ON CLASSICAL SOLUTIONS OF THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH VACUUM

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Abstract. In this paper, we consider the 3-D compressible isentropic MHD equations with infinity electric conductivity. The existence of unique local classical solutions with vacuum is firstly established when the initial data is arbitrarily large, contains vacuum and satisfies some initial layer compatibility condition. The initial mass density needs not be bounded away from zero, it may vanish in some open set or decay at infinity. Moreover, we prove that the $L^\infty$ norm of the deformation tensor of velocity gradients controls the possible blow-up (see [16][22]) for classical (or strong) solutions, which means that if a solution of the compressible MHD equations is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by the losing the bound of the deformation tensor as the critical time approches. Our result (see (1.12)) is the same as Ponce’s criterion for 3-D incompressible Euler equations [15] and Huang-Li-Xin’s blow-up criterion for the 3-D compressible Navier-stokes equations [9].

1. Introduction

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. In 3-D space, the compressible isentropic magnetohydrodynamic equations in a domain $\Omega$ of $\mathbb{R}^3$ can be written as

\[
\begin{align*}
H_t - \text{rot}(u \times H) &= -\text{rot}\left(\frac{1}{\sigma}\text{rot}H\right), \\
\text{div} H &= 0, \\
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div} \mathbb{T} + \mu_0 \text{rot}H \times H. 
\end{align*}
\]

(1.1)

In this system, $x \in \Omega$ is the spatial coordinate; $t \geq 0$ is the time; $H = (H^{(1)}, H^{(2)}, H^{(3)})$ is the magnetic field; $0 < \sigma \leq \infty$ is the electric conductivity coefficient; $\rho$ is the mass...
density; \( u = (u^{(1)}, u^{(2)}, u^{(3)}) \) ∈ \( \Omega \) is the velocity of fluids; \( P \) is the pressure law satisfying
\[
P = A\rho^\gamma, \quad \gamma > 1,
\]
where \( A \) is a positive constant and \( \gamma \) is the adiabatic index; \( \nabla \) is the stress tensor given by
\[
\nabla = 2\mu D(u) + \lambda \text{div} u \mathbb{I}, \quad D(u) = \frac{\nabla u + (\nabla u)^\top}{2},
\]
where \( D(u) \) is the deformation tensor, \( \mathbb{I} \) is the \( 3 \times 3 \) unit matrix, \( \mu \) is the shear viscosity coefficient, \( \lambda \) is the bulk viscosity coefficient, \( \mu \) and \( \lambda \) are both real constants,
\[
\mu > 0, \quad \lambda + \frac{2}{3} \mu \geq 0,
\]
which ensures the ellipticity of the Lamé operator. Although the electric field \( E \) doesn’t appear in system (1.1), it is indeed induced according to a relation \( E = -\mu_0 u \times H \) by moving the conductive flow in the magnetic field.

However, in this paper, when \( \sigma = +\infty \), system (1.1) can be written into
\[
\begin{align*}
H_t - \text{rot}(u \times H) &= 0, \\
\text{div} H &= 0, \\
\rho_t + \text{div}(\rho u) &= 0, \\
(pu)_t + \text{div}(pu \otimes u) + \nabla P &= \text{div} \nabla + \mu_0 \text{rot} H \times H
\end{align*}
\]
with initial-boundary conditions
\[
(H, \rho, u)|_{t=0} = (H_0(x), \rho_0(x), u_0(x)), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0,
\]
\[
(H(t, x), \rho(t, x), u(t, x), P(t, x)) \to (0, \overline{\rho}, 0, \overline{P}) \quad \text{as} \quad |x| \to \infty, \quad t > 0,
\]
where \( \overline{\rho} \geq 0 \) and \( \overline{P} = A\overline{\rho}^\gamma \) are both constants, and \( \Omega \) can be a bounded domain in \( \Omega \) with smooth boundary or the whole space \( \mathbb{R}^3 \). We have to point out that, if \( \Omega \) is a bounded domain (or \( \mathbb{R}^3 \)), then the condition (1.7) at infinity (or the boundary condition in (1.6) respectively) should be neglected.

Throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:
\[
D^{k,r} = \{ f \in L^1_{\text{loc}}(\Omega) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty \}, \quad D^k = D^{k,2},
\]
\[
D^0_\Omega = \{ f \in L^2(\Omega) : |f|_{D^1} = |\nabla f|_{L^2} < \infty \text{ and } f|_{\partial\Omega} = 0 \},
\]
\[
||f||_s = ||f||_{H^s(\Omega)}, \quad ||f||_p = ||f||_{L^p(\Omega)}, \quad ||f||_{D^k} = ||f||_{D^k(\Omega)}, \quad A : B = (a_{ij}b_{ij})_{3 \times 3},
\]
\[
f \cdot \nabla g = \sum_{i=1}^3 f_i \partial_i g, \quad f \cdot (\nabla g) = (\sum_{i=1}^3 f_i \partial_1 g_i, \sum_{i=1}^3 f_i \partial_2 g_i, \sum_{i=1}^3 f_i \partial_3 g_i)^\top,
\]
where \( f = (f_1, f_2, f_3)^\top \in \mathbb{R}^3 \) or \( f \in \mathbb{R} \), \( g = (g_1, g_2, g_3)^\top \in \mathbb{R}^3 \) or \( g \in \mathbb{R} \), \( X \) is some Sobolev space, \( A = (a_{ij})_{3 \times 3} \) and \( B = (b_{ij})_{3 \times 3} \) are both \( 3 \times 3 \) matrixes. A detailed study of homogeneous Sobolev space may be found in [6].

As has been observed in [5], which proved the existence of unique local strong solution with initial vacuum, in order to make sure that the Cauchy problem or IBVP (1.5)-(1.7)
with vacuum is well-posed, the lack of a positive lower bound of the initial mass density \( \rho_0 \) should be compensated with some initial layer compatibility condition on the initial data \((H_0, \rho_0, u_0, P_0)\). For classical solution, it can be shown as

**Theorem 1.1.** Let constant \( q \in (3, 6] \). If the initial data \((H_0, \rho_0, u_0, P_0)\) satisfies

\[
(H_0, \rho_0 - \mathbf{p}, P_0 - \mathbf{P}) \in H^2 \cap W^{2,q}, \quad \rho_0 \geq 0, \quad u_0 \in D_0^1 \cap D^2,
\]

and the compatibility condition

\[
Lu_0 + \nabla P_0 - \text{rot} H_0 \times H_0 = \sqrt{\rho_0} g_1
\]

for some \( g_1 \in L^2 \), where \( L \) is the Lamé operator defined via

\[
Lu = -\mu \Delta u - (\lambda + \mu) \nabla \text{div} u.
\]

Then there exists a small time \( T^* \) and a unique solution \((H, \rho, u, P)\) to IBVP \((\ref{eq:ibvp_1}), (\ref{eq:ibvp_2})\) satisfying

\[
(H, \rho - \mathbf{p}, P - \mathbf{P}) \in C([0, T^*]; H^2 \cap W^{2,q}),
\]

\[
u \in C([0, T^*]; D_0^1 \cap D^2) \cap L^2([0, T^*]; D^3) \cap L^{p_0}([0, T^*]; D^{3,q}), \quad u_t \in L^2([0, T^*]; D_0^1),
\]

\[
\sqrt{\rho} u_t \in L^\infty([0, T^*]; L^2), \quad t^{\frac{3}{2}} u_t \in L^\infty([0, T^*]; D^3), \quad t^{\frac{5}{2}} \sqrt{\rho} u_{tt} \in L^2([0, T^*]; L^2),
\]

\[
tu_t \in L^\infty([0, T^*]; D^3), \quad tu_{tt} \in L^2([0, T^*]; D_0^1), \quad t \sqrt{\rho} u_{tt} \in L^\infty([0, T^*]; L^2)
\]

where \( p_0 \) is a constant satisfying \( 1 \leq p_0 \leq \frac{4q}{3q-6} \in (1, 2) \).

**Remark 1.1.** The solution we obtained in Theorem \((\ref{thm:ibvp_1})\) becomes a classical one for positive time. Some similar results have been obtained in \([5, 12]\), which give the local existence of strong solutions. So, the main purpose of this theorem is to give a better regularity for the solutions obtained in \([5, 12]\) when the initial mass density is nonnegative.

Though the smooth global solution near the constant state in one-dimensional case has been studied in \([10]\), however, in 3-D space, the non-global existence has been proved for the classical solution to isentropic magnetohydrodynamic equations in \([16]\) as follows:

**Theorem 1.2.** \([16]\) Assume that \( \gamma \geq \frac{6}{5} \), if the momentum \( \int_\Omega \rho u dx \neq 0 \), then there exists no global classical solution to \((\ref{eq:ibvp_1}), (\ref{eq:ibvp_2})\) with conserved mass and total energy.

So, naturally, we want to make sure the mechanism of blow-up and the structure of possible singularities: what kinds of singularities will form in finite time and what is the main mechanism of possible breakdown of smooth solutions for the 3-D compressible MHD equations? Therefore, it is an interesting question to ask whether the same blow-up criterion in terms of \( D(u) \) in \([9, 15]\) still holds for the compressible MHD equations or not. However, the similar result has been obtained in Xu-Zhang \([21]\) for strong solutions obtained in \([5]\), which is in terms of \( \nabla u \):

\[
\lim \sup_{T \to T^*} ||| \nabla u |||_{L^1([0, T^*]; L^\infty(\Omega))} = \infty.
\]

Based on a subtle estimate for the magnetic field, our main result in this paper answered this question for classical (or strong) solutions positively, which can be shown as
Theorem 1.3 (Blow-up criterion for the IBVP \((1.5)-(1.7)\).
Assume that \(\Omega\) is a bounded domain and the initial data \((H_0, \rho_0, u_0, P_0)\) satisfies \((1.8)-(1.9)\). Let \((H, \rho, u, P)\) is a classical solution to IBVP for \((1.5)-(1.7)\). If \(0 < T < \infty\) is the maximal time of existence, then
\[
\limsup_{T \to T} |D(u)|_{L^1([0,T];L^\infty(\Omega))} = \infty.
\tag{1.12}
\]
Moreover, our blow-up criterion also holds for the strong solutions obtained in \([5]\).

Remark 1.2. When \(H \equiv 0\) in 3-D space, the existence of unique local strong solution with vacuum has been solved by many papers, we refer to readers to \([2,3,1]\). Huang-Li-Xin obtained the well-posedness of classical solutions with small energy but possibly large oscillations and vacuum for Cauchy problem \([7]\) or IBVP \([8]\).

However, for compressible non-isentropic Navier-Stokes equations, the finite time blow-up has been proved in Olga \([17]\) for classical solutions \((\rho, u, S)\) (S is the entropy) with highly decreasing at infinity for the compressible non-isentropic Navier-stokes equations, but the local existence for the corresponding smooth solution is still open.

Recently, Xin-Yan \([23]\) showed that if the initial vacuum only appear in some local domain, the smooth solution \((\rho, \theta, u)\) to the Cauchy problem \((1.5)-(1.7)\) will blow-up in finite time regardless of the size of initial data, which has removed the key assumption that the vacuum must appear in the far field in \([22]\).

Sun-Wang-Zhang \([20,21]\) established a Beal-Kato-Majda blowup criterion in terms of the upper bound of density for the strong solution with vacuum in 3-D or 2-D space, which is more weaker than the blow-up criterions obtained in \([9,15]\). Then our result can not replace \(\int_0^T |D(u)|_{\infty} dt\) by \(|\rho|_\infty\) because of the coupling of \(u\) and \(H\) in magnetic equation and the lack of smooth mechanism of \(H\).

Moreover, these results presented above are essentially dependent of the strong ellipticity of Lamé operator. Compared with Euler equations \([14]\), the velocity \(u\) of fluids satisfies \(Lu_0 = 0\) in the vacuum domain naturally due to the constant viscosity coefficients which makes sure that \(u\) is well defined in the vacuum points without other assumptions as \([14]\).

Recently, Li-Pan-Zhu \([11]\) proved the local existence of regular solutions for the 2-D Shallow water equations with \(T = \rho \nabla u\) when initial mass density decays to zero, and the corresponding Beal-Kato-Majda blow-up criterion is also obtained.

The rest of this paper is organized as follows. In Section 2, we give some important Lemmas which will be used frequently in our proof. In Section 3, via establishing a priori estimate (for the approximation solutions) which is independent of the lower bound of the initial mass density \(\rho_0\), we can obtain the existence of unique local classical solution by the approximation process from non-vacuum to vacuum. In Section 4, we give the proof for the blow-up criterion \((1.12)\) for the classical solutions obtained in Section 3. Firstly in Section 4.1, via assuming that the opposite of \((1.12)\) holds, we show that the solution in \([0, \overline{T}]\) has the regularity that the strong solution has to satisfy obtained in \([5]\). Then secondly in Section 4.2, based on the estimates shown in Section 4.1, we improve the regularity of \((H, \rho, u, P)\) to make sure that it is also a classical one in \([0, \overline{T}]\), which contradicts our assumption.
2. Preliminary

Now we give some important Lemmas which will be used frequently in our proof.

**Lemma 2.1.** [13] Let constants $l$, $a$ and $b$ satisfy the relation $\frac{1}{l} = \frac{1}{a} + \frac{1}{b}$ and $1 \leq a$, $b$, $l \leq \infty$. \( \forall s \geq 1 \), if \( f, g \in W^{s,a}(\Omega) \cap W^{s,b}(\Omega) \), then we have
\[
|D^s(fg) - fD^s g|_l \leq C_s(|\nabla f|_a|D^{s-1}g|_b + |D^s f|_b|g|_a),
\]
(2.1)
\[
|D^s(fg) - fD^s g|_l \leq C_s(|\nabla f|_a|D^{s-1}g|_b + |D^s f|_a|g|_b),
\]
(2.2)
where \( C_s > 0 \) is a constant only depending on \( s \).

