ON LINEAR TIME-VARYING SYSTEM CHARACTERIZATIONS

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ABSTRACT

We outline the Laplace transform approach for obtaining the response of linear time-varying (LTV) networks and systems. Systems with variable parameters are widely used in the automatic control and communications fields. It is shown that the frequency-domain representation of such systems using extensions of the Laplace transform known as the iterated and the two-dimensional Laplace transform (2DLT) is feasible. The frequency-domain characterization has a large significant advantage over the time-domain representation.

Index Terms— Time-varying circuits, system analysis and design, linear systems, transfer functions

1. INTRODUCTION

With the advent of digital signal processing we witnessed a rapid growth in application of time-varying systems as well as a notable shift in emphasis from the continuous-time to the discrete-time representation. Although the digital system realization had a profound impact on constructing a model of physical dynamic systems, but has caused decreased interest in resolving fundamental problems associated with analog systems, such as the frequency-domain analysis and synthesis, stability, and the filtering and prediction. The outstanding problem is that of system characterization, which in any case is a basic prerequisite for analysis and synthesis by either computer-aided or analytical methods. In the past decades, a large number of textbooks and works have been published on the theory and application of the transform methods for signals and systems; but works on the application of two- and multidimensional Laplace transform have been dormant, or at least rare, in the literature. In addition, the birth and growth of digital filters and systems have also had an inverse impact on the development of analog system characterizations.

Considering this background, the authors feel the necessity to attempt one more time to lay down the fundamental concept for transformation of variable networks and systems [1-2]. This paper is intended to present the extension of Laplace transform - as the most basic transformation known insofar - to include characterization of LTV systems the same as LTI (linear time-invariant) systems.

2. TIME CHARACTERIZATION

It is customary to characterize the relation between the input and output of a dynamic time-varying system based on the use of differential equations. In this paper, we use a different approach. We examine the fundamental mathematical relation between the input and output of a single-input single-output (SISO) system of finite order given by a time function \( h(t) \), and represent it as a generalized delay system. This time-varying system represented by its input-output linear relationship

\[
y(t, \tau) = h^{-1}(\tau)x(t)u(t, \tau).
\]

(1)

where \( x(t) \) denotes the input to the system and \( y(t, \tau) \) is the response of the system (at rest) to \( x(t) \) and \( u(t, \tau) \) is a unit-step function in the \( (t, \tau) \) - plane defined as

\[
u(t, \tau) = \begin{cases} 
1 & \text{for all } t, \tau \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

(2)

Notice that based on this definition \( u(t, \tau) \) can readily be written as \( u(t, \tau) = u(t)u(\tau) \). In system representation (1) we made no distinction between the input and system function. In other words, \( x(t) \) and \( h(\tau) \) are interchangeable. Using relationship (1) the common product of two functions of \( t \) can operationally be written as

\[
y(t,t) = h^{-1}(t)x(t)u(t,t),
\]

(3)

with understanding that (3) is valid for all \( t \geq 0 \) only and \( y(t) = 0 \) for \( t < 0 \). With no loss of generality, the response (3) may be written as

\[
y(t) = e^{-\ln h(t)}x(t)u(t),
\]

(4)

where it is further assumed that the system function \( h(t) \) is bounded by a finite number \( M \); i.e., \( |h(t)| \leq M \) for all
$0 \leq t < \infty$. If $h(t)$ is a (piecewise) continuous function of $t$, its first- and higher-order derivatives exist. Thus, Eq. (4) can be written equivalently as

$$y(t) = e^{-g(t)} x(t) u(t)$$  \hspace{1cm} (5)$$

where $h'(.)$ is the first derivative of $h(.)$. Furthermore, we take $t_o = 0$ and may assume $\lim_{t \to 0} h(t_o) = 1$. Eq. (5) describes the relation between the input and output of a LTV operational form as variable digital filters derived from analog prototypes [3-4].

The relationship (5) may be written more compactly in an operational form as

$$y(t) = e^{-g(t)} x(t) u(t).$$  \hspace{1cm} (6)$$

With the understanding that (6) is simply an abbreviation of (5), representing a generalized delay operator. The significance of (6) is that it characterizes a general LTV system as a variable-delay system. This representation is said, in general, to be reciprocal in the sense that the input and output can be interchanged. Thus, $g(t)$ might be either a positive or negative quantity. The product of two operators $\exp[-g_2(.)]$ and $\exp[-g_1(.)]$ is an operator $\exp[-g_3(.)]$ where

$$g_3(t) = g_2(t) + g_1(t)$$  \hspace{1cm} (7)$$

such that the result of operating with $\exp[-g_2(.)]$ on $\exp[-g_1(.)] x(t)$ is identical with that of operating with $\exp[-g_3(.)]$ on $x(t)$ [5]. This approach yields the system response in terms of a product of two or more generalized delay operators, such that each of them represents a part of the overall system transition.

