conditions for a renewal type equation, where

$$w(T, \tau) = G_1(T) \left\{ \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^\tau F_1(t) dt] + c_4 \right\}. \quad (14)$$

Therefore, the solution of equation (13) is given by the following
\[ k_1(T, \tau) = w(T, \tau) + \int_0^T w(T - y, \tau) dV_1(y) \]

\[ C_iG_1(T) + C_i \int_0^T G_1(T - y) dV_1(y) = C_i \{ G_1(T) + \int_0^T G_1(T - y) dV_1(y) \} \]

\[ C_i \{ G_1(T) + \int_0^T V_1(T - y) dG_1(y) \} = C_i V_1(T). \]

The s-expectation warranty cost of the RAP under this policy is given by

\[ k(T, \tau) = E[K(T, \tau)] = \]

\[ = V_1(T) \left\{ \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^T \frac{1}{F_1(t)} dt] + c_4 \right\} + (c_3 + c_4) \int_0^T r_2(\tau + u) du , \]

and

\[ \frac{\partial}{\partial \tau} k(T, \tau) = \]

\[ = C_i \frac{\partial}{\partial \tau} V_1(T) + V_1(T) \left[ C_i r_1(\tau) - c_4 r_1(\tau) + c_2 \right] + (c_3 + c_4) [r_2(\tau + T) - r_2(\tau)]. \]

**Remark 4.** The RAP warranty cost model combines the warranty for the repairable product with the non-repairable product model considered by Nguyen and Murthy [13].

(1) \( r_1(t) = 0 \) indicates the case of repairable products as considered by Nguyen and Murthy [13]. In this case, it is evident that

\[ k(T, \tau) = (c_3 + c_4) \int_0^T r_2(\tau + u) du = (c_3 + c_4) [\wedge_2(\tau + T) - \wedge_2(\tau)] \]

and

\[ \frac{\partial}{\partial \tau} k(T, \tau) = (c_3 + c_4) [r_2(\tau + T) - r_2(\tau)] \]

which are identical to (12) and (13) in [13].

(2) \( r_2(t) = 0 \) indicates the case of non-repairable products as considered by Nguyen and Murthy [13]. In this case,

\[ k(T, \tau) = V_1(T) \left\{ \frac{1}{F_1(\tau)} [c_{01} + c_1 + c_2 \int_0^\tau \frac{1}{F_1(t)} dt] + c_4 \right\} \]
which is the same as (14) in [13].

When \( \frac{r_1(t)}{r_1(t) + r_2(t)} = p \), indicates the general repairable products as considered by Sheu [20]. In this case, the product is mentioned as one item. \( k(T, \tau) \) and is to be modified as (14) in Sheu [20]

\[
w(T, \tau) = V_r(T)\{[C_4 + \nu(\tau)] + (C_3 + C_4)\left[\frac{1}{p} - 1\right]\}.
\]

\( \nu(\tau) \) is the same as \( v(\tau) \), and

\[
v(\tau) = \frac{1}{F_1(\tau)}[c_{01} + c_1 + c_2 \int_0^{\tau} \frac{F_1(t)}{f_1(t)} dt].
\]

### 6.1.2. Renewing

We can define \( \eta_2 - 1 \) to be the random variable which is the number of major repairs until the first item I surviving the warranty period is obtained. It is evident that the random variable \( \eta_2 \) has a geometric distribution given by

\[
Pr\{\eta_2 = k\} = G_1(T)G_1^{k-1}(T), \quad \forall k \geq 1.
\]  

(17)

The warranty cost of item I during the \([0, T]\) must be equal to \( \sum_{i=1}^{\eta_2-1} C_i \), and the warranty cost of item II is \( (c_3 + c_4) \int_0^T r_2(\tau + u) du \).

Accordingly, the total warranty cost \( K(T, \tau) \) of the RAP is given by

\[
K(T, \tau) = \sum_{i=1}^{\eta_2-1} C_i + (c_3 + c_4) \int_0^T r_2(\tau + u) du,
\]  

(18)

where \( C_i = \frac{1}{F_1(\tau)}[c_{01} + c_1 + c_2 \int_0^{\tau} \frac{F_1(t)}{f_1(t)} dt] + c_4 \).