The proof can be seen in Majda [13], here we omit it. The following one is some Sobolev inequalities obtained from the well-known Gagliardo-Nirenberg inequality:

**Lemma 2.2.** For \( n \in (3,\infty) \), there exists some generic constant \( C > 0 \) that may depend \( n \) such that for \( f \in D^0_0(\Omega) \), \( g \in D^0_0 \cap D^2(\Omega) \) and \( h \in W^{1,n}(\Omega) \), we have
\[
|f|_6 \leq C|f|_{D^0_0}, \quad |g|_\infty \leq C|g|_{D^0_0 \cap D^2}, \quad |h|_\infty \leq C\|h\|_{W^{1,n}}.
\]
(2.3)

The next lemma is important in the derivation of our local a priori estimate for the higher order term of \( u \), which can be seen in the Remark 1 of [1].

**Lemma 2.3.** If \( h(t,x) \in L^2(0,T;L^2) \), then there exists a sequence \( s_k \) such that
\[
s_k \to 0, \quad \text{and} \quad s_k|h(s_k, x)|^2 \to 0, \quad \text{as} \quad k \to \infty.
\]

Based on Harmonic analysis, we introduce a regularity estimate result for Lamé operator
\[
-\mu \Delta u - (\mu + \lambda)\nabla \text{div} u = Lu = F \quad \text{in} \quad \Omega, \quad u \to 0 \quad \text{as} \quad |x| \to \infty.
\]
(2.4)
We define \( u \in D^{1,q}_0(\Omega) \) means that \( u \in D^{1,q}(\Omega) \) with \( u|_{\partial \Omega} = 0 \).

**Lemma 2.4.** [19] Let \( u \in D^{1,l}_0 \) with \( 1 < l < \infty \) be a weak solution to system (2.4), if \( \Omega = \mathbb{R}^3 \), we have
\[
|u|_{D^{k+2,l}(\mathbb{R}^3)} \leq C|F|_{D^{k,l}(\mathbb{R}^3)};
\]
if \( \Omega \) is a bounded domain with smooth boundary, we have
\[
|u|_{D^{k+2,l}(\Omega)} \leq C\left(|F|_{D^{k,l}(\Omega)} + |u|_{D^{l}_0(\Omega)}\right),
\]
where the constant \( C \) depending only on \( \mu, \lambda \) and \( l \).

**Proof.** The proof can be obtained via the classical estimates from Harmonic analysis, which can be seen in [2] [19] or [20].

We also show some results obtained via the Aubin-Lions Lemma.

**Lemma 2.5.** [18] Let \( X_0, X \) and \( X_1 \) be three Banach spaces with \( X_0 \subset X \subset X_1 \). Suppose that \( X_0 \) is compactly embedded in \( X \) and that \( X \) is continuously embedded in \( X_1 \).

\[ I \] Let \( G \) be bounded in \( L^p(0,T; X_0) \) where \( 1 \leq p < \infty \), and \( \frac{\partial G}{\partial t} \) be bounded in \( L^1(0,T; X_1) \). Then \( G \) is relatively compact in \( L^p(0,T; X) \).

\[ II \] Let \( F \) be bounded in \( L^\infty(0,T; X_0) \) and \( \frac{\partial F}{\partial t} \) be bounded in \( L^l(0,T; X_1) \) with \( l > 1 \). Then \( F \) is relatively compact in \( C(0,T; X) \).
Finally, for \((H, u) \in C^2(\Omega)\), there are some formulas based on \(\text{div}H = 0\):
\[
\begin{aligned}
\text{rot}(u \times H) &= (H \cdot \nabla)u - (u \cdot \nabla)H - H\text{div}u, \\
\text{rot}H \times H &= \text{div}(H \otimes H - \frac{1}{2}|H|^2 I_3) = -\frac{1}{2}\nabla|H|^2 + H \cdot \nabla H.
\end{aligned}
\]  
(2.5)

3. Well-posedness of classical solutions

In order to prove the local existence of classical solutions to the original nonlinear problem, we need to consider the following linearized problem:
\[
\begin{aligned}
H_t + v \cdot \nabla H + (\text{div}I_3 - \nabla v)H &= 0 \quad \text{in } (0, T) \times \Omega, \\
\text{div}H &= 0 \quad \text{in } (0, T) \times \Omega, \\
\rho_t + \text{div}(\rho v) &= 0 \quad \text{in } (0, T) \times \Omega, \\
\rho u_t + \rho v \cdot \nabla v + \nabla P + Lu &= \mu_0\text{rot}H \times H \quad \text{in } (0, T) \times \Omega, \\
(H, \rho, u)|_{t=0} &= (H_0(x), \rho_0(x), u_0(x)) \quad \text{in } \Omega, \\
(H, \rho, u, P) &\to (0, \overline{\rho}, 0, \overline{\mu}) \quad \text{as } |x| \to \infty, \quad t > 0,
\end{aligned}
\]  
(3.1)

where \((H_0(x), \rho_0(x), u_0(x))\) satisfies (1.8)-(1.9) and \(v(t, x) \in \mathbb{R}^3\) is a known vector \(v \in C([0, T]; D^1_v) \cap L^2([0, T]; D^3_v) \cap L^2([0, T]; D^3_v), v_t \in L^2([0, T]; D^1_v), \)
\[
\begin{aligned}
t \frac{1}{2}v &\in L^\infty([0, T]; D^3), \quad t \frac{1}{2}v_t \in L^\infty([0, T]; D^1_v) \cap L^2([0, T]; D^2), \\
vt &\in L^\infty([0, T]; D^2), \quad tv \in L^\infty([0, T]; D^3), \quad tv_t \in L^2([0, T]; D^1_v), \quad v(0, x) = u_0.
\end{aligned}
\]  
(3.2)

3.1. Unique solvability of (3.1) away from vacuum.

First we give the following existence of classical solution \((H, \rho, u)\) to (3.1) by the standard methods at least for the case that the initial mass density is away from vacuum.

**Lemma 3.1.** Assume in addition to (1.8)-(1.9) that \(\rho_0 \geq \delta\) for some constant \(\delta > 0\). Then there exists a unique classical solution \((H, \rho, u)\) to (3.1) such that
\[
\begin{aligned}
(H, \rho - \overline{\rho}, P - \overline{P}) &\in C([0, T]; H^2 \cap W^{2,3}), \quad (H_t, \rho_t, P_t) \in C([0, T]; H^1), \\
t \frac{1}{2}(H_t, \rho_t, P_t) &\in L^\infty([0, T]; D^1_v) \cap L^2([0, T]; D^3) \cap L^p([0, T]; D^3), \\
u_t &\in L^2([0, T]; D^1_v) \cap L^\infty([0, T]; D^3), \\
t \frac{1}{2}u_t &\in L^\infty([0, T]; D^1_v), \quad t \frac{1}{2}u_{tt} \in L^2([0, T]; D^2), \quad tu \in L^\infty([0, T]; D^3), \\
tu_t &\in L^\infty([0, T]; D^2), \quad tu_{tt} \in L^2([0, T]; D^1_v) \cap L^\infty([0, T]; D^2), \quad tu_{tt} \in L^\infty([0, T]; H^{-1}),
\end{aligned}
\]
and \(\rho \geq \tilde{\delta}\) on \([0, T] \times \mathbb{R}^3\) for some positive constant \(\tilde{\delta}\).

**Proof.** Firstly, we observe the magnetic equations (3.1), it has the form
\[
H_t + \sum_{j=1}^3 A_j \partial_j H + BH = 0,
\]  
(3.3)
where $A_j = v_j I_3$ $(j = 1, 2, 3)$ are symmetric and $B = \text{div} v I_3 - \nabla v$. According to the regularity of $v$ and the standard theory for positive and symmetric hyperbolic system, we easily have the desired conclusions.

Secondly, the existence and regularity of a unique solution $\rho$ to (3.1) can be obtained essentially according to Lemma 1 in [4]. Due to pressure $P$ satisfies the following problem

$$P_t + v \cdot \nabla P + \gamma P \text{div} v = 0, \quad P_0 - \mathbf{P} \in H^2 \cap W^{2,q},$$

(3.4)

so we easily have the same conclusions for $P$ via the similar argument as $\rho$.

Finally, the momentum equations (3.1) can be written into

$$\rho u_t + Lu = -\nabla P - \rho v \cdot \nabla v + \mu_0 \text{rot} H \times H,$$

(3.5)

then from Lemma 3 in [4], the desired conclusions is easily obtained. \hfill \square

3.2. A priori estimate to the linearized problem away from vacuum.

Now we want to get some a priori estimate for the classical solution $(H, \rho, u)$ to (3.1) obtained in Lemma 3.1, which is independent of the lower bound of the initial mass density $\rho_0$. For simplicity, we first fix a positive constant $c_0$ sufficiently large that

$$2 + \mathbf{P} + \| (\rho_0 - \mathbf{P}, P_0 - \mathbf{P}, H_0) \|_{H^2 \cap W^{2,q}} + |u_0|_{D^3_0 \cap D^2} + |g_1|_2 \leq c_0,$$

(3.6)

and

$$\sup_{0 \leq t \leq T^*} |v(t)|_{D^3_0 \cap D^2}^2 + \int_0^{T^*} \left( |v|_{D^3_0}^2 + |v|_{D^3_0}^2 + |v_t|_{D^3_0}^2 \right) dt \leq c_1,$$

$$\text{ess sup}_{0 \leq t \leq T^*} \left( t^2 |v(t)|_{D^3_0}^2 + t |v(t)|_{D^3_0}^2 \right) + \int_0^{T^*} t |v_t|_{D^2}^2 dt \leq c_2,$$

(3.7)

$$\text{ess sup}_{0 \leq t \leq T^*} \left( t^2 |v(t)|_{D^3_0}^2 + t^2 |v(t)|_{D^3_0}^2 \right) + \int_0^{T^*} t^2 |v_t|_{D^3_0}^2 dt \leq c_3$$

for some time $T^* \in (0, T)$ and constants $c_i$'s with $1 < c_0 \leq c_1 \leq c_2 \leq c_3$. Throughout this and next two sections, we denote by $C$ a generic positive constant depending only on fixed constants $\mu$, $\mu_0$, $T$ and $\lambda$.

Now we give some estimates for the magnetic field $H$.

**Lemma 3.2 (Estimates for magnetic field $H$).**

$$\|H(t)\|_{H^2 \cap W^{2,q}}^2 + \|H_t(t)\|^2 \leq Cc_1^3, \quad \int_0^t |H_{tt}|^2 \, ds \leq Cc_2^3, \quad t |H_t(t)|_{D^3_{1,q}}^2 \leq Cc_2^3$$

(3.8)

for $0 \leq t \leq T_1 = \min(T^*, (1 + c_1)^{-1})$.

**Proof.** Firstly, let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ $(|\alpha| \leq 2)$ and $\alpha_i = 0, 1, 2$, differentiating (3.1) $\alpha$ times with respect to $x$, we have

$$D^\alpha H_t + \sum_{j=1}^3 A_j \partial_j D^\alpha H + BD^\alpha H$$

$$=(D^\alpha (BH) - BD^\alpha H) + \sum_{j=1}^3 (D^\alpha (A_j \partial_j H) - A_j \partial_j D^\alpha H) = \Theta_1 + \Theta_2.$$
Then multiplying (3.9) by $rD^a H|D^a H|^r$ ($r \in [2, q]$) and integrating over $\Omega$, we have

$$
\frac{d}{dt}|D^a H|^r \leq \left( \sum_{j=1}^{3} |\partial x_j A_j|_{\infty} + |B|_{\infty} \right) |D^a H|^r + |\Theta_1|^r + |\Theta_2|^r |D^a H|^r.
$$

(3.10)

Secondly, let $l = r = a$, $b = \infty$ and $s = |\alpha| = 1$ in (2.2) of Lemma 2.1 we easily have

$$
|\Theta_1|^r = |D^a (BH) - BD^a H|^r \leq C|\nabla^2 v|^r |H|^r \leq C|\nabla^2 v|^r |H|^r.
$$

(3.11)

let $l = r = a$, $b = \infty$ and $s = |\alpha| = 1$ in (2.2) of Lemma 2.1 we have

$$
|\Theta_1|^r = |D^a (BH) - BD^a H|^r \leq C( |\nabla^2 v|^r |\nabla H|_\infty + |\nabla^3 v|^r |H|_\infty ) \leq C|\nabla^2 v|^r |H|^r.
$$

(3.12)

And similarly, let $b = \infty$, $l = r = a$ and $s = |\alpha| = 1$ in (2.2) of Lemma 2.1 we have

$$
|D^a (A_j \partial_j H) - A_j \partial_j D^a |^r \leq C( |\nabla v|_\infty |\nabla^2 H|^r |\nabla v|_\infty |\nabla H|_\infty ) \leq C|\nabla v|^r |H|^r.
$$

(3.13)

let $a = \infty$, $l = r = b$ and $s = |\alpha| = 1$ in (2.1) of Lemma 2.1 we have

$$
|D^a (A_j \partial_j H) - A_j \partial_j D^a |^r \leq C( |\nabla v|_\infty |\nabla^2 H|^r |\nabla v|_\infty |\nabla H|_\infty ) \leq C|\nabla v|^r |H|^r.
$$

(3.14)

Then combining (3.10)-(3.14), according to Gronwall’s inequality, we have

$$
|H|^r |H|_{L^2} \leq C|H_0|^r |H|_{L^2} \exp \left( \int_0^t |\nabla v|^r |H|^r ds \right) \leq C_{0}.
$$

(3.15)

for $0 \leq t \leq T_1$, where we have used the fact

$$
\int_0^t |v(s)|_{D^3} ds \leq \frac{1}{q_0} \left( \int_0^t |v(s)|^{2q_0} ds \right)^{\frac{1}{q_0}} \leq C_{1}, \quad \text{and}
$$

$$
\int_0^t |\nabla v|^r |H|^r ds \leq t^{\frac{1}{2}} \left( \int_0^t |\nabla v|^r |H|^r ds \right)^{\frac{1}{r}} \leq C_{1} t + (c_1 t)^{\frac{1}{2}} \leq C_{1},
$$

(3.16)

and $\frac{1}{q_0} + \frac{1}{q_0} = 1$. Finally, from the magnetic field equations (3.1):

$$
\dot{H} = -v \cdot \nabla H - (\operatorname{div} I_3 - \nabla v) H,
$$

we quickly get the desired estimates for $H_t$ and $H_{tt}$.