### 2.1. LTI System

Consider a basic LTI element $N$. Let $h(t) = h$, then the input-output relationship for this LTI system $N$ is

$$y(t, \tau) = h x(t) u(t) u(\tau).$$  \hspace{1cm} (8)$$

That means the system $N$ is turned-on at $\tau = 0$. The impulse response for this first-order system $N$ is

$$D_1 y(t, \tau) = h [D x(t) u(t) + x(t) \delta(t)] u(\tau) \hspace{1cm} (9 \text{-a})$$

$$D_1 y(t, \tau) = h x(t) u(t) \delta(\tau) \hspace{1cm} (9 \text{-b})$$

where $D_x(.)$ and $D_t(.)$ denote the derivatives with respect to $t$ and $\tau$, respectively. For example, for a LTI resistor $R$, the impulse response obtained from (8) is

$$v(t, \tau) = R i(t) u(t) \delta(\tau).$$  \hspace{1cm} (10)$$

Equations (9-a) and (9-b) yield the voltage-current relationships for a LTI inductor $L$ as

$$v_i(t, \tau) = L \frac{dt}{dt} u(t) + i(t) \delta(t) \hspace{1cm} u(\tau), \hspace{1cm} (11 \text{-a})$$

$$v_r(t, \tau) = L i(t) u(t) \delta(\tau). \hspace{1cm} (11 \text{-b})$$

Similarly, a LTI capacitor may be expressed by

$$i_r(t, \tau) = C \frac{dv}{dt} u(t) + v(t) \delta(t) \hspace{1cm} u(\tau), \hspace{1cm} (12 \text{-a})$$

$$i_c(t, \tau) = C v(t) u(t) \delta(\tau). \hspace{1cm} (12 \text{-b})$$

### 2.2. LTV System

Consider a two-terminal variable network element $N$. Let $r(t), c(t)$, and $l(t)$ represent a variable resistor, capacitor, and inductor in the time domain, respectively. The input-output relationships for these elements in terms of their corresponding electrical quantities are

$$v(t, \tau) = r(t) i(t) u(t) u(\tau) \hspace{1cm} (13 \text{-a})$$

$$\varphi(t, \tau) = l(t) i(t) u(t) u(\tau) \hspace{1cm} (13 \text{-b})$$

$$q(t, \tau) = c(t) v(t) u(t) u(\tau) \hspace{1cm} (13 \text{-c})$$

where $v(t, \tau), \varphi(t, \tau)$, and $q(t, \tau)$ are the voltage, flux, and charge across the terminals of $N$ and $i(t)$ is the input current flowing through $N$. This characterization is dependent on an assumed mode of excitation and response for a general network. It is valid only for the simple case that the excitation is a function of a single time variable $t$. In this case, the voltage-current relationships for a variable inductor are

$$v_r(t, \tau) = l(t) \frac{dt}{dt} u(t) + i(t) \delta(t) \hspace{1cm} u(\tau), \hspace{1cm} (14 \text{-a})$$

$$v_c(t, \tau) = i(t) \frac{d}{d\tau} u(\tau) + l(\tau) \delta(\tau) \hspace{1cm} u(t). \hspace{1cm} (14 \text{-b})$$

Similar relationships can be written for a variable capacitor.

The hybrid situation where the input is a bivariate function of the input time $t$ and the system time $\tau$ is referred to by the term mixed-excitation. A single-input single-output LTV system element $N$ under mixed-excitation is characterized by a more general relationship

$$y(t, \tau) = h(t, \tau) x(t, \tau) u(t, \tau)$$  \hspace{1cm} (15)$$

where $h(t, \tau)$ is a circularly symmetric function; i.e., $h(t, \tau) = h(\tau, t)$ [2]. If this system element is turned-on at $t = \tau$, the response is the commonly known impulse response given by

$$y(t, \tau) = h(t, \tau).$$  \hspace{1cm} (16)$$

From an operational point of view, one may write a circularly symmetric system function $h(t, \tau)$ as

$$h(t, \tau) = h\left(\sqrt{t^2 + \tau^2}\right) \delta(t, \tau) u(t, \tau).$$  \hspace{1cm} (17)$$
where it can readily be shown that \( \delta(t, \tau) = \delta(t)\delta(\tau) \).

Based on this discussion, only system responses to a zero-, a first-, and a second-order delay operator are needed to be determined since higher-order delays can be factored into products of lower-order delays.

It can readily be seen that the above operational relationships are, in general, Laplace-transformable subject to convergence conditions. We shall proceed to obtain these Laplace transforms in the next section.