The s-expectation warranty cost of the RAP under this policy is given by

\[
k(T, \tau) = E[K(T, \tau)] = \frac{1}{G_1(T)} C_i - C_i + (c_3 + c_4) \int_0^T r_2(\tau + u) du
\]  

(19)

and

\[
\frac{\partial}{\partial \tau} k(T, \tau) = \frac{F_1(\tau)}{F_1(\tau + T)} \{C_i r_1(\tau + T) - c_4 r_1(\tau) + c_2\}
\]

\[
-(C_i - c_4) r_1(\tau) - c_2 + (c_3 + c_4)[r_2(\tau + T) - r_2(\tau)].
\]  

(20)
6.2. Rebate Policy

Under the rebate policy for a RAP, if a type I failure occurs in the warranty period \([0, T]\) the manufacturer refunds a proportion of the sales price \(c_p\) to the customer and takes back the failure RAP, which has a residual value \(p_1\). The amount of rebate, \(R(t)\), is a function of the type I failure time \(t\). This study assumes that \(R(t)\) is a linear function of \(t\), i.e.,

\[
R(T) = \begin{cases} 
\alpha c_p (1 - \frac{t\rho}{T}) & \text{for } 0 \leq t \leq T, \\
0 & \text{for } t > T, 
\end{cases}
\]

(21)

where \(0 < \alpha \leq 1, 0 \leq \rho \leq 1\).

**Remark 5.** Two special case are considered as follows:

(i) \(\alpha = 1, \rho = 0\) full refund policy.

(ii) \(\alpha = 1, \rho = 1\) prorated rebate policy.

Under this policy, during the warranty period, the manufacturer must rectify type II failures free of charge before addressing a type I failure. The s-expectation warranty cost of the RAP under this policy is given by

\[
k(T, \tau) = \left\{ \frac{1}{G_1(T)} \int_0^T \left[ (c_3 + c_4) \int_0^\tau r_2(\tau + u)du + R(t) - p_1 \right] dG_1(t) \right\} P(Y_{1,1} \leq T) \\
+ \left[ (c_3 + c_4) \int_0^T r_2(\tau + u)du + 0 \right] P(Y_{1,1} > T) \\
= (c_3 + c_4) \int_0^T \overline{G_1(t)} r_2(\tau + t)dt + \int_0^T (R(t) - p_1) dG_1(t) \\
= (c_3 + c_4) \int_0^T \overline{G_1(t)} r_2(\tau + t)dt + (\alpha c_p - p_1)[G_1(T) - G_1(0)] - \frac{\alpha c_p}{T} \int_0^T \overline{G_1(t)}dt \\
= \left( (c_3 + c_4) \cdot T \cdot \int_0^T \overline{F_1(\tau + t)} \cdot r_2(\tau + t)dt \\
+ T \cdot (\alpha c_p - p_1)[\overline{F_1(\tau)} - \overline{F_1(\tau + T)}] - \alpha c_p \int_0^T \overline{F_1(\tau + t)}dt \right) (T \cdot \overline{F_1(\tau)})^{-1}. 
\]

(22)

and

\[
\frac{\partial}{\partial \tau} k(T, \tau)
\]
\[= (T \cdot F_1(\tau))^{-1}\{(c_3 + c_4)T \int_0^T [r_1(\tau) - r_1(\tau + t) + \frac{\partial}{\partial \tau}r_2(\tau + t)]F_1(\tau + t)dt
\]
\[+(\alpha c_p - p_1)T[r_1(\tau + T) - r_1(\tau)]F_1(\tau + T) - \alpha c_p \rho \int_0^T [r_1(\tau) - r_1(\tau + t)]F_1(\tau + t)dt\} \]

(23)

7. Optimization Model

Let \(C(T, \tau)\) denote the expected total cost per unit sold, for a RAP with burn-in time \(\tau\) and warranty period \(T\); and let \(C(T,0)\) represent the corresponding cost without burn-in.

Then:
\[C(T, \tau) = v(\tau) + k(T, \tau), \quad C(T, 0) = v(0) + k(T, 0),\]
when \(\tau \to 0^+\) and \(\tau \neq 0\), the set up cost \(c_1 > 0\), so
\[\lim_{\tau \to 0^+} C(T, \tau) > C(T, 0).\]

For a specific warranty period \(T\), the manufacturer needs to decide the optimal burn-in time \(\tau^*\), when
\[C(T, 0) > C(T, \tau) \quad \text{for} \quad \tau \in (0, t_{m1}], \quad \text{(see Appendix B)},\]
where \(t_{m1} = \max (t_{11}, t_{21}^*)\).