Next we give the estimates for the mass density $\rho$ and pressure $P$.

**Lemma 3.3 (Estimates for the mass density $\rho$ and pressure $P$).**

$$
\| (\rho - \overline{\rho}, P - \overline{P}) (t) \|_{L^2} \leq C_{0}^3,
$$

$$
\int_0^t \| (\rho_t, P_t) \|^2 ds \leq C_{1}^3, \quad t \| (\rho_t, P_t) \|^2 |_{D^1} \leq C_{2}^3
$$

for $0 \leq t \leq T_1 = \min(T^*, C(1 + c_1)^{-1})$. 

\[\square\]
Proof. From (3.13) and the standard energy estimate shown in [3], for \(2 \leq r \leq q\), we have
\[
\|\rho(t) - \overline{\rho}\|_{W^{2,r}} \leq \left( \|\rho_0 - \overline{\rho}\|_{W^{2,r}} + \overline{\rho} \int_0^t \|\nabla v(s)\|_{W^{2,r}} \, ds \right) \exp \left( C \int_0^t \|\nabla v(s)\|_{H^2;\cap W^{2,q}} \, ds \right).
\]
(3.17)
Then from (3.16), the desired estimate for \(\|\rho(t)\|_{H^2;\cap W^{2,q}}\) can be easily obtained via (3.17):
\[
\|\rho(t) - \overline{\rho}\|_{H^2;\cap W^{2,q}} \leq C_0, \quad \text{for } 0 \leq t \leq T_1 = \min(T^*, (1 + c_1)^{-1}).
\]
(3.18)
Secondly, the estimates for \((\rho_t, \rho u)\) follows immediately from the continuity equation
\[
\rho_t = -\rho \text{div} v - v \cdot \nabla \rho.
\]
(3.19)
Finally, due to the material pressure \(P\) satisfies (3.4), then the corresponding estimates for \(P\) can be obtained via the same method as \(\rho\).

Now we give the estimates for the lower order terms of the velocity \(u\).

**Lemma 3.4 (Lower order estimate of the velocity \(u\)).**

\[
|u(t)|_{D_0}^2 + |\sqrt{\rho} u_t(t)|_2^2 + \int_0^t \left( |u|_{D_0}^2 + |u_t|_{D_0}^2 \right) \, ds \leq C c_1^2
\]
for \(0 \leq t \leq T_2 = \min(T^*, C(1 + c_1)^{-8})\).

**Proof.** Step 1: Multiplying (3.1) by \(u_t\) and integrating over \(\Omega\), we have
\[
\int_{\Omega} \rho |u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\nabla u|^2 + (\lambda + \mu) (\text{div} u)^2 \right) \, dx = \int_{\Omega} \left( -\nabla P - \rho v \cdot \nabla v + (\text{rot} H \times H) \right) \cdot u_t \, dx = \frac{d}{dt} \Lambda_1(t) - \Lambda_2(t),
\]
(3.20)
where
\[
\Lambda_1(t) = \int_{\Omega} \left( (P - \overline{\rho}) \text{div} u + (\text{rot} H \times H) \cdot u \right) \, dx,
\]
\[
\Lambda_2(t) = \int_{\Omega} \left( P \text{div} u + \rho (v \cdot \nabla v) \cdot u_t + (\text{rot} H \times H)_t \cdot u \right) \, dx.
\]
According to Lemmas 3.2,3.3, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we easily deduce that
\[
\Lambda_1(t) \leq C(|\nabla u|_2|P - \overline{\rho}|_2 + |\nabla H|_2|H|_3|\nabla u|_2) \leq \frac{\mu}{10}|\nabla u|_2^2 + C c_1^8,
\]
\[
\Lambda_2(t) \leq C(|\nabla u|_2|P_t|_2 + |\rho|_{\infty}^\frac{1}{2}|\sqrt{\rho} u_t|_2|v|_\infty) |\nabla v|_2 + \|H\|_2\|H_t\|_1 |\nabla u|_2)
\]
\[
\leq C|\nabla u|_2^2 + \frac{1}{10} |\sqrt{\rho} u_t|^2 + C c_1^8
\]
for \(0 < t < T_1\). Then integrating (3.20) over \((0, t)\) with respect to \(t\), we have
\[
\int_0^t \left| \sqrt{\rho} u_t(s) \right|^2 \, ds + |\nabla u(t)|_2^2 \leq C \int_0^t |\nabla u(s)|_2^2 \, ds + C c_1^8
\]
for $0 \leq t \leq T_1$, via Gronwall’s inequality, we have
\[
\int_0^t |\bar{\rho}u_t(s)|^2 \, ds + |\nabla u(t)|^2 \leq \frac{\mu}{10} |\nabla u_t|^2 + Cc_1^8, \quad 0 \leq t \leq T_1.
\] (3.21)
Combining Lemmas 3.2-3.3 and Lemma 2.4, we easily have
\[
\int_0^t |u|^2 \, ds \leq C \int_0^t \left( |\rho u_t + \rho v \cdot \nabla v|^2 + |\nabla P|^2 + |\text{rot} H \times H|^2 + |u|_{D_1}^2 \right) \, ds \leq Cc_1^{10}. \quad (3.22)
\]
\[\text{Step 2: Differentiating (3.23) with respect to } t, \text{ we have}
\]
\[\rho u_{tt} + Lu_t = -\nabla P_t - \rho_t u_t - (\rho v \cdot \nabla v)_t + (\text{rot} H \times H)_t. \quad (3.23)\]

Multiplying (3.23) by $u_t$ and integrating (3.23) over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\rho| u_t|^2 \, dx + \int_{\Omega} (\mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2) \, dx
\]
\[\quad = \int_{\Omega} \left( -\nabla P_t - (\rho v \cdot \nabla v)_t - \frac{1}{2} \rho_t u_t + (\text{rot} H \times H)_t \right) \cdot u_t \, dx \equiv: \sum_{i=1}^4 I_i. \quad (3.24)\]

According to Lemmas 3.2-3.3 Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that
\[I_1 = \int_{\Omega} P_t \text{div} u_t \, dx \leq C |P_t|_2 |\nabla u_t|_2 \leq \frac{\mu t}{10} |\nabla u_t|^2 + Cc_1^8, \]
\[I_2 \leq C |\rho|^{\frac{1}{2}} |\nabla v_t|_2 |\nabla v|_3 |\nabla u_t|_2 + |\rho|^{\frac{1}{2}} |v|_\infty |\nabla v_t|_2 |\nabla u_t|_2 + C |\rho|_3 |v|_\infty |\nabla v_t|_2 |u_t|_6 \leq C |\nabla u_t|^2 + \frac{\mu t}{10} |\nabla u_t|^2 + Cc_1^8 (1 + |\nabla v_t|^2), \] (3.25)
\[I_3 = -\frac{1}{2} \int_{\Omega} \rho u_t |u_t|^2 \, dx = \int_{\Omega} \rho |u_t|^2 |\nabla u_t| \leq C |\rho|^{\frac{1}{2}} |v|_{D_1}^2 |\nabla u_t| |\nabla v_t|_2 \leq Cc_1^8 |\nabla u_t|^2 + \frac{\mu t}{10} |\nabla u_t|^2, \]
\[I_4 = \int_{\Omega} \text{div} \left( H \otimes H - \frac{1}{2} |H|^2 I_3 \right) \cdot u_t \, dx = -\int_{\Omega} (H \otimes H - \frac{1}{2} |H|^2 I_3) : \nabla u_t \, dx \leq C |\nabla u_t|_2 |H|_2 |H|_\infty \leq Cc_1^8 + \frac{\mu t}{10} |\nabla u_t|^2. \]

Then combining the above estimate (3.25) and (3.24), we have
\[\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\rho| u_t|^2 \, dx + \int_{\Omega} |\nabla u_t|^2 \, dx \leq Cc_1^8 |\nabla u_t|^2 + Cc_1^8 |\nabla v_t|^2 + Cc_1^8. \quad (3.26)\]

Integrating (3.26) over $(\tau, t)$ $(\tau \in (0, t))$, for $\tau \leq t \leq T_1$, we have
\[|\sqrt{\rho} u_t(t)|^2 + \int_{\tau}^t |\nabla u_t|^2 \, ds \leq |\sqrt{\rho} u_t(\tau)|^2 + Cc_1^8 \int_{\tau}^t |\nabla u_t|^2 \, ds + Cc_1^8. \quad (3.27)\]

From the momentum equations (3.14), we easily have
\[|\sqrt{\rho} u_t(\tau)|^2 \leq C \int_{\Omega} |\rho| |v|^2 |\nabla v|^2 \, dx + C \int_{\Omega} \frac{|\nabla P + Lu - \text{rot} H \times H|^2}{\rho} \, dx, \quad (3.28)\]
due to the initial layer compatibility condition (1.9), letting $\tau \to 0$ in (3.28), we have
\[
\limsup_{\tau \to 0} \left| \sqrt{\rho} u_t(\tau) \right|^2_2 \leq C \int_\Omega \rho_0 |u_0|^2 |\nabla u_0|^2 \, dx + C \int_\Omega |g_1|^2 \, dx \leq C c_0^4.
\] (3.29)

Then, letting $\tau \to 0$ in (3.27), we have
\[
|\sqrt{\rho} u_t(t)|^2_2 + \int_0^t |\nabla u_t|^2_2 \, ds \leq C c_1^8 + C c_1^8 \int_0^t |\sqrt{\rho} u_t|^2_2 \, ds.
\] (3.30)

From Gronwall’s inequality, we deduce that
\[
|\sqrt{\rho} u_t(t)|^2_2 + \int_0^t |\nabla u_t|^2_2 \, ds \leq C c_1^8 \exp(\widetilde{C} c_1^8 t) \leq C c_1^8, \quad 0 \leq t \leq T_2.
\] (3.31)

Finally, due to Lemmas 3.2-3.3 and Lemma 2.4 for $0 \leq t \leq T_2$, we easily have
\[
\int_0^t |u_t|^2_2 \, dt \leq \left( |\rho u_t(t) + \rho v \cdot \nabla v(t) + \nabla P(t)|_2 + |\nabla H \times H(t)|_2 + |u(t)|_{D_1} \right) \leq C c_1^5,
\]
\[
\int_0^t |u|^2_3 \, ds \leq C \int_0^t \left( |\rho u_t + \rho v \cdot \nabla v|_{D_1}^2 + |\nabla P|^2_{D_1} + |\nabla H \times H|^2_{D_1} + |u|^2_{D_0} \right) \, ds \leq C c_1^{12}.
\]

Now we will give some estimates for the higher order terms of the velocity $u$ in the following three Lemmas.

**Lemma 3.5 (Higher order estimate of the velocity $u$).**
\[
t |u_t(t)|^2_{D_1} + t |u(t)|^2_{D_3} + \int_0^t s \left( |u_t|^2_{D_2} + |\sqrt{\rho} u_t|^2_2 \right) \, ds \leq C c_2^{24}, \quad 0 \leq t \leq T_2.
\]

**Proof.** Multiplying (3.23) by $u_{tt}$ and integrating over $\Omega$, we have
\[
\int_\Omega \rho |u_{tt}|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \left( \mu |\nabla u_t|^2 + (\lambda + \mu) (\text{div} u_t)^2 \right) \, dx
= \int_\Omega \left( - \nabla P_t - (\rho v \cdot \nabla v)_t - \rho u_t + (\text{rot} H \times H)_t \right) \cdot u_{tt} \, dx = \frac{d}{dt} \Lambda_3(t) + \Lambda_4(t),
\] (3.32)

where
\[
\Lambda_3(t) = \int_\Omega \left( P_t \text{div} u_t - \rho_t (v \cdot \nabla v)_t \cdot u_t - \frac{1}{2} \rho_t |u_t|^2 + (\text{rot} H \times H)_t \cdot u_t \right) \, dx,
\]
\[
\Lambda_4(t) = \int_\Omega \left( - \rho_t \text{div} u_t - \rho (v \cdot \nabla v)_t \cdot u_t + \rho u_t (v \cdot \nabla v)_t \cdot u_t + \rho t (v \cdot \nabla v)_t \cdot u_t \right) \, dx
+ \int_\Omega \left( \frac{1}{2} \rho |u_t|^2 - (\text{rot} H \times H)_t \cdot u_t \right) \, dx \equiv \sum_{i=5}^{10} I_i.
\]

Then almost same to (3.25), we also have
\[
\Lambda_3(t) \leq \frac{\mu}{10} |\nabla u_t|^2_2 + C c_1^8 |\sqrt{\rho} u_t|^2_2 + C c_1^8 \leq \frac{\mu}{10} |\nabla u_t|^2_2 + C c_1^{30}, \quad 0 \leq t \leq T_2.
\] (3.33)

Let us denote
\[
\Lambda^*(t) = \frac{1}{2} \int_\Omega \mu |\nabla u_t|^2 + (\lambda + \mu) (\text{div} u_t)^2 \, dx - \Lambda_3(t),
\]
then from (3.33), for $0 \leq t \leq T_2$, we quickly have
\[ C|\nabla u_t|^2 - Cc_1^{20} \leq \Lambda^*(t) \leq C|\nabla u_t|^2 + Cc_1^{20}. \] (3.34)

