LOWER ORDER FINITE ELEMENT APPROXIMATIONS OF
SYMMETRIC TENSORS ON SIMPLICIAL GRIDS IN $\mathbb{R}^n$

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ABSTRACT. In this paper, we construct, in a unified fashion, lower order finite element subspaces of spaces of symmetric tensors with square-integrable divergence on a domain in any dimension. These subspaces are essentially the symmetric $H(\text{div}) - P_k$ ($1 \leq k \leq n$) tensor spaces, enriched, for each $n-1$ dimensional simplex, by $\frac{(n+1)n}{2} H(\text{div}) - P_{k+1}$ bubble functions when $1 \leq k \leq n-1$, and by $\frac{(n-1)n}{2} H(\text{div}) - P_{k+1}$ bubble functions when $k = n$. These spaces can be used to approximate the symmetric matrix field in a mixed formulation problem where the other variable is approximated by discontinuous piecewise $P_{k-1}$ polynomials. This in particular leads to first order mixed elements on simplicial grids with total degrees of freedom per element 18 plus 3 in 2D, 48 plus 6 in 3D. The previous record of the degrees of freedom of first order mixed elements is, 21 plus 3 in 2D, and 156 plus 6 in 3D, on simplicial grids. We also derive, in a unified way and without using any tools like differential forms, a family of auxiliary mixed finite elements in any dimension. One example in this family is the Raviart-Thomas elements in one dimension, the second example is the mixed finite elements for linear elasticity in two dimensions due to Arnold and Winther, the third example is the mixed finite elements for linear elasticity in three dimensions due to Arnold, Awanou and Winther.

Keywords. mixed finite element, symmetric finite element, first order system, conforming finite element, simplicial grid, inf-sup condition.

AMS subject classifications. 65N30, 73C02.

1. INTRODUCTION

The constructions, using polynomial shape functions, of stable pairs of finite element spaces for approximating the pair of spaces $H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^n)$ in first order systems are a long-standing, challenging and open problem, see [4, 6]. For mixed finite elements of linear elasticity, many mathematicians have been working on this problem and compromised to weakly symmetric or composite elements, cf. [3, 7, 8, 32, 34, 36, 37, 38]. It is not until 2002 that Arnold and Winther were able to propose the first family of mixed finite element spaces with polynomial shape functions in two dimensions [10]. Such a two dimensional family was extended to a three dimensional family of mixed elements [6], while the lowest order element with $k = 2$ was first proposed in [2]. We refer interested readers to [2, 5, 6, 10, 12, 14, 11, 21, 27, 33, 40, 9, 13, 19, 23, 24, 29, 28], for recent progress on mixed finite elements for linear elasticity.

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In very recent papers [30] and [31], Hu and Zhang attacked this open and challenging problem by initially proposing new ideas to design discrete stress spaces and analyze the discrete inf-sup condition. In particular, they were able to construct suitable $H(\text{div}, \Omega; S) - P_k$ space, with $k \geq 3$ for 2D, and $k \geq 4$ for 3D, finite element spaces for the stress discretization in both two and three dimensions. In [20], Hu constructed, in a unified fashion, suitable $H(\text{div}, \Omega; S) - P_k$ space with $k \geq n + 1$, and proposed a set of degrees of freedom for the shape function space, in any dimension.

The purpose of this paper is to extend those elements in [20] to lower order cases where $1 \leq k \leq n$. Since it is, at moment, very difficult to prove that the pair of $H(\text{div}) - P_k$ and $L^2 - P_{k-1}$ spaces is stable, the $H(\text{div}) - P_k$ space has to be enriched by some higher order polynomials whose divergence are in $P_{k-1}$. Thanks to [20], it suffices to control the piecewise rigid motion space. Hence, we only need to add, for each $n - 1$ dimensional simplex, $\frac{(n+1)n}{2} H(\text{div}) - P_{n+1}$ bubble functions when $2 \leq k \leq n - 1$, and $\frac{(n-1)n}{2} H(\text{div}) - P_{n+1}$ bubble functions when $k = n$. This in particular leads to first order mixed elements on simplicial grids with total degrees of freedom per element $18 + 3$ in 2D, $48 + 6$ in 3D. The previous record of the degrees of freedom of first order mixed elements is, $21 + 3$ in 2D, and $156 + 6$ in 3D, on simplicial grids. These enriched bubble functions belong to the lowest order space from a family of auxiliary discrete stress spaces which, together with the $P_{k-1}$ space, form a stable pair of spaces for first order systems. Note that these spaces in this auxiliary family are constructed in a unified and direct way and that no tools like differential forms are used. One example in this auxiliary family is the Raviart–Thomas elements in one dimension, the second example is the mixed finite elements for linear elasticity in two dimensions due to Arnold and Winther [10], the third example is the mixed finite elements for linear elasticity in three dimensions due to Arnold, Awanou and Winther [6].