The \(t_{m1}\) is the first change point of the RAP. Mi (1996) proved the optimal time \(\tau^* \leq \max (t_{11}^*, t_{21}^*)\) for the bathtub-shape model. There are two methods to minimize \(C(T, \tau)\). One approach is the numeric method, another approach is to dissolve the differential equation
\[\frac{d}{d\tau}C(T, \tau) = \frac{d}{d\tau}v(\tau) + \frac{d}{d\tau}k(T, \tau) = 0\]
for \(\tau^*\) and \(\frac{d^2}{d\tau^2}C(T, \tau^*) > 0\).

Under the failure-free policy, the manufacturing cost, \(v(\tau)\), increases with the burn-in time, and the warranty cost, \(k(T, \tau)\), does not decrease when \(\tau > t_{m1}\).

That is \(\frac{d}{d\tau}C(T, \tau) > 0\) for \(\tau > t_{m1}\).
Theorem 1. If \( \frac{d}{d\tau} C(T, \tau) < 0 \) for \( \tau = 0 \), then there exists an optimal solution \( \tau^* > 0 \) for a failure-free policy to satisfy the differential equation
\[
\frac{d}{d\tau} C(T, \tau) = \frac{d}{d\tau} v(\tau) + \frac{d}{d\tau} k(T, \tau) = 0.
\]
(1) under the non-renewing condition \( \frac{d}{d\tau} C(T, \tau) < 0 \) for \( \tau = 0 \), that is:
\[
r_2(0) > (r_1(0)(c_{01} + c_1) \cdot [1 + V_1(T)] + c_2[2 + V_1(T)] + (c_3 + c_4)r_2(T)
+ (c_{01} + c_1 + c_4) \frac{d}{d\tau} V_1(T)_{\tau=0}) (c_4)^{-1}.
\]
(2) under the renewing condition, that is:
\[
r_1(0) > \frac{[(c_{01} + c_1 + c_4)r_1(T) + c_2] + F_1(T)(c_3 + c_4)[r_2(T) - r_2(0)]}{c_4 + F_1(T)(c_{01} + c_1)}.
\]

Appendix A

Lemma 1. If a RAP is assembled by \( n_1 \) major items and \( n_2 \) minor items, then the failure rate of a major item \( i \) is \( r_{1i}(t) \), the failure rate of a minor item \( j \) is \( r_{2j}(t) \) and the repair cost of the major item \( i \) is \( c_{1i} \), the repair cost of the minor item \( j \) is \( c_{2j} \) (\( c_{2j} \ll c_{1i} \)). Therefore the repair cost of a RAP is
\[
c_0 = c_{01} \int_0^t r_1(u)du + c_{02} \int_0^t r_2(u)du,
\]
and the failure rate of the RAP is
\[
r_0(t) = r_1(t) + r_2(t), \quad r_1(t) = \sum_{i=1}^{n_1} r_{1i}(t), \quad r_2(t) = \sum_{j=1}^{n_2} r_{2j}(t).
\]

Proof. Step 1.
\[
c_0 = \sum_{i=1}^{n_1} c_{1i} \int_0^t r_{1i}(u)du + \sum_{j=1}^{n_2} c_{2j} \int_0^t r_{2j}(u)du.
\]
Step 2. Let
\[
c_{01} = \sum_{i=1}^{n_1} c_{1i} \int_0^t r_{1i}(u)du / \int_0^t \sum_{i=1}^{n_1} r_{1i}(u)du
\]
\[= \sum_{i=1}^{n_1} c_{1i} \int_0^t r_{1i}(u)du / \int_0^t r_1(u)du,
\]
\[
c_{02} = \sum_{j=1}^{n_2} c_{2j} \int_0^t r_{2j}(u)du / \int_0^t \sum_{j=1}^{n_2} r_{2j}(u)du
\]
\[= \sum_{j=1}^{n_2} c_{2j} \int_0^t r_{2j}(u)du / \int_0^t r_2(u)du.
\]

Step 3.

\[
c_0 = \sum_{i=1}^{n_1} c_{1i} \int_0^t r_{1i}(u)du + \sum_{j=1}^{n_2} c_{2j} \int_0^t r_{2j}(u)du
\]
\[= c_{01} \int_0^t r_1(u)du + c_{02} \int_0^t r_2(u)du.
\]