Similarly, from Holder’s inequality and Gagliardo-Nirenberg inequality, for $0 < t \leq T_2$, we deduce that
\[ I_5 \leq C|P\nabla u_t|^2, \quad I_6 \leq |\rho|^{\frac{1}{2}} |\nabla u_t|^2 (|v|_\infty |\nabla v_t|^2 + |\nabla v|_3 |\nabla v_t|^2), \]
\[ I_7 \leq C|\rho u_t|^2 |\nabla u_t|^2 |\nabla v|_3 |v|_\infty, \]
\[ I_8 \leq C|\rho u_t|^2 |v_t|^2 |\nabla u_t|^2 + C|v|_\infty |v_t|^2 |\nabla u_t|^2 |\rho u_t|, \]
\[ I_9 \leq C|\rho u_t|^2 |v_t|^2 |u_t|^2 + C|\rho|^{\frac{1}{2}} |\nabla u_t|^2 |v_t|^2 |\nabla u_t|^2, \] (3.35)

where we have used the facts $\rho_t = -\text{div}(\rho v)$, and
\[ I_{10} = - \int_\Omega (\nabla \times H)_{tt} \cdot u_t dx = \int_\Omega (H \otimes H - \frac{1}{2} |H|^2 I_3)_{tt} : \nabla u_t dx \]
\[ \leq C|\nabla u_t|^2 |H_{tt}|^2 + C|\nabla u_t|^2 |H_{tt}|^2 |H|_\infty. \] (3.36)

Combining (3.35) - (3.36) and Lemmas 3.2 - 3.4 from Young’s inequality, we have
\[ \Lambda_4(t) \leq \frac{1}{2} |\nabla u_t|^2 + Cc_1^{\xi} (1 + |v_t|^2 |D_0^2 + Cc_1^{4} (1 + |P_{tt}|^2 + |\rho u_t|^2 + |H_{tt}|^2) + Cc_1^{\xi} |v_t|^2 |D_0^2. \] (3.37)

Then multiplying (3.32) with $t$ and integrating over $(\tau, t)$ ($\tau \in (0, t)$), from (3.34) and (3.37), we have
\[ \int_\tau^t s^2 |\nabla u_t|^2 ds + t \int \nabla u_t(t)^2 \]
\[ \leq \tau |u_t(\tau)|^2 |D_0^2 + Cc_1^{\xi} \int_\tau^t s(1 + |\nabla v_t|^2)^2 |\nabla u_t|^2 ds + Cc_1^{20} \] (3.38)

for $\tau \leq t \leq T_2$. From Lemma 3.4 we have $\nabla u_t \in L^2([0, T_2]; L^2)$, then according to Lemma 2.3 there exists a sequence $s_k$ such that
\[ s_k \to 0, \quad \text{and} \quad s_k \nabla u_t(s_k)^2 \to 0, \quad \text{as} \quad k \to \infty. \]

Therefore, letting $\tau = s_k \to 0$ in (3.38), we conclude that
\[ \int_0^t s^2 |\nabla u_t|^2 ds + t \int \nabla u_t(t)^2 \]
\[ \leq Cc_1^{\xi} \int_0^t s(1 + |\nabla v_t|^2)^2 ds + Cc_1^{20}. \] (3.39)

Then from Gronwall’s inequality, we have
\[ \int_0^t s^2 |\nabla u_t|^2 ds + t \int u(t)^2 |D_0^2 \leq Cc_2^{20} \exp \left( Cc_1^{\xi} \int_0^t s(1 + |\nabla v_t|^2)^2 ds \right) \leq Cc_3^{20}. \]

Finally, from Lemma 2.4 for $0 \leq t \leq T_2$, we immediately have
\[ t|u(t)|^2 \leq t (|\rho u_t + \rho v \cdot \nabla v_t|^2 |D_0^2 + |\nabla P_t^2 |D_0^2 + |\nabla H_t|_2^2 + |u|^2 |D_0^2 \leq Cc_4^{24}, \]
and similarly,
\[ \int_0^t s^2 |u(t)|^2 ds \leq C \int_0^t s \left( (|\rho u_t + \rho v \cdot \nabla v_t)|^2 + |\nabla P_t|^2 + |\nabla H_t|^2 + |u|^2 |D_0^2 \right) ds \leq Cc_5^{22}. \]
Lemma 3.6 (Higher order estimate of the velocity $u$).

\[
\int_0^t |u(s)|_{L^3}^5 \, ds \leq Cc_2^{54} \quad \text{for} \quad 0 \leq t \leq T_2.
\]

**Proof.** From (3.1), via Lemma 2.1, Hölder’s inequality and Gagliardo-Nirenberg inequality, we easily deduce that

\[
|u|_{D^3} \leq (|pu_t + \rho v \cdot \nabla v|_{D^3} + |\nabla P|_{D^3} + |\text{rot} \times H|_{D^3} + |u|_{D^3})
\]

\[
\leq C(c_1^5 + c_1^5 |u|_\infty + c_1^5 |\nabla u_t|_q + c_1^5 |v|_{D^2}) \tag{3.40}
\]

Due to the Sobolev inequality and Young’s inequality, we have

\[
\begin{aligned}
|u_t|_\infty &\leq C|u_t|_{L^q}^{1-\frac{3}{q}} |u_t|_{H^{1,q}}^{\frac{3}{q}} \leq C|\nabla u_t|_2 + C|\nabla u_t|_q, \quad \text{when} \; \Omega \; \text{is bounded}, \\
|u_t|_\infty &\leq C|u_t|_{L^6}^{\frac{6(q-3)}{3q}} |\nabla u_t|_q^{\frac{3q}{3q-6}} \leq C|\nabla u_t|_2 + C|\nabla u_t|_q, \quad \text{when} \; \Omega = \mathbb{R}^3.
\end{aligned}
\]

Then we quickly obtain

\[
|u(t)|_{D^3} \leq Cc_2^5 (|\nabla u_t|_2 + |\nabla u_t|_q) + Cc_1^3 |v|_{D^2} + Cc_1^6.
\]

According to Lemmas 3.2-3.5, we have

\[
\int_0^t |u|_{D^3}^5 \, ds \leq Cc_2^{12} + Cc_1^6 \int_0^t (|v|_{D^2}^5 + |\nabla u_t|_2^5 + |\nabla u_t|_q^5) \, ds
\]

\[
\leq Cc_2^{12} + Cc_1^6 \int_0^t |\nabla u_t|_2^{\frac{p_0(6-q)}{4q}} |\nabla u_t|_q^{\frac{p_0(6-q)}{4q}} \, ds
\]

\[
\leq Cc_2^{12} + Cc_1^6 \int_0^t \sup_{[0,T]} s |\nabla u_t|_2^{\frac{p_0(6-q)}{4q}} \int_0^t s^{-\frac{p_0(6-q)}{4q}} (s |u_t|_{D^2}^2) \frac{p_0(3q-6)}{4q} \, ds
\]

\[
\leq Cc_2^{12} + Cc_1^6 \left( \left( \int_0^t s^{-\frac{2p_0(6-q)}{4q-p_0(3q-6)}} ds \right) \frac{4q-p_0(3q-6)}{4q} \left( \int_0^t s |u_t|_{D^2}^2 ds \right) \frac{p_0(3q-6)}{4q} \right)
\]

\[
\leq Cc_2^{54}
\]

due to $0 < \frac{2p_0(6-q)}{4q-p_0(3q-6)} < 1$ and $0 < \frac{p_0(3q-6)}{4q} < 1$. \(\square\)

Lemma 3.7 (Higher order estimate of the velocity $u$).

\[
t^2 |u(t)|_{D^3} + t^2 |u_t(t)|_{D^2}^2 + t^2 |\sqrt{\rho} u_{tt}(t)|_2^2 + \int_0^t s^2 |u_t(s)|_{D^3}^2 \, ds \leq Cc_3^{34}
\]

for $0 \leq t \leq T_3 = \min(T^*, (1 + c_3)^{-8})$.

**Proof.** Differentiating the equations (3.23) with respect to $t$, we have

\[
\rho u_{tt} + Lu_{tt} = -\nabla P_{tt} - \rho \cdot \nabla v)_{tt} - 2\rho_t (v \cdot \nabla v + u_t)_t - \rho_t (v \cdot \nabla v + u_t) + (\text{rot} \times H)_{tt} \tag{3.42}
\]
Multiplying (3.43) by $u_t$ and integrating over $\Omega$, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 dx + \int_{\Omega} (\mu |\nabla u_t|^2 + (\lambda + \mu) (\text{div} u_t)^2) dx \\
= \int_{\Omega} (P_t \text{div} u_t - \rho (v \cdot \nabla v) u_t - 2 \rho_t (v \cdot \nabla v) u_t - \rho_t (v \cdot \nabla v) u_t) dx \\
+ \int_{\Omega} \left(- \frac{3}{2} \rho_t |u_t|^2 - \rho_t u_t \cdot u_t + (\text{rot} H \times H) u_t \right) dx = \Lambda_5(t) \equiv \sum_{i=1}^{17} I_i.
\]
From Lemmas 3.2-3.6, Hölder’s inequality and Gagliardo-Nirenberg inequality, we obtain
\[
I_{11} \leq C |P_t||u_t||u_t||, \quad I_{12} \leq C |\rho|^\frac{3}{2} |\nabla u_t||v_t|| |D_0|^\frac{1}{2} |\nabla v_t||, \\
I_{13} \leq C |\rho_t||v_t|| |D_0|^\frac{1}{2} |\nabla u_t||, \quad I_{14} \leq C |\rho_t|^2 |\nabla v_t||^2 |\nabla u_t||, \\
I_{15} \leq C |\rho|^\frac{3}{2} |\nabla v_t|| |D_0|^\frac{1}{2} |\nabla u_t||, \quad I_{16} \leq C |\rho_t|^2 |\nabla v_t|| |D_0|^\frac{1}{2} |\nabla u_t|| \\
+ C |\rho|^\frac{3}{2} |v_t|| |D_0|^\frac{1}{2} |\nabla u_t|| + |u_t|| |D_0|^\frac{1}{2} |\nabla u_t||,
\]
where we have used the fact that $\rho_t = \text{div}(\rho v)$, and
\[
I_{17} = - \int_{\Omega} (\text{rot} H \times H) u_t dx = \int_{\Omega} (H \otimes H - \frac{1}{2} |H|^2 I_3) u_t : \nabla u_t dx \\
\leq C |\nabla u_t||H||_4^2 + C |\nabla u_t||H_t||_2 H||_\infty.
\]
Then from Young’s inequality, the above estimates (3.44)–(3.45) imply that
\[
t^2 \Lambda_5(t) \leq \frac{H^2}{2} |u_t|^2 |D_0| + C (c_6^3 + c_7^3 |v_t|^2 |D_0|)^2 |\nabla u_t|^2 + C c_3^2 (\rho_t^2 + |P_t|^2) \\
+ C c_3^2 |H_t|^2 + C t^2 (|v_t|^2 |D_0| + |v_t|^2 |D_2|) + C c_3^2 |u_t|^2 |D_2| + C c_3^3.
\]
Then multiplying (3.43) by $t^2$ and integrating over $(\tau, t)$ ($\tau \in (0, t)$), we obtain
\[
t^2 |\nabla u_t|^2 + \int_{\tau}^{t} s^2 |\nabla u_t|^2 ds \\
\leq t^2 |\nabla u_t(\tau)|^2 + C \int_{\tau}^{t} (c_6^3 + c_7^3 |v_t|^2 |D_0|) s^2 |\nabla u_t|^2 ds + C c_3^3
\]
for $\tau \leq t \leq T_2$. Due to Lemma 3.5, we have $t^2 |\nabla u_t| \in L^2([0, T_2]; L^2)$, then from Lemma 2.3 there exists a sequence $s_k$ such that
\[
s_k \to 0, \quad \text{and} \quad s_k^2 |\nabla u_t(s_k)|^2 \to 0, \quad \text{as} \quad k \to \infty.
\]
Therefore, letting $\tau = s_k \to 0$ in (3.47), we conclude that
\[
t^2 |\nabla u_t|^2 + \int_{0}^{t} s^2 |\nabla u_t|^2 ds \leq C \int_{\tau}^{t} (c_6^3 + c_7^3 |v_t|^2 |D_0|) s^2 |\nabla u_t|^2 ds + C c_3^3.
\]
Via the Gronwall’s inequality, for $0 \leq t \leq T_3$, we have
\[
t^2 |\nabla u_t|^2 + \int_{0}^{t} s^2 |\nabla u_t|^2 ds \leq C c_3^3 \exp \left( \int_{\tau}^{t} (c_6^3 + c_7^3 |v_t|^2 |D_0|) ds \right) \leq C c_3^3.
\]
Moreover, from Lemma 2.4 and (3.40), we quickly have
\[ t^2 |u_t|^2_{D^2} \leq C t^2 \left( (|p_{tt} + pv \cdot \nabla v|^2 + |\nabla P_t|^2 + |(\text{rot} H \times H)|^2 + |u_t|^2_{D^2} \right) \leq C t^2, \]
\[ t^2 |u|_{D^3} \leq C t^2 \left( c_1^2 + c_2^2 |u_t|_{\infty} + c_3^2 |\nabla u_t|_q + Cc_1^2 |v|_{D^2,q} \right) \leq C t^2. \]

Then combining the above lemmas, for \( 0 \leq t \leq T_* = \min(T^*, (1 + c_3)^{-8}) \), we have the following a priori estimate:
\[ \| H(t) \| _{H^2 \cap W^{2,q}} + \| (H_t, \rho_t, P_t(t)) \| _{H^2 \cap L^q} \leq C t^2, \]
\[ \int_0^t |(H_t, \rho_t, P_t)|^2_2 ds + t |(H_t, \rho_t, P_t(t))|_{D^1,q}^2 \leq C t^2, \]
\[ |u(t)|^2_{D^2} + \| \sqrt{\rho} u_t(t) \|_2^2 + \int_0^t \left( |u_t|^2_{D^1} + |u_t|^2_{D^2} \right) ds \leq C t^2, \]
\[ t |u(t)|^2_{D^1} + t^2 |u|_{D^2}^2 + \| \sqrt{\rho} u_t(t) \|_2^2 + \int_0^t s^2 |u_{tt}|_{D^3}^2 ds \leq C t^2. \]

3.3. Unique solvability of the IBVP (3.1) and (3.6)-(3.7) with vacuum.

In this section, we will construct a sequence of approximation solutions to the linearized problem (3.1) with vacuum.