We end this section by introducing first order systems and related notations. We consider mixed finite element methods of first order systems with symmetric tensors: Find $(\sigma, u) \in \Sigma \times V := H(\text{div}, \Omega; S) \times L^2(\Omega; \mathbb{R}^n)$, such that

\begin{align}
&\begin{cases}
(A\sigma, \tau) + (\text{div} \tau, u) = 0 & \text{for all } \tau \in \Sigma, \\
(\text{div} \sigma, v) = (f, v) & \text{for all } v \in V.
\end{cases}
\end{align}

Here the symmetric tensor space for the stress $\Sigma$ is defined by

\begin{align}
H(\text{div}, \Omega; S) := \left\{ \tau = \begin{pmatrix}
\tau_{11} & \cdots & \tau_{1n} \\
\vdots & \ddots & \vdots \\
\tau_{n1} & \cdots & \tau_{nn}
\end{pmatrix} \in H(\text{div}, \Omega; \mathbb{R}^{n \times n}) \middle| \tau^T = \tau \right\},
\end{align}

and the space for the vector displacement $V$ is

\begin{align}
L^2(\Omega; \mathbb{R}^n) := \left\{ (u_1, \cdots, u_n)^T \middle| u_i \in L^2(\Omega), i = 1, \cdots, n \right\}.
\end{align}

This paper denotes by $H^k(T; X)$ the Sobolev space consisting of functions with domain $T \subset \mathbb{R}^n$, taking values in the finite-dimensional vector space $X$, and with all derivatives of order at most $k$ square-integrable. For our purposes, the range space $X$ will be either $\mathbb{S}$, $\mathbb{R}^n$, or $\mathbb{R}$. Let $\| \cdot \|_{k,T}$ be the norm of $H^k(T)$, $\mathbb{S}$ denote the space of symmetric tensors, $H(\text{div}, T; S)$ consist of square-integrable symmetric
matrix fields with square-integrable divergence. The $H(\text{div})$ norm is defined by
\[ \|\tau\|_{H(\text{div},T)}^2 := \|\tau\|_{0,T}^2 + \|\text{div}\tau\|_{0,T}^2. \]
$L^2(T;\mathbb{R}^n)$ is the space of vector-valued functions which are square-integrable. Here, the compliance tensor $A = A(x) : \mathbb{S} \to \mathbb{S}$, characterizing the properties of the material, is bounded and symmetric positive definite uniformly for $x \in \Omega$.

The rest of the paper is organized as follows. In the next section, we present some preliminary results from [26]; see also [30] and [31], for the cases $n = 2$ and $n = 3$, respectively. In Section 3, based on these preliminary results, we propose a family of auxiliary mixed finite elements. In Section 4, we present lower order mixed finite elements and analyze the well–posedness of the discrete problem and error estimates of the approximation solution. In Section 5, we present the first order mixed elements. In Section 6, we give a way to construct these added bubble functions for each face in three dimensions. The paper ends with Section 6 which lists some numerics.

2. Preliminary results

Suppose that the domain $\Omega$ is subdivided by a family of shape regular simplicial grids $\mathcal{T}_h$ (with the grid size $h$). We introduce the finite element space of order $k \geq 1$ on $\mathcal{T}_h$.

\begin{equation}
\Sigma_{k,h} := \left\{ \sigma \in H(\text{div}, \Omega; \mathbb{S}), \sigma|_K \in P_k(K; \mathbb{S}) \quad \forall K \in \mathcal{T}_h \right\},
\end{equation}

where $P_k(K; \mathbb{X})$ denotes the space of polynomials of degree $\leq k$, taking value in the space $\mathbb{X}$.