Appendix B

**Lemma 2.** If a RAP is the parent of major item \(i\) (\(i = 1, 2, \ldots, n\)) and minor item \(j\) (\(j = 1, 2, \ldots, m\)) with item \(i\) (\(j\)) having a bathtub-shaped failure rate \(r_i(t), h_j(t)\), then the first change point of \(r_i(t)\) is \(t_{11}\), the second change point of \(r_i(t)\) is \(t_{22}\) and the first change point of \(h_j(t)\) is \(k_{11}\), the second change point of \(h_j(t)\) is \(k_{22}\), the failure rate \(r^*_i(t)\) of the major items is a bathtub-shaped and the first change point of \(r^*_i(t)\) will be \(t^*_{11} = \max(t_{11}, \ldots, t_{n1})\), and the second change point of \(r^*_i(t)\) will be \(t^*_{12} = \min(t_{12}, \ldots, t_{n2})\).

The failure rate \(h^*_j(t)\) of the minor items is a bathtub-shape, and the first change point of \(h^*_j(t)\) will be \(t^*_{21} = \max(k_{11}, \ldots, k_{m1})\), and the second change point of \(h^*_j(t)\) will be \(t^*_{22} = \min(k_{12}, \ldots, k_{m2})\).

**Proof.** Step 1. By recursion, we proof only when \(n=2\) (\(m=2\)), \(r_s(t) = r_1(t) + r_2(t)\).

(a) \(r_s(t)\) is strictly decreasing in the interval \([0, t_{s1}]\). Let \(0 < t_1 < t_2 \leq t_{s1}\).

(1) If \(t_{s1} = \max(t_{11}, t_{21}) = t_{11}\), then \(r_1(t_1) > r_1(t_2) \geq r_1(t_{11}), r_2(t_1) \geq r_2(t_2) \geq r_2(t_{11})\), we get \(r_s(t_1) > r_s(t_2) \geq r_s(t_{s1})\).
(2) If \( s_1 = \max(t_{11}, t_{21}) = t_{21} \), it is evident that \( r_s(t_1) > r_s(t_2) \geq r_s(s_1) \).

(b) \( r_s(t) \) is strictly increasing in the interval \([t_{s2}, \infty)\). Let \( t_{s2} < t_1 < t_2 \leq \infty \).

(1) If \( t_{s2} = \min(t_{12}, t_{22}) = t_{12} \), then \( r_1(t_2) > r_1(t_1) > r_1(t_{12}) \), \( r_2(t_2) \geq r_2(t_{12}) \) then \( r_s(t_2) > r_s(t_1) > r_s(t_{12}) \).

(2) If \( t_{s2} = \min(t_{12}, t_{22}) = t_{22} \), it is evident that: \( r_s(t_2) > r_s(t_1) > r_s(t_{12}) \).

(c) \( r_s(t) \) is the constant in the interval \([t_{s1}, t_{s2}]\). It then is easy to prove that \( r_s(t_1) = \lambda_1 + \lambda_2 \), for \( t_1 \in [t_{s1}, t_{s2}] \).

Step 2. The same approach gets the results.

**Appendix C**

**Proof of Theorem 1.** Step 1. \( \frac{d}{d\tau} C(T, \tau) < 0 \) for \( \tau = 0 \), and \( \frac{d}{d\tau} C(T, \tau) > 0 \) for \( \tau > t_{m1} \), then the equation \( \frac{d}{d\tau} C(T, \tau) = \frac{d}{d\tau} v(\tau) + \frac{d}{d\tau} k(T, \tau) \) has a solution \( \tau = \tau^* \) and \( \tau^* \epsilon [0, t_{m1}] \).

(1) Under non-renewing condition. From equation (16) \( \frac{d}{d\tau} C(T, \tau) < 0 \) for \( \tau = 0 \), we can get the result

\[
r_2(0) > (r_1(0)(c_{01} + c_1) \cdot [1 + V_1(T)] + c_2[2 + V_1(T)] + (c_3 + c_4)r_2(T)
\]

\[
+ (c_{01} + c_1 + c_4) \frac{d}{d\tau} V_1(T \tau=0) \bigg) (c_4)^{-1}.
\]

(2) Under renewing condition. From equation (20) \( \frac{d}{d\tau} C(T, \tau) < 0 \) for \( \tau = 0 \), we can get the result

\[
r_1(0) > \frac{[c_0 + c_4]r_1(T) + c_2]}{c_4 + F_1(T)(c_0 + c_1)} \bigg) (c_4)^{-1}.
\]

**References**