Lemma 3.8. Let (3.2) and (3.6)-(3.7) hold. Assume \((H_0, \rho_0, u_0)\) satisfies (3.9). Then there exists a unique classical solution \((H, \rho, u)\) to (3.1) satisfying
\[ (H, \rho - \bar{\rho}, P - \bar{P}) \in C([0, T_*]; H^2 \cap W^{2,q}), \]
\[ u \in C([0, T_*]; D^0_0 \cap D^2) \cap L^2([0, T_*]; D^3) \cap L^{p_0}([0, T_*]; D^{3,q}), \]
\[ \sqrt{\rho} u_t \in L^\infty([0, T_*]; L^2), \]
\[ t^2 u_t \in L^\infty([0, T_*]; D^2), \]
\[ t^2 \sqrt{\rho} u_{tt} \in L^2([0, T_*]; L^2), \]
\[ t^2 \sqrt{\rho} u_{tt} \in L^\infty([0, T_*]; D^{3,q}), \]
\[ tu_t \in L^\infty([0, T_*]; D^2), \]
\[ tu_{tt} \in L^\infty([0, T_*]; D^2), \]
\[ t \sqrt{\rho} u_{tt} \in L^\infty([0, T_*]; L^2). \]
Moreover, the solution \((H, \rho, u)\) also satisfies the estimate (3.50).

Proof. Step 1: Existence. We define \( \rho_0 = \rho_0 + \delta \) for each \( \delta \in (0, 1) \). Then from the compatibility condition (1.9), we have
\[ Lu_0 + \nabla P(\rho_0^\delta) - \mu_0 \text{rot} H_0 \times H_0 = (\rho_0^\delta)^{\frac{5}{2}} g_1^\delta, \]
where
\[ g_1^\delta = \left( \frac{\rho_0}{\rho_0^\delta} \right)^{\frac{5}{2}} g_1 + \frac{\nabla P(\rho_0^\delta) - P(\rho_0)}{(\rho_0^\delta)^{\frac{5}{2}}}. \]

Then according to assumption (3.6), for sufficiently small \( \delta > 0 \), we have
\[ 1 + \bar{\rho} + \delta + \| (\rho_0^\delta - (\bar{\rho} + \delta), P(\rho_0^\delta - P(\bar{\rho} + \delta), H_0) \|_{H^2 \cap W^{2,q}} + |u_0|_{D^2_0 \cap D^2} + |g_1^\delta|_2 \leq c_0. \]
Therefore, corresponding to \((H_0, \rho_0^\delta, P(\rho_0^\delta), u_0)\), there exists a unique classical solution \((H^\delta, \rho^\delta, P^\delta, u^\delta)\) satisfying \((3.50)\). Then there exists a subsequence of solutions \((H^\delta, \rho^\delta, P^\delta, u^\delta)\) converges to a limit \((H, \rho, P, u)\) in weak or weak* sense. And for any \(R > 0\), due to Lemma 2.5, there exists a subsequence of solutions \((H^\delta, \rho^\delta, P^\delta, u^\delta)\) satisfying
\[
(H^\delta, \rho^\delta, P^\delta, u^\delta) \rightarrow (H, \rho, P, u) \text{ in } C([0,T_*]; H^1(\Omega_R)),
\]
where \(\Omega_R = \Omega \cap B_R\). Combining the lower semi-continuity of norms and \((3.52)\), we know that \((H, \rho, P, u)\) also satisfies the local estimates \((3.50)\). So it is easy to show that \((H, \rho, P, u)\) is a solution in distribution sense and satisfies the regularity
\[
(H, \rho - \overline{\rho}, P - \overline{P}) \in L^\infty([0,T_*]; H^2 \cap W^{2,q}),
\]
\[
u \in L^\infty([0,T_*]; D_0^1 \cap D^2) \cap L^2([0,T_*]; D^3) \cap L^{p_0}([0,T_*]; D^{3,q}),
\]
\[
 u_t \in L^2([0,T_*]; D_0^0), \quad \sqrt{\rho} u_t \in L^\infty([0,T_*]; L^2),
\]
\[
 t^{3/2} u_t \in L^\infty([0,T_*]; D^3), \quad t^{1/2} \sqrt{\rho} u_{tt} \in L^2([0,T_*]; L^2),
\]
\[
 t^{1/2} u_{tt} \in L^\infty([0,T_*]; D_0^1) \cap L^2([0,T_*]; D^2), \quad tuberculosis \in L^\infty([0,T_*]; D^{3,q}),
\]
\[
 tu_{tt} \in L^2([0,T_*]; D_0^0), \quad tu_t \in L^\infty([0,T_*]; D^2), \quad t^{1/2} \rho u_{tt} \in L^\infty([0,T_*]; L^2).
\]

Step 2: Uniqueness. Let \((H_1, \rho_1, u_1)\) and \((H_2, \rho_2, u_2)\) be two solutions. Due to Lemma 3.1 in Section 3.1, we know \(\rho_1 = \rho_2\) and \(H_1 = H_2\). For the momentum equations \((3.14)\), let \(\overline{u} = u_1 - u_2\), we have
\[
\rho \overline{u}_t - \mu \overline{\Delta u} - (\lambda + \mu) \nabla \text{div} \overline{u} = 0,
\]
(3.54)

because we do not know whether \(\sqrt{\rho} \overline{u} \in L^\infty([0,T_*]; L^2(\Omega))\) or not, so we consider this equation in bounded domain \(\Omega_R\). We define \(\varphi^R(x) = \varphi(x/R)\), where \(\varphi \in C^\infty_c(B_1)\) is a smooth cut-off function such that \(\varphi = 1\) in \(B_{1/2}\). Let \(\overline{u}^R = \varphi^R(t,x)u(t,x)\), we have
\[
\rho \overline{u}_t^R - \mu \varphi^R \overline{\Delta u} - \varphi^R(\lambda + \mu) \nabla \text{div} \overline{u} = 0.
\]

Therefore, multiplying \((3.55)\) by \(\overline{u}^R\) and integrating over \([0,t] \times \Omega_R\) \((t \in (0,T_*))\), we have
\[
\frac{1}{2} \int_{\Omega_R} \rho |\overline{u}^R|^2(t) dx + \int_0^t \int_{\Omega_R} \left( \mu (\varphi^R)^2 |\nabla \overline{u}|^2 + (\lambda + \mu)(\varphi^R)^2 |\text{div} \overline{u}|^2 \right) dx dt,
\]
(3.56)

\[
= \int_0^t \int_{\Omega_R} \rho \nu \cdot \nabla \overline{u}^R \cdot \overline{u}^R dx dt - 2\mu \int_0^t \int_{\Omega_R} \varphi^R (\nabla \overline{u} \cdot \nabla \varphi^R) dx dt
\]
\[
- 2 \int_0^t \int_{\Omega_R} (\lambda + \mu) \varphi^R \text{div} \overline{u} \nabla \varphi^R \cdot \overline{u}^R dx dt = A_1 + A_2 + A_3.
\]
From Holder’s inequality and Sobolev’s imbedding theorem, we have
\[
|A_1| \leq \int_0^t \int_{\Omega_R} |\varphi^R \rho v \cdot \nabla u| \, dx \, ds + \int_0^t \int_{\Omega_R} |\bar{\rho} u|^2 \, ds \, dx \, ds \\
\leq C \int_0^t |\sqrt{\rho} u|^2_2 \, ds + \int_0^t \frac{1}{2} (\rho^R)^2_2 \, ds + \frac{C}{R^2} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\bar{\rho} u|^2_2 \, dx \, ds,
\]
\[
|A_2| \leq \frac{C}{R^2} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\bar{\rho} u|^2_2 \, dx \, ds + C \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\nabla \bar{\rho}|_2^2 \, ds \, dx \, ds \\
\leq \frac{C}{R^2} \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\nabla \bar{\rho}|_2^2 \, dx \, ds \, dx \, ds + C \int_0^t \int_{(\Omega_R \setminus B_{R/2})} |\nabla \bar{\rho}|_2^2 \, ds \, dx \, ds \\
\leq C \int_0^{T_\ast} \int_{(\Omega_R \setminus B_{R/2})} |\nabla \bar{\rho}(s)|_{L^2(\Omega_R \setminus B_{R/2})}^2 \, ds \to 0 \quad \text{as} \quad R \to \infty.
\]
Similarly, we can also obtain that
\[
|A_3| \leq C \int_0^{T_\ast} \int_{(\Omega_R \setminus B_{R/2})} |\nabla \bar{\rho}(s)|_{L^2(\Omega_R \setminus B_{R/2})}^2 \, ds \to 0 \quad \text{as} \quad R \to \infty.
\]
Then from the above estimates, we deduce that
\[
\frac{1}{2} \int_{\Omega_R} \rho |\bar{\rho}^R|^2(t) \, dx + \int_0^t \int_{\Omega_R} \mu (\rho^R)^2_2 \, ds \, dx \, ds \leq C \int_0^t |\sqrt{\rho} u|^2_2 \, ds + Q_R, \tag{3.57}
\]
where \(Q_R \to 0\) as \(R \to \infty\). Then letting \(R \to \infty\) in \(3.57\), via Gronwall’s inequality, we derive that \(\bar{\rho} \equiv 0\), which means that \(u_1 = u_2\).

**Step 3: Time-continuity of the solution \((H, \rho, u, P)\).** Firstly, the time-continuity of \(\rho, P\) and \(H\) can be obtained by Lemma 3.1. Secondly, from a classical embedding result (see \(6\)), we have \(u \in C([0, T_\ast]; D_0^1) \cap C([0, T_\ast]; D^2 - \text{weak})\). From the momentum equations \(3.4\), we know that \((\rho u_t)^i \in L^2([0, T_\ast]; H^{-1})\). Due to \(\rho u_t \in L^2([0, T_\ast]; D_0^1)\), we have immediately that \(\rho u_t \in C([0, T_\ast]; D_0^1)\). Similarly, from the following equations,
\[
Lu = -\rho u_t - \rho (v \cdot \nabla)v - \nabla P + \text{rot} \, H \times H \equiv F,
\]
where \(F \in C([0, T_\ast]; L^2)\), we can obtain \(u \in C([0, T_\ast]; D^2)\).

**3.4. Proof of Theorem 1.1.**

Based on Lemma 3.8, now we give the proof of Theorem 1.1. We first fix a positive constant \(c_0\) sufficiently large such that
\[
2 + \bar{\rho} + \|(|\rho_0 - \bar{\rho}, P_0 - \overrightarrow{P}, H_0)\|_{H^2 \cap W^{2,q}} + |u_0|_{D_0^1 \cap D^2} + |g_1|_2 \leq c_0. \quad \tag{3.58}
\]
Then let \(u^0 \in C([0, +\infty); D_0^1 \cap D^2) \cap L^p([0, +\infty); D^{3,q})\) be the unique solution to the following linear parabolic problem
\[
h_t - \Delta h = 0 \quad (0, +\infty) \times \Omega \quad \text{and} \quad h(0) = u_0 \quad \text{in} \quad \Omega.
\]
Then taking a small time $T^c \in (0, T_*)$, we have
\[
\sup_{0 \leq t \leq T^c} |u^0(t)|^2_{D_0^3} + \int_0^{T^c} \left( |u^0|^2_{D_0^3} + |u^0|^2_{D_3^4} + |u^0|^2_{D_0^1} \right) dt \leq c_1,
\]
\[
\text{ess sup}_{0 \leq t \leq T^c} \left( t |u^0(t)|^2_{D_0^3} + t |u^0(t)|^2_{D_3^4} + \int_0^{T^c} t |u^0(t)|^2_{D_0^1} dt \right) \leq c_2,
\]
\[
\text{ess sup}_{0 \leq t \leq T^c} \left( t^2 |u^0(t)|^2_{D_0^3} + t^2 |u^0(t)|^2_{D_3^4} + \int_0^{T^c} t^2 |u^0(t)|^2_{D_0^1} dt \right) \leq c_3
\]
for constants $c_i's$ with $1 < c_0 \leq c_1 \leq c_2 \leq c_3$.

**Proof.** From Lemma 3.8, we know that there exists a unique classical solution $(H^1, \rho^1, P^1, u^1)$ to the linearized problem (3.1) with $v$ replaced by $u^0$, which satisfies the estimate (3.50). Similarly, we construct approximate solutions $(H^{k+1}, \rho^{k+1}, P^{k+1}, u^{k+1})$ inductively, as follows: assuming that $u^k$ was defined for $k \geq 1$, let $(H^{k+1}, \rho^{k+1}, P^{k+1}, u^{k+1})$ be the unique classical solutions to the problem (3.1) with $v$ replaced by $u^k$ as following

\[
\begin{cases}
H_{t}^{k+1} + u^k \cdot \nabla H^{k+1} + (\text{div} u^k I_3 - \nabla u^k) H^{k+1} = 0, \\
\text{div} H^{k+1} = 0, \\
\rho_{t}^{k+1} + \text{div} (\rho^{k+1} u^k) = 0, \\
\rho^{k+1} u_{t}^{k+1} + \rho^{k+1} u^k \cdot \nabla u^k + \nabla P^{k+1} + L^{k+1} u = \mu_{0} \text{rot} H^{k+1} \times H^{k+1}, \\
(H^{k+1}, \rho^{k+1}, u^{k+1})|_{t=0} = (H_0(x), \rho_0(x), u_0(x)) \quad x \in \Omega, \\
(H^{k+1}, \rho^{k+1}, u^{k+1}, P^{k+1}) \to (0, \overline{\rho}, 0, \overline{P}) \quad \text{as} \quad |x| \to \infty, \quad t > 0.
\end{cases}
\] (3.59)

Then from Lemma 3.8 that $(H^k, \rho^k, P^k, u^k)$ satisfies (3.50). Next, we show that $(H^k, \rho^k, P^k, u^k)$ converges to a limit $(H, \rho, P, u)$ in a strong sense. But this can be done by a slight modification of the arguments in [3]. We omits its details. Then adapting the proof of Lemma 3.8, we can easily show that $(H, \rho, P, u)$ is a solution to (1.12)-(1.14). The proof for uniqueness and time-continuity is also similar to those in [3][5] and so omitted. \(\square\)

**Remark 3.1.** For the case $0 < \sigma < +\infty$, if we add $H|_{\partial \Omega} = 0$ to (1.12)-(1.14), then the similar existence result can be obtained via the similar argument used in this Section.