To define the degrees of freedom for the shape function space $P_k(K; \mathbb{S})$, let $x_0, \cdots, x_n$ be the vertices of simplex $K$. The referencing mapping is then
\[ x := F_K(\hat{x}) = x_0 + (x_1 - x_0, \cdots, x_n - x_0) \hat{x}, \]
mapping the reference tetrahedron $\hat{K} := \{0 \leq \hat{x}_1, \cdots, \hat{x}_n, 1 - \sum_{i=1}^{n} \hat{x}_i \leq 1\}$ to $K$.

Then the inverse mapping is
\begin{equation}
\hat{x} := \begin{pmatrix}
\nu_1^T \\
\vdots \\
\nu_n^T
\end{pmatrix} (x - x_0),
\end{equation}

where
\begin{equation}
\begin{pmatrix}
\nu_1^T \\
\vdots \\
\nu_n^T
\end{pmatrix} = (x_1 - x_0, \cdots, x_n - x_0)^{-1}.
\end{equation}

By (2.2), these normal vectors are coefficients of the barycentric variables:
\[ \lambda_1 := \nu_1 \cdot (x - x_0), \]
\[ \vdots \]
\[ \lambda_n := \nu_n \cdot (x - x_0), \]
\[ \lambda_0 := 1 - \sum_{i=1}^{n} \lambda_i. \]
For any edge $x_i x_j$ of element $K$, $i \neq j$, let $t_{i,j}$ denote associated tangent vectors, which allow for us to introduce the following symmetric matrices of rank one

\begin{equation}
T_{i,j} := t_{i,j} t_{i,j}^T, \quad 0 \leq i < j \leq n.
\end{equation}

For these matrices of rank one, we have the following result from \cite{26}; see also \cite{30} and \cite{31}, for the cases $n = 2$ and $n = 3$, respectively.

Lemma 2.1. The symmetric tensors $T_{i,j}$ in (2.4) are linearly independent, and form a basis of $\mathcal{S}$.

With these symmetric matrices $T_{i,j}$ of rank one, we define a $H(\text{div}, K; \mathbb{S})$ bubble function space

\begin{equation}
\Sigma_{K,k,b} := \sum_{0 \leq i < j \leq n} \lambda_i \lambda_j P_{k-2}(K; \mathbb{R}) T_{i,j}
\end{equation}

Define the full $H(\text{div}, K; \mathbb{S})$ bubble function space consisting of polynomials of degree $\leq k$

\begin{equation}
\Sigma_{\partial K,k,0} := \{ \tau \in H(\text{div}, K; \mathbb{S}) \cap P_k(K; \mathbb{S}), \tau \nu|_{\partial K} = 0 \}.
\end{equation}

Here $\nu$ is the normal vector of $\partial K$. We have the following result due to \cite{26}.

Lemma 2.2. It holds that

\begin{equation}
\Sigma_{K,k,b} = \Sigma_{\partial K,k,0}.
\end{equation}

We need an important result concerning the divergence space of the bubble function space. To this end, we introduce the following rigid motion space on each element $K$.

\begin{equation}
R(K) := \{ v \in H^1(K; \mathbb{R}^n), (\nabla v + \nabla v^T)/2 = 0 \}.
\end{equation}

It follows from the definition that $R(K)$ is a subspace of $P_1(K; \mathbb{R}^n)$. For $n = 1$, $R(K)$ is the constant function space over $K$. The dimension of $R(K)$ is $\frac{n(n+1)}{2}$. For two dimensions, the rigid motion space $R(K)$ is

\begin{equation}
R(K) := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + b \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, a_1, a_2, b \in \mathbb{R} \right\};
\end{equation}

for three dimensions, the rigid motion space $R(K)$ reads

\begin{equation}
R(K) := \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + b_1 \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} + b_2 \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix}, a_i, b_i \in \mathbb{R}, i = 1, 2, 3 \right\}.
\end{equation}

This allows for defining the orthogonal complement space of $R(K)$ with respect to $P_{k-1}(K; \mathbb{R}^n)$ by

\begin{equation}
R^\perp(K) := \{ v \in P_{k-1}(K; \mathbb{R}^n), (v, w)_K = 0 \text{ for any } w \in R(K) \},
\end{equation}

where the inner product $(v, w)_K$ over $K$ reads $(v, w)_K = \int_K v \cdot w dx$.