### 3. FREQUENCY CHARACTERIZATION

To formally derive the one-sided (unilateral) Laplace transform of \((1)\), we consider the system response to the unit-impulse \( \delta(t - \tau) \) as

\[
y(t, \tau) = h(\tau)\delta(t - \tau)u(t, \tau) \tag{18}
\]

where \( t \) is the observation time of input, \( \tau \) is the application time of system, and \( \delta(t - \tau) \) is a unit impulse occurring at \( t = \tau \). The ordinary (one-dimensional) Laplace transform of \((18)\) is obtained as

\[
\int_{0}^{\infty} y(t, \tau) e^{-st} dt = h(\tau)e^{-st} u(\tau). \tag{19}
\]

Multiplying both sides by \( e^{st} \), we obtain

\[
Y(s; \tau) \equiv \int_{0}^{\infty} y(t, \tau) e^{-st} dt = h(\tau)u(\tau), \tag{20}
\]

where the lower limit of the integral is actually \( \tau \) since the impulse response of a nonanticipative system is zero prior to application of the input; i.e., \( y(t, \tau) \equiv 0 \) for \( t < \tau \). \( Y(s; \tau) \), which is the ratio of the system output to the exponential input \( e^{st} \), is commonly referred to as the Zadeh’s transformation \([6]\). More specifically, using \((20)\), the system response at \( \tau = 0 \) is

\[
Y(s) = \int_{0}^{\infty} y(t) e^{-st} dt = h(0)u(0), \tag{21}
\]

where, for convenience, \( h(0) \) is assumed to be unity. The input \( x(t) \) is assumed to be zero for \( t < 0 \) and applied to the system at \( t \geq 0 \). The lower limit of integration in \((21)\) takes on the usual value of \( 0^- \), as is customary for one-sided Laplace transform.

It is seen that the integral of \((20)\) should be evaluated in the region \( t \geq \tau \) in the \((t, \tau)\)-plane. Considering this fact in evaluating the integral of \((20)\), it can be written as:

\[
\int_{0}^{\infty} y(t, \tau) e^{-st} dt = h(\tau)u(\tau), \tag{22}
\]

by a simple change of variable \( t - \tau \to t \) \([7]\). Taking a second Laplace transform of \((22)\), we obtain

\[
H(s) = \int_{0}^{\infty} \int_{0}^{\infty} h(t, \tau) e^{-st(t+\tau)} dt \, d\tau, \tag{23}
\]

where \( h(t, \tau) \) is the impulse response corresponding to the situation where both observation time and application time are set equal to zero; i.e., \( t = \tau = 0 \). This iterated Laplace transform is a special form of the more general two-dimensional Laplace transform \((2DLT)\).

### 4. FIRST-ORDER SYSTEM CHARACTERIZATION

The major advantage of characterizing the response of a LTV network by \((1)\) is that it allows application of 2DLT for analysis and synthesis problems in the frequency-domain. In addition, this will treat the LTI and LTV systems in a similar manner.

Application of the iterated Laplace transform to a variable resistor whose voltage-current relationship is given by \((13-a)\) yields

\[
L_2 \{ v(t, \tau) \} = R_1(s)I_1(s), \tag{24-a}
\]

where \( L_2 \{ \} \) denotes, in general, a two-dimensional Laplace transformation. In \((24)\), \( L_2 \{ \} \) represents the co-called iterated transformation. The subscript 1 in \( R_1(s) \) and \( I_1(s) \) indicates the usual one-dimensional Laplace transform. We choose to indicate a 2DLT by a subscript 2 in the remainder of this paper. Thus, \((24-a)\) can symbolically be written as

\[
V_2(s) = R_1(s)I_1(s). \tag{24-b}
\]

It should be noted that in derivation of \((24-a)\) it is assumed that the frequency \( s \) is associated with the observation time \( t \). In that sense, there is a distinction between the observation and application time of a system element. Using the input time as the reference, the voltage-current relationships for variable inductor, given by \((14-a)\) and \((14-b)\), can easily be obtained as

\[
V_2(s) = sL_1(s)I_1(s), \tag{25}
\]

and for a variable capacitor as

\[
I_2(s) = sC_1(s)V_1(s). \tag{26}
\]
It is seen that \( R_1(s), L_1(s), \) and \( C_1(s) \) given by (24), (25), and (26) are essentially the system functions connecting the corresponding electrical quantities. It is seen that the frequency-domain representation of a LTI resistor \( r(\tau) = R \) is \( V_2(s) = R \frac{I_1(s)}{s} \), which is the characterization of a fixed resistor \( R \) turned-on at \( t = \tau \) in a circuit. Furthermore, it is seen that, through an iterated Laplace transform, the product of two time-functions that exist only for \( t, \tau \geq 0 \) is transformed into the product of the same two functions in the frequency domain.

5. HIGHER-ORDER SYSTEM CHARACTERIZATION

In practice, in analysis and synthesis of more general systems, we deal with hybrid situations, where the input-output relationship is given by (15). In this case, the frequency characterization can be obtained using the general 2DLT approach. The application of 2DLT is further discussed in [7].

6. CONCLUDING REMARKS

The approach used in this paper is based on the use of operational analysis of LTV system to show that Laplace transform techniques are applicable to variable systems. We showed that the time-varying differential equations describing a LTV system element can be transformed into the frequency-domain using the iterated Laplace transform.

7. REFERENCES