## 4. Blow-up criterion for classical solutions

Now we prove (1.14). Let $(H, \rho, u)$ be the unique classical solution to IBVP (1.5)-(1.7). We assume that the opposite holds, i.e.,

\[
\limsup_{T \to T^c} |D(u)|_{L^1([0,T];L^\infty(\Omega))} = C_0 < \infty.
\] (4.1)

Due to $P = A\rho^\gamma$, we quickly know that $P$ satisfies

\[
P_t + u \nabla P + \gamma P \text{div} u = 0, \quad P_0 \in H^2 \cap W^{2,q}.
\] (4.2)

We first give the standard energy estimate that
Lemma 4.1.

\[
(|\sqrt{\rho}u(t)|_2^2 + |H|_2^2 + |P|_1) + \int_0^T |\nabla u(t)|_2^2 dt \leq C, \quad 0 \leq t < T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, \overline{T}] \)).

Proof. We first show that

\[
\frac{d}{dt} \int_\Omega \left( \frac{1}{2} \rho |u|^2 + \frac{P}{\gamma - 1} + \frac{1}{2} H^2 \right) dx + \int_\Omega (\mu |\nabla u|^2 + (\lambda + \mu)(\text{div} u)^2) dx = 0.
\]

(4.3)

Actually, (4.3) is classical, which can be shown by multiplying (1.5) by \( u \), (1.5) by \( |u|^2 \) and (1.5) by \( H \), then summing them together and integrating the result equation over \( \Omega \) by parts, where we have used the fact

\[
\int_\Omega \text{rot} H \times H \cdot H dx = -\int_\Omega -\text{rot}(u \times H) \cdot H dx.
\]

(4.4)

Let \( f = (f^1, f^2, f^3)^\top \in \mathbb{R}^3 \) and \( g = (g^1, g^2, g^3)^\top \in \mathbb{R}^3 \), we denote \( (f \otimes g)_{ij} = (f_i g_j) \). Next we need to show some lower order estimate for our classical solution \((H, \rho, u)\), which is the same as the regularity that the strong solution obtained in [5] has to satisfy.

4.1. Lower order estimate.

By assumption (4.1), we first show that both \( H \) and \( \rho \) are both uniform bounded.

Lemma 4.2.

\[
(|\rho(t)|_\infty + |H(t)|_\infty) \leq C, \quad 0 \leq t < T,
\]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, \overline{T}] \)).

Proof. Multiplying (1.5) by \( q |H|^{q-2} H \) and integrating over \( \Omega \) by parts, then we have

\[
\frac{d}{dt} |H|^q = q \int_\Omega (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \cdot H |H|^{q-2} dx
\]

\[
= q \int_\Omega (H \cdot D(u) - u \cdot \nabla H - H \text{div} u) \cdot H |H|^{q-2} dx.
\]

(4.5)

By integrating by parts, the second term on the right-hand side can be written as

\[
-q \int_\Omega (u \cdot \nabla H) \cdot H |H|^{q-2} dx = \int_\Omega \text{div} u |H|^q dx,
\]

(4.6)

which, together with (4.5), immediately yields

\[
\frac{d}{dt} |H|^q \leq (2q + 1) \int_\Omega |D(u)| |H|^q dx \leq (2q + 1)|D(u)|_\infty |H|_q^q,
\]

(4.7)

which means that

\[
\frac{d}{dt} |H|^q \leq \frac{(2q + 1)}{q} |D(u)|_\infty |H|_q,
\]

(4.8)

hence, it follows from (4.1) and (4.8) that

\[
\sup_{0 \leq t \leq T} |H|_q \leq C, \quad 0 \leq T < \overline{T},
\]
where $C > 0$ is independent of $q$. Therefore, letting $q \to \infty$ in the above inequality leads to the desired estimate of $|H|_\infty$. In the same way, we also obtains the boundeness of $|\rho|_\infty$ which indeed depends only on $\|\text{div} u\|_{L^1([0,T], L^\infty(\Omega))}$.

\[\square\]

The next lemma will give a key estimate on $\nabla H$, $\nabla \rho$ and $\nabla u$.

**Lemma 4.3.**

\[
\sup_{0 \leq t \leq T} (|\nabla u|^2 + |\nabla \rho|^2 + |\nabla H|^2) + \int_0^T |\nabla^2 u|^2 dt \leq C, \quad 0 \leq T < T,
\]

where $C$ only depends on $C_0$ and $T$.

**Proof.** Firstly, multiplying (4.10) by $\rho^{-1}(-Lu - \nabla P - \nabla|H|^2 + H \cdot \nabla H)$ and integrating the result equation over $\Omega$, then we have

\[
\frac{1}{2} \frac{d}{dt} (\frac{\mu}{2} |\nabla u|^2 + \frac{\mu + \lambda}{2} |\text{div} u|^2) + \int_\Omega \rho^{-1}(-Lu - \nabla P - \nabla|H|^2 + H \cdot \nabla H)^2 dx
\]

\[
= -\mu \int_\Omega (u \cdot \nabla u) \cdot \nabla \cdot (\text{rot} u) dx + (2\mu + \lambda) \int_\Omega (u \cdot \nabla u) \cdot \text{div} u dx
\]

\[
- \int_\Omega (u \cdot \nabla u) \cdot \nabla P(\rho) dx - \int_\Omega (u \cdot \nabla u)\left(\frac{1}{2} \nabla|H|^2 - H \cdot \nabla H\right) dx
\]

\[
- \int_\Omega u_t \cdot \nabla P(\rho) dx - \int_\Omega u_t \cdot \frac{1}{2} \nabla|H|^2 - H \cdot \nabla H dx \equiv \sum_{i=1}^6 L_i,
\]

where we have used the fact that $\Delta u = \nabla \text{div} u - \nabla \times \text{rot} u$.

We now estimate each term in (4.9). Due to the fact that $\rho^{-1} \geq C^{-1} > 0$, we find the second term on the left hand side of (4.9) admits

\[
\int_\Omega \rho^{-1}|-Lu + \nabla P + \nabla|H|^2 - H \cdot \nabla H|^2 dx \geq C^{-1}|Lu|^2_2 - C(|\nabla P|_2^2 + |\nabla u|^2_2 + |H|^2_\infty |\nabla H|^2_2)
\]

\[
\geq C^{-1}|u|^2_2 - C(|\nabla \rho|^2_2 + |\nabla u|^2_2 + |\nabla H|^2_2),
\]

where we have used the standard $L^2$-theory of elliptic system and Lemma 4.2. Note that $L$ is a strong elliptic operator. Next according to

\[
\begin{align*}
\{ & u \times \text{rot} u = \frac{1}{2} \nabla (|u|^2) - u \cdot \nabla u, \\
\n\n\n\{ & \nabla \times (a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + (\text{div} b)a - (\text{div} a)b,
\end{align*}
\]

and Holder's inequality, Gagliardo-Nirenberg inequality and Young's inequality, we deduce

\[
|L_1| = \mu \left| \int_\Omega (u \cdot \nabla u) \cdot \nabla \times (\text{rot} u) dx \right| = \mu \left| \int_\Omega \nabla \times (u \cdot \nabla u) \cdot \text{rot} u dx \right|
\]

\[
= \mu \left| \int_\Omega \nabla \times (u \times \text{rot} u) \cdot \text{rot} u dx \right| = \mu \frac{1}{2} \int_\Omega (\text{rot} u)^2 dx - \int_\Omega \text{rot} u \cdot D(u) \cdot \text{rot} u dx \leq C|D(u)|_\infty |\nabla u|^2_2,
\]

\[\square\]
\[ |L_2| = (2\mu + \lambda) \int_{\Omega} (u \cdot \nabla u) \cdot \nabla \text{div} u \, dx \]
\[ = (2\mu + \lambda) \left| - \int_{\Omega} \nabla u : (\nabla u)^T \, \text{div} u + \frac{1}{2} \int_{\Omega} (\text{div} u)^3 \, dx \right| \]
\[ \leq C |D(u)|_{\infty} |\nabla u|^2, \]

\[ L_3 = - \int_{\Omega} (u \cdot \nabla u) \cdot \nabla P \, dx \leq C |\nabla u|_2 |\nabla u|_3 |\nabla P|_2 \]
\[ \leq C(\epsilon)(|\nabla \rho|^2_2 + 1)|\nabla u|^4 + \epsilon |u|^2_{D^2}, \]

\[ L_4 = - \int_{\Omega} (u \cdot \nabla u) \left( \frac{1}{2} |\nabla H|^2 - H \cdot \nabla H \right) \, dx \leq C |\nabla H|_2 |\nabla u|_3 |u|_6 \]
\[ \leq C(\epsilon)|\nabla H|^2 \frac{|\nabla u|^2}{2} + \epsilon |\nabla u|^2 \leq C(\epsilon)(|\nabla H|^2 + 1)|\nabla u|^2 + \epsilon |u|^2_{D^2}, \]

\[ L_5 = - \int_{\Omega} u_t \cdot \nabla P \, dx = \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx - \int_{\Omega} P_t \text{div} u \, dx \]
\[ = \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx + \int_{\Omega} (u \cdot \nabla P \text{div} u + (\gamma - 1)P(\text{div} u)^2) \, dx \]
\[ \leq \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx + C |\nabla P|_2 |u|_6 |\nabla u|_3 + C |P|_{\infty} |\nabla u|^2 \]
\[ = \frac{d}{dt} \int_{\Omega} P \text{div} u \, dx + C(\epsilon)|\nabla u|^2(1 + |\nabla \rho|^2) + \epsilon |u|^2_{D^2}, \]

\[ L_6 = - \int_{\Omega} u_t \cdot \left( \frac{1}{2} |\nabla H|^2 - H \cdot \nabla H \right) \, dx \]
\[ = \frac{1}{2} \frac{d}{dt} \int_{\Omega} |H|^2 \text{div} u \, dx - \frac{d}{dt} \int_{\Omega} H \cdot \nabla u \cdot H \, dx \]
\[ - \int_{\Omega} \text{div} u H \cdot H_t \, dx + \int_{\Omega} H_t \cdot \nabla u \cdot H \, dx + \int_{\Omega} H \cdot \nabla u \cdot H_t \, dx. \]

where we have used the fact \( \text{div} H = 0 \) and \( \epsilon > 0 \) is a sufficiently small constant. To deal with the last three terms on the right-hand side of \( L_6 \), we need to use \( H_t = H \cdot \nabla u - u \cdot \nabla H - H \text{div} u \).

Hence, similar to the proof of the above estimates for \( L_i \), we also have

\[ \int_{\Omega} H_t \cdot \nabla u \cdot H \, dx = \int_{\Omega} -\text{div} u H \cdot (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \, dx \]
\[ \leq C |H|^2_{\infty} |\nabla u|^2 + |D(u)|_{\infty} |\nabla H|_2 |u|_6 |H|_3 \]
\[ \leq C(|D(u)|_{\infty} + 1)(|\nabla u|^2 + |\nabla H|^2), \]

\[ \int_{\Omega} H_t \cdot \nabla u \cdot H \, dx + \int_{\Omega} H \cdot \nabla u \cdot H_t \, dx \]
\[ \leq \int_{\Omega} |(H \cdot \nabla u - u \cdot \nabla H - H \text{div} u) \cdot \nabla u \cdot H| \, dx \]
\[ \leq C |H|^2_{\infty} |\nabla u|^2 + |u|_{\infty} |\nabla u|_2 |\nabla H|_2 |H|_{\infty} \]
\[ \leq C(\epsilon)(|\nabla H|^2 + 1)|\nabla u|^2 + \epsilon |u|^2_{D^2}. \]
Then integrating (4.15) over Ω, we have
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 - (P + \frac{1}{2} |H|^2) \text{div} u H - \nabla u \cdot H) \, dx + C |\nabla^2 u|^2 \]
\[ \leq C (|\nabla u|^2 + |\nabla H|^2 + 1) (|\nabla u|^2 + |D(u)|_\infty + 1). \]  
(4.14)

Secondly, applying \( \nabla \) to (1.5) and multiplying the result equation by 2\( \nabla \rho \), we have
\[ (|\nabla \rho|^2)_t + \text{div}(|\nabla \rho|^2 u) + |\nabla \rho|^2 \text{div} u = -2(\nabla \rho) \cdot \nabla \rho - 2\rho \nabla \rho \cdot \nabla \text{div} u \]
\[ = -2(\nabla \rho) \cdot D(u) \nabla \rho - 2\rho \nabla \rho \cdot \nabla \text{div} u. \]  
(4.15)

Then integrating (4.15) over \( \Omega \), we have
\[ \frac{d}{dt} |\nabla \rho|^2 \leq C (|D(u)|_\infty + 1) |\nabla \rho|^2 + \epsilon |\nabla^2 u|^2. \]  
(4.16)

Thirdly, applying \( \nabla \) to (1.5)\(_1\), due to
\[ A = \nabla (H \cdot \nabla u) = (\partial_j H \cdot \nabla u^i)_{(ij)} + (H \cdot \nabla \partial_j u^i)_{(ij)}, \]
\[ B = \nabla (u \cdot \nabla H) = (\partial_j u \cdot \nabla H^i)_{(ij)} + (u \cdot \nabla \partial_j H^i)_{(ij)}, \]
\[ C = \nabla (H \text{div} u) = \nabla H \text{div} u + H \otimes \nabla \text{div} u, \]  
(4.17)

then multiplying the result equation \( \nabla (1.5) \) by 2\( \nabla H \), we have
\[ (|\nabla H|^2)_t - 2 A : \nabla H + 2 B \nabla H - 2 C : \nabla H = 0. \]  
(4.18)

Then integrating (4.18) over \( \Omega \), due to
\[
\int_\Omega A : \nabla H \, dx \\
= \int_\Omega \sum_{j=1}^3 \left( \sum_{i=1}^3 \sum_{k=1}^3 \partial_j H^k \partial_k u^i \partial_j H^i \right) dx + \int_\Omega \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 H^k \partial_k u^i \partial_j H^i dx \\
= \int_\Omega \sum_{j=1}^3 \left( \sum_{i,k} \partial_j H^k \left( \partial_k u^i + \partial_k u^j \right) / 2 \partial_j H^i \right) dx + \int_\Omega \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 H^k \partial_k u^i \partial_j H^i dx \\
\leq C |D(u)|_\infty |\nabla H|^2 + C |H|_\infty |\nabla H|^2 |u|_{D^2}, \]  
(4.19)

\[
\int_\Omega B : \nabla H \, dx \\
= \int_\Omega \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 \partial_j \partial_k u^k \partial_j H^i \partial_j H^i dx + \int_\Omega \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 u^k \partial_k H^i \partial_j H^i dx \\
= \int_\Omega \sum_{j,k} \left( \sum_{i,k} \partial_j H^i \left( \partial_j u^k + \partial_k u^j \right) / 2 \partial_j H^i \right) dx + \frac{1}{2} \int_\Omega \sum_{j,k} \left( \sum_{i,k} u^k \partial_k (\partial_j H^i)^2 \right) dx \\
\leq C |D(u)|_\infty |\nabla H|^2, \]
\[
\int_{\Omega} C : \nabla H \, dx = \int_{\Omega} \left( \text{div} u |\nabla H|^2 + (H \otimes \nabla \text{div} u) : \nabla H \right) \, dx \\
\leq C |D(u)|_\infty |\nabla H|_2^2 + C |H|_\infty |\nabla H|_2 |u|_{D^2},
\]
we quickly have the following estimate from (4.18)-(4.20):
\[
\frac{d}{dt} |\nabla H|_2^2 \leq C (|D(u)|_\infty + 1) |\nabla H|_2^2 + \epsilon |\nabla^2 u|_2^2.
\] (4.21)

Adding (4.16) and (4.21) to (4.14), from Gronwall’s inequality we immediately obtain
\[
|\nabla u(t)|_2^2 + |\nabla \rho(t)|_2^2 + |\nabla H(t)|_2^2 + \int_0^t |\nabla^2 u(s)|_2^2 \, dt \leq C, \quad 0 \leq t < T.
\]

Next, we proceed to improve the regularity of \( \rho, H \) and \( u \). To this end, we first drive some bounds on derivatives of \( u \) based on estimates above. Now we give the estimates for the lower order terms of the velocity \( u \).

**Lemma 4.4 (Lower order estimate of the velocity \( u \)).**
\[
|u(t)|_{D^2}^2 + |\sqrt{\rho} u_t(t)|_2^2 + \int_0^T |u_t|_{D^2}^2 \, dt \leq C, \quad 0 \leq t \leq T,
\]
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, \bar{T}) \)).

**Proof.** Via (1.34) and Lemmas 2.4, 4.1-4.3, we show that
\[
|u|_{D^2} \leq C (|\sqrt{\rho} u_t|_2 + 1).
\] (4.22)

Differentiating (1.54) with respect to \( t \), we have
\[
\rho u_t + Lu_t = -\rho_t u_t - \rho u \cdot \nabla u_t - \rho \nabla u - \rho u_t \cdot \nabla u - \nabla P_t + (\text{rot} H \times H)_t.
\] (4.23)

Multiplying (4.23) by \( u_t \) and integrating over \( \Omega \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} (\mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2) \, dx \\
= -\int_{\Omega} \rho u \cdot \nabla |u_t|^2 \, dx - \int_{\Omega} \rho u \nabla u \cdot u_t \, dx - \int_{\Omega} \rho u_t \cdot \nabla u \cdot u_t \, dx + \int_{\Omega} P_t \nabla u_t \, dx \\
+ \int_{\Omega} H \cdot H_t \, dx - \int_{\Omega} \left( H \cdot \nabla u_t \cdot H_t + H_t \nabla u_t \cdot H \right) \, dx \equiv \sum_{i=7}^{12} L_i,
\] (4.24)
where we have used the fact \( \text{div} H = 0 \).
According to Lemmas 4.1-4.3, Holder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we deduce that

\begin{align}
L_7 = & - \int_{\Omega} \rho u \cdot \nabla |u_t|^2 \, dx 
\leq C|\rho|_{\infty}^{\frac{1}{2}} C|\rho|_{\infty}^{\frac{1}{2}} |u_t|_{2} |\nabla u|_{2} 
\leq C |\nabla u|_{2}^{2} + |u_t|_{2}^{2} + |\nabla u|_{2}^{2},
\end{align}

\begin{align}
L_8 = & - \int_{\Omega} \rho u \nabla (u \cdot \nabla u_t) \, dx 
\leq C \int_{\Omega} (|u| |\nabla u_t|^2 + |u_t|^2 |\nabla u| + |u|^2 |\nabla u_t| |\nabla u|) \, dx 
\leq C |u|_{6}^{2} |\nabla u|_{2}^{2} + C |u_t|_{6}^{2} |\nabla u|_{2}^{2} + C |u|_{6}^{2} |\nabla u|_{2}^{2} 
\leq C |\nabla u|_{1} |\nabla u_t|_{2} \leq \epsilon |\nabla u_t|_{2}^{2} + C(\epsilon) |\nabla u|_{1}^{2},
\end{align}

where we have used the fact that

\begin{align}
|u|_{2}^{2} \leq C |u|_{6}^{2} \leq C |\nabla u|_{2}^{2}, \quad |\nabla u|_{2}^{2} \leq C |\nabla u|_{2} |\nabla u|_{2} \leq C |\nabla u|_{2} |\nabla u|_{1}.
\end{align}

And similarly, we also have

\begin{align}
L_9 = & - \int_{\Omega} \rho u_t \cdot \nabla u - u_t \, dx 
\leq \epsilon |\nabla u_t|_{2}^{2} + C(\epsilon) |\nabla u|_{1}^{2},
\end{align}

\begin{align}
L_{10} = & \int_{\Omega} P \text{div} u_t \, dx 
\leq C |\nabla u|_{2}^{2} + C |\nabla u_t|_{2}^{2},
\end{align}

\begin{align}
L_{11} + L_{12} = & \int_{\Omega} H \cdot H_t \text{div} u_t \, dx + \int_{\Omega} (H \cdot \nabla u_t \cdot H_t + H_t \nabla u_t \cdot H) \, dx 
\leq C |\nabla u_t|_{2} |H|_{2} |\nabla H|_{2} \leq C |\nabla u_t|_{2} |H|_{2} |\nabla H|_{2} 
\leq C |\nabla u_t|_{2}^{2} + C(\epsilon) |\nabla u|_{1}^{2}.
\end{align}

Then combining the above estimate (4.25)-(4.27), from (4.24), we have

\begin{align}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} |\nabla u_t|^2 \, dx \leq C (\sqrt{\rho u_t} + 1) (|\nabla u_t|^2 + 1).
\end{align}

Then integrating (4.28) over \((\tau, t) \ (\tau \in (0, t))\), for \(\tau \leq t \leq T\), we have

\begin{align}
|\sqrt{\rho u_t(t)}|_{2}^{2} + t \int_{\tau}^{t} |\nabla u_t|_{2}^{2} \, ds \leq \frac{1}{2} |\sqrt{\rho u_t(t)}|_{2}^{2} + t \int_{\tau}^{t} (|\nabla u|^2 + 1) \sqrt{\rho u_t}^2 \, ds + C.
\end{align}

From the momentum equations (1.5), we easily have

\begin{align}
|\sqrt{\rho u_t(t)}|_{2}^{2} \leq C \int_{\Omega} \rho |u_t|^2 |\nabla u|^2 \, dx + C \int_{\Omega} \frac{\nabla P + Lu - \text{rot} H \times H}{\rho} \, dx,
\end{align}

due to the initial layer compatibility condition (1.9), letting \(\tau \to 0\) in (4.30), we have

\begin{align}
\limsup_{\tau \to 0} |\sqrt{\rho u_t(t)}|_{2}^{2} \leq C \int_{\Omega} \rho_0 |u_0|^2 |\nabla u_0|^2 \, dx + C |g_1|^2 \, dx \leq C.
\end{align}
Then, letting $\tau \to 0$ in (4.29), from Gronwall’s inequality and (4.22), we deduce that

$$|\sqrt{\rho}u_t(t)|^2 + |u(t)|_{D^2} + \int_0^t |\nabla u_t|^2_{D^1} ds \leq C, \ 0 \leq t \leq T.$$  (4.32)

Finally, the following lemma gives bounds of $\nabla \rho$, $\nabla H$ and $\nabla^2 u$.

**Lemma 4.5.**

$$\left(\|(\rho, H, P)(t)\|_{W^{1,q}} + |(\rho_t, H_t, P_t)(t)|_q\right) + \int_0^T |u(t)|^2_{D^{2,1}} dt \leq C, \ 0 \leq t < T,$$  (4.33)

where $C$ only depends on $C_0$ and $T$ (any $T \in (0, T]$), and $q \in (3, 6]$.

**Proof.** Via (1.5) and Lemmas 2.4, 4.1-4.4, we show that

$$|\nabla^2 u|_q \leq C(\|\rho u_t\|_q + |\rho u \cdot \nabla u|_q + |\nabla P|_q + |\text{rot} H \times H|_q + |u|_{D^{1,q}})$$

$$\leq C(1 + |\nabla u_t|_2 + |\nabla P|_q + |\nabla H|_q).$$  (4.34)

Firstly, applying $\nabla$ to (1.5), multiplying the result equations by $q|\nabla \rho|^{q-2} \nabla \rho$, we have

$$(|\nabla \rho|^q)_t + \text{div}(|\nabla \rho|^{q} u) + (q-1)|\nabla \rho|^{q} \text{div}$$

$$= -q|\nabla \rho|^{q-2}(\nabla \rho)^\top D(u)(\nabla \rho) - q|\nabla \rho|^{q-2} \nabla \rho \cdot \nabla \text{div} u.$$  (4.35)

Then integrating (4.35) over $\Omega$, we immediately obtain

$$\frac{d}{dt}|\nabla \rho|_q \leq C|D(u)|_\infty |\nabla \rho|_q + C|\nabla^2 u|_q.$$  (4.36)

Secondly, applying $\nabla$ to (1.5), multiplying the result equations by $q|\nabla H|^{q-2} \nabla H$, we have

$$(|\nabla H|^2)_t - qA : \nabla H|\nabla H|^{q-2} + qB \nabla H|\nabla H|^{q-2} + qC : \nabla H|\nabla H|^{q-2} = 0.$$  (4.37)

Then integrating (4.37) over $\Omega$, due to

$$\int_{\Omega} A : \nabla H|\nabla H|^{q-2} dx$$

$$= \int_{\Omega} \sum_{j=1}^3 \left( \sum_{i,k} \partial_j H_k \partial_k u^i \partial_j H^i \right) |\nabla H|^{q-2} dx + \int_{\Omega} \sum_{j=1}^3 \sum_{i=1}^3 \sum_{k=1}^3 H_k \partial_j u^i \partial_j H^i |\nabla H|^{q-2} dx$$

$$\leq C|D(u)|_\infty |\nabla H|_q^q + C|H|_\infty |\nabla H|_q^{q-1} |u|_{D^{2,q}},$$  (4.38)
\[ \int_{\Omega} B : \nabla H |\nabla H|^{q-2} dx \]

\[ = \int_{\Omega} \sum_{j=1}^{3} \sum_{i=1}^{3} \sum_{k=1}^{3} \partial_j u^k \partial_k H^i \partial_j H^i |\nabla H|^{q-2} dx + \int_{\Omega} \sum_{j=1}^{3} \sum_{i=1}^{3} u^k \partial_k H^i \partial_j H^i |\nabla H|^{q-2} dx \]

\[ = \int_{\Omega} \sum_{i=1}^{3} \left( \sum_{j,k} \partial_j u^k \partial_k H^i \partial_j H^i \right) |\nabla H|^{q-2} dx + \frac{1}{2} \int_{\Omega} \sum_{k=1}^{3} u^k \left( \sum_{j,i} \partial_k |\partial_j H|^2 |\nabla H|^{q-2} \right) dx \]

\[ = \int_{\Omega} \sum_{i=1}^{3} \left( \sum_{j,k} \partial_k H^i \partial_j u^k \partial_j H^i \right) |\nabla H|^{q-2} dx + \frac{1}{2} \int_{\Omega} \sum_{k=1}^{3} u^k \left( \sum_{j,i} \partial_k |\nabla H|^2 |\nabla H|^{q-2} \right) dx \]

\[ = \int_{\Omega} \sum_{i=1}^{3} \left( \sum_{j,k} \partial_k H^i \partial_j u^k \partial_j H^i \right) |\nabla H|^{q-2} dx + \frac{1}{q} \int_{\Omega} \sum_{k=1}^{3} u^k \partial_k |\nabla H|^q dx \]

\[ \leq C |D(u)|_\infty |\nabla H|_q^q, \]

\[ \int_{\Omega} C : \nabla H |\nabla H|^{q-2} dx = \int_{\Omega} (\text{div} u |\nabla H|^q + (H \otimes \nabla \text{div} u) : \nabla H |\nabla H|^{q-2}) dx \]

\[ \leq C |D(u)|_\infty |\nabla H|_q^q + C |H|_\infty |\nabla H|_q^{q-1} |u|_{D^{2,q}}, \quad (4.39) \]

we quickly obtain the following estimate:

\[ \frac{d}{dt} |\nabla H|_q \leq C (|D(u)|_\infty + 1) |\nabla H|_q + C |u|_{D^{2,q}}. \quad (4.40) \]

Then from (4.34), (4.36), (4.40) and Gronwall’s inequality, we immediately have

\[ (|\nabla \rho(t)|_q + |\nabla H(t)|_q) \leq C \exp \left( \int_0^t (1 + |D(u)|_\infty) ds \right) \leq C, \quad 0 \leq t \leq T. \]

Finally, via (4.34) and Lemma 4.1, we easily have

\[ \int_0^t |u(s)|^2_{D^{2,q}} ds \leq C \int_0^t (1 + |u_t(s)|^2_s) ds \leq C, \quad 0 \leq t \leq T. \quad (4.41) \]

\[ \square \]

4.2. Improved regularity.

In this section, we will get some higher order regularity of \( H, \rho \) and \( u \) to make sure that this solution is a classical one in \([0, T]\). Based on the estimates obtained in the above section, in truth, we have already proved that \( \int_0^T |\nabla u|_\infty^2 ds \leq C \).

**Lemma 4.6** (Higher order estimate).

\[ (|\rho, P, H(t)|^2_{D^2} + ||\rho_t, P_t, H_t(t)||_2^2) + \int_0^T \left( |u|^2_{D^3} + |(\rho u_t, P_{tt}, H_{tt})|^2_2 \right) dt \leq C, \quad 0 \leq t < T, \]

where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T) \)).

**Proof.** Via (1.3) and Lemmas 2.4, 3.1, 3.5 we show that

\[ |u|^2_{D^3} \leq C (|\rho u_t|_{D^1} + |\rho u \cdot \nabla u|_{D^1} + |\nabla P|_{D^1} + |\text{rot} H \times H|_{D^1}) \]

\[ \leq C (1 + |u_t|_{D^1} + |P|_{D^2} + |H|_{D^2}). \quad (4.42) \]
Firstly, applying $\nabla^2$ to (1.5), and multiplying the result equation by $2\nabla^2 \rho$, integrating over $\Omega$ we easily deduce that
\[
\frac{d}{dt} |\rho|_{D^2}^2 \leq C |\nabla u|_{\infty} |\rho|_{D^2}^2 + C |\rho|_{\infty} |u|_{D^3} |\rho|_{D^2} + |\nabla \rho|_3 |\nabla^2 \rho||\nabla^2 u|_1, \tag{4.43}
\]
which, together with (4.42),
\[
\frac{d}{dt} |\rho|_{D^2} \leq C (|\nabla u|_{\infty} + 1)(1 + |\rho|_{D^2} + |P|_{D^2} + |H|_{D^2}) + C |\nabla u_t|_2^2. \tag{4.44}
\]
And similarly, we have
\[
\begin{align*}
\frac{d}{dt} |H|_{D^2} &\leq C (|\nabla u|_{\infty} + 1)(1 + |P|_{D^2} + |H|_{D^2}) + C |\nabla u_t|_2^2, \\
\frac{d}{dt} |P|_{D^2} &\leq C (|\nabla u|_{\infty}^2 + 1)(1 + |P|_{D^2} + |H|_{D^2}) + C |\nabla u_t|_2^2. 
\end{align*} \tag{4.45}
\]
So combining (4.44)–(4.45), we quickly have
\[
\frac{d}{dt} (|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) 
\leq C (1 + |\nabla u|_{\infty})(|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2}) + C (1 + |\nabla u_t|_2^2). \tag{4.46}
\]
Then via Gronwall’s inequality and (4.46), we obtain
\[
|\rho|_{D^2} + |H|_{D^2} + |P|_{D^2} + \int_0^t |u(s)|_{D^3}^2 dt \leq C, \quad 0 \leq t \leq T.
\]
Finally, due to the following relation
\[
\begin{align*}
H_t &= H \cdot \nabla u - u \cdot \nabla H - H \text{div} u, \\
\rho_t &= -u \cdot \nabla \rho - \rho \text{div} u, \quad P_t = -u \cdot \nabla P - \gamma P \text{div} u, \tag{4.47}
\end{align*}
\]
we immediately get the desired conclusions. □

Now we will give some estimates for the higher order terms of the velocity $u$ in the following three Lemmas.

**Lemma 4.7 (Higher order estimate of the velocity $u$).**

\[
t |u_t(t)|_{D^1}^2 + t |u(t)|_{D^3}^2 + \int_0^T t (|u_t|_{D^2}^2 + |\sqrt{\rho} u_{tt}|_2^2) dt \leq C, \quad 0 \leq t \leq T,
\]
where $C$ only depends on $C_0$ and $T$ (any $T \in (0, \overline{T})$).

**Proof.** Firstly, multiplying (4.23) by $u_{tt}$ and integrating over $\Omega$, we have
\[
\int_\Omega \rho |u_{tt}|^2 dx + \frac{1}{2} \frac{d}{dt} \int_\Omega (\mu |\nabla u_t|^2 + (\lambda + \mu)(\text{div} u_t)^2) dx 
= \int_\Omega \left( - \nabla P_t - (\rho u \cdot \nabla) u_t - \rho_t u_t + (\text{rot} H \times H_t) \right) \cdot u_{tt} dx = \frac{d}{dt} \Phi_1(t) + \Phi_2(t), \tag{4.48}
\]
Then multiplying (4.48) by $s$, there exists a sequence $\{t_\rho\}$ where we have used the facts for $\tau$ from (4.49), for $0 \leq t \leq T$, we have

Let we denote \( \Phi(t) = \frac{1}{2} \int_\Omega |\nabla u_t|^2 + (\lambda + \mu)(\text{div}u)^2 dx - \Lambda_3(t), \)

then from (4.49), for $0 \leq t \leq T$, we quickly have

\[
C|\nabla u_t|_2^2 - C \leq \Phi(t) \leq C|\nabla u_t|_2 + C. \tag{4.50}
\]

Similarly, according to Lemmas [4.2, 4.6] Holder’s inequality and Gagliardo-Nirenberg inequality, for $0 < t \leq T$, we deduce that

\[
L_{13} \leq C|P_t|_2|\nabla u_t|_2, \quad L_{14} \leq |\rho_3|_{\infty} \sqrt{\mu} |u_t|_2(|u|_\infty |\nabla u_t|_2 + |\nabla u_3|_2 |u_t|_2),
\]

\[
L_{15} \leq C|P_t|_2|\nabla u_t|_2 |\nabla u_3|_2 |u_t|_\infty, \quad L_{16} \leq C|\rho_3|_2 |\nabla u_6|_2 |\nabla u_t|_2 + C|u|_\infty |u_6|_2 |\nabla u_t|_2 |\rho_3|_3, \tag{4.51}
\]

\[
L_{17} \leq C|\rho_3|_3 |\nabla u_t|_2 |u|_\infty |u_6|_2 + C|\rho_3|_\infty \sqrt{\mu} |u_t|_2 |\nabla u_t|_2, \]

where we have used the facts $\rho_t = -\text{div}(\rho u)$, and

\[
L_{18} = -\int_\Omega \left( \text{rot}H \times H \right)_t \cdot dx = \int_\Omega (H \otimes H - \frac{1}{2} |H|^2 I_3)_t : \nabla u_t dx \leq C|\nabla u_t|_2 |H_t|_4^2 + C|\nabla u_t|_2 |H_{tt}|_2 |H|_\infty \tag{4.52}\]

Combining (4.51)–(4.52), from Young’s inequality, we have

\[
\Phi(t) \leq \frac{1}{2} |\sqrt{\rho} u_t(t)|_2^2 + C(1 + |\nabla u_1|_2^2 |\nabla u_t|_2^2 |\nabla u_2|_\infty^2 + C(|P_t|_2^2 + |\rho t|_2^2 + |H_{tt}|_2^2. \tag{4.53}
\]

Then multiplying (4.48) by $t$ and integrating the result inequality over $\tau, t$ (\( \tau \in (0, t) \)), from (4.50) and (4.53), we have

\[
\int_\tau^t s|\sqrt{\rho} u_{tt}(s)|_2^2 ds + t|\nabla u_t(t)|_2^2 \leq t|u_t(\tau)|_2^2 + C \int_\tau^t s(1 + |\nabla u_1|_2^2 |\nabla u_t|_2^2 ds + C \tag{4.54}
\]

for $t \leq t \leq T$. From Lemma [4.4] we have $\nabla u_t \in L^2([0, T]; L^2)$, then according to Lemma [2.3], there exists a sequence $s_k$ such that

\[
s_k \to 0, \quad \text{and} \quad s_k |\nabla u_t(s_k)|_2^2 \to 0, \quad \text{as} \quad k \to \infty.
\]
Therefore, letting \( \tau = s_k \to 0 \) in \((4.54)\), from Gronwall's inequality, we have
\[
\int_0^t s|\sqrt{\rho_u}u(s)|_2^2 ds + t|u(t)|_{D^1}^2 \leq C \exp \left( \int_0^t (1 + |\nabla u(t)|_2^2) ds \right) \leq C.
\]
From \((4.42)\) \((4.54)\), Lemmas 2.4 and 4.1-4.6 we immediately have
\[
t|u(t)|_{D^2}^2 + \int_0^t s|\rho_u|_2^2 ds \leq C(t|u(t)|_{D^0} + 1) + C \int_0^t s(1 + |\sqrt{\rho_u}u|_2^2) ds \leq C.
\]

\[\square\]

**Lemma 4.8 (Higher order estimate of the velocity \( u \)).**
\[
(|\rho, P, H|)(t)|_{D^{2,q}} + t(|\rho_t, P_t, H_t|(t)|_{D^{1,q}}) + \int_0^T |u|_{D^{3,q}}^{p_0} dt \leq C,
\]
where \( C \) only depends on \( C_0 \) and \( T \) (any \( T \in (0, T) \)).

**Proof.** From Lemmas 2.4 and 4.1-4.6 we easily obtain
\[
|u|_{D^{3,q}} \leq C(|\rho u_t + \rho \cdot \nabla u|_{D^{1,q}} + |\text{rot} H \times H|_{D^{1,q}} + |P|_{D^{2,q}})
\leq C(|u_t|_\infty + |\nabla u_t|_q + |u|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}}).
\]

Due to the Sobolev inequality, Poincare inequality and Young's inequality, we have
\[
|u_t|_\infty \leq C|u_t|_q^{1 - \frac{2}{q}}\|u_t\|_{W^{1,q}}^\frac{2}{q} \leq C|\nabla u_t|_2 + C|\nabla u_t|_q,
\]
then we have
\[
|u(t)|_{D^{3,q}} \leq C(|\nabla u_t|_2 + |\nabla u_t|_q + |u|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}}).
\]
According to Lemmas 4.3-4.7 via the completely same argument in \((3.31)\), we have
\[
\int_0^t C(|\nabla u_t|_2 + |\nabla u_t|_q + |u|_{D^{2,q}})^{p_0} ds \leq C.
\]

Then, applying \( \nabla^2 \) to \((1.53)\), and multiplying the result equation by \( q|\nabla^2 \rho||\nabla^2 \rho|^{q-2} \), integrating over \( \Omega \) we easily deduce that
\[
\frac{d}{dt}|\rho|_{D^{2,q}}^q \leq C(|\nabla u|_\infty|\rho|_{D^{2,q}}^q + C|\rho|_\infty|u|_{D^{2,q}}|\rho|_{D^{2,q}}^{q-1} + |\nabla \rho|_\infty|u|_{D^{2,q}}|\rho|_{D^{2,q}}^{q-1},
\]
which, together with \((4.42)\),
\[
\frac{d}{dt}|\rho|_{D^{2,q}} \leq C(|\nabla u|_\infty + F + 1)(1 + |\rho|_{D^{2,q}} + |P|_{D^{2,q}} + |H|_{D^{2,q}}) + CF,
\]
where \( F = |\nabla u_t|_2 + |\nabla u_t|_q + |u|_{D^{2,q}}. \) And similarly, we have
\[
\begin{aligned}
\frac{d}{dt}|H|_{D^{2,q}} &\leq C(|\nabla u|_\infty + F + 1)(1 + |\rho|_{D^{2,q}} + |P|_{D^{2,q}} + |H|_{D^{2,q}}) + F, \\
\frac{d}{dt}|P|_{D^{2,q}} &\leq C(|\nabla u|_\infty + F + 1)(1 + |\rho|_{D^{2,q}} + |P|_{D^{2,q}} + |H|_{D^{2,q}}) + F.
\end{aligned}
\]
So we combining \((4.58)\) - \((4.59)\), we quickly have
\[
\frac{d}{dt}(|\rho|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}})
\leq C(1 + |\nabla u|_\infty + F)(1 + |\rho|_{D^{2,q}} + |P|_{D^{2,q}} + |H|_{D^{2,q}}) + C(1 + F).
\]
Then via Gronwall’s inequality, \((4.56)\) and \((4.60)\), we obtain
\[
|\rho|_{D^{2,q}} + |H|_{D^{2,q}} + |P|_{D^{2,q}} + \int_0^t |u(s)|_{D^{3,q}}^2 \, ds \leq C, \quad 0 \leq t \leq T.
\]

Finally, due to relation \((4.47)\), we immediately get the desired conclusions.

Finally, we have

**Lemma 4.9 (Higher order estimate of the velocity \(u\)).**
\[
t^2|u(t)|_{D^{3,q}} + t^2|u_t(t)|_{D^{2}} + t^2|\sqrt{\rho}u_H(t)|_{D^{0}}^2 + \int_0^T s^2|u_H(s)|_{D^{0}}^2 \, ds \leq C
\]

where \(C\) only depends on \(C_0\) and \(T\) (any \(T \in (0, T]\)).

This lemma can be easily proved via the method used in Lemma \([4.7]\) here we omit it. And this will be enough to extend the regular solutions of \((H, \rho, u, P)\) beyond \(t \geq T\).

In truth, in view of the estimates obtained in Lemmas \([4.1]-[4.8]\) we quickly know that the functions \((H, \rho, u, P)|_{t=T} = \lim_{t \to T}(H, \rho, u, P)\) satisfies the conditions imposed on the initial data \([1.8]-[1.9]\). Therefore, we can take \((H, \rho, u, P)|_{t=T}\) as the initial data and apply the local existence Theorem \([1.1]\) to extend our local classical solution beyond \(t \geq T\). This contradicts the assumption on \(T\).

**References**


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