Lemma 2.3. For any $K \in \mathcal{T}_h$, it holds that

\begin{equation}
\text{div} \Sigma_{K,k,b} = R^\perp(K).
\end{equation}

Proof. The proof can be found in \cite{26}; see also \cite{30} and \cite{31}, for the cases $n = 2$ and $n = 3$, respectively.
We need a classical result and its variant.

**Lemma 2.4.** It holds the following Chu-Vandermonde combinatorial identity and its variant

\begin{align}
\sum_{\ell=0}^{n} C_{n+1}^{\ell+1} C_{k-1}^{\ell} &= \sum_{\ell=0}^{n} C_{n+1}^{n-\ell} C_{k-1}^{\ell} = C_{n+k}^{n}, \\
\sum_{\ell=0}^{n} C_{n+1}^{\ell+1} C_{k-1}^{\ell} C_{\ell+1}^{2} &= \frac{(n+1)n}{2} C_{n+k-2}^{n},
\end{align}

where the combinatorial number $C_{n}^{m} = \frac{n! \cdots (n-m+1)!}{m! \cdots 1}$ for $n \geq m$ and $C_{n}^{m} = 0$ for $n < m$.

### 3. A family of auxiliary mixed elements in any dimension

#### 3.1. The lowest order auxiliary mixed elements.

To define lower order mixed finite elements with $k \leq n$, we need a family of auxiliary mixed elements. To this end, we introduce the following divergence free space for element $K \in \mathcal{T}_{h}$,

\begin{equation}
\Sigma_{3 \to n+1,DF}(K;S) := \{ \tau \in P_{n+1}(K;S) \setminus P_{2}(K;S), \text{div}\tau = 0 \}.
\end{equation}

It is straightforward to see that the dimension of the space $\Sigma_{3 \to n+1,DF}(K;S)$ reads

\begin{equation}
\left( \frac{(2n+1)!}{n!(n+1)!} - \frac{(n+2)!}{2!n!} \right) \frac{n(n+1)}{2} - n \frac{(2n)!}{n!n!} + n(n+1).
\end{equation}

Here $\left( \frac{(2n+1)!}{n!(n+1)!} - \frac{(n+2)!}{2!n!} \right) \frac{n(n+1)}{2}$ is the dimension of the space $P_{n+1}(K,S) \setminus P_{2}(K,S)$, and $n \left( \frac{(2n)!}{n!n!} - (n+1) \right)$ is the number of constraints by the divergence free. Then we can define the following enriched $P_{2}(K;S)$ space

\begin{equation}
P_{2}^{*}(K;S) := P_{2}(K;S) + \Sigma_{3 \to n+1,DF}(K;S).
\end{equation}

It follows that the dimension of $P_{2}^{*}(K;S)$ is equal to

\begin{equation}
\text{the dimension of } P_{2}(K;S) \text{ + the dimension of } \Sigma_{3 \to n+1,DF}(K;S) = \frac{(2n+1)!}{n!(n+1)!} \frac{n(n+1)}{2} - n \frac{(2n)!}{n!n!} + n(n+1).
\end{equation}

To present the degrees of freedom of $P_{2}^{*}(K;S)$, we define

\begin{equation}
M_{2}(K) := \{ \tau \in P_{2}^{*}(K;S), \text{div}\tau = 0 \text{ and } \tau\nu|_{\partial K} = 0 \},
\end{equation}

where $\nu$ is the normal vector of $\partial K$. For the space $M_{2}(K)$, we have the following important result.

**Lemma 3.1.** The dimension of $M_{2}(K)$ is

\begin{equation}
\frac{(2n-1)!}{n!(n-1)!} \frac{n(n+1)}{2} + \frac{n(n+1)}{2} - n \frac{(2n)!}{n!n!}.
\end{equation}
Proof. The dimension of the space $\Sigma_{K,n,1,b}$ reads
\begin{equation}
\frac{(2n-1)!}{n!(n-1)!} \frac{n(n+1)}{2}.
\end{equation}
Since the dimension of $R(K)$ is $\frac{n(n+1)}{2}$, the dimension of $R^\top(K)$ (with respect to $P_n(K,\mathbb{R}^n)$) is
\begin{equation}
\frac{n(2n)!}{n!n!} - \frac{n(n+1)}{2}.
\end{equation}
It follows from the definition of $P^*_2(K;\mathbb{S})$ and Lemma 2,2 that $M_2(K)$ contains all divergence free tensor-value functions of $\Sigma_{K,n,1,b}$. Then the desired result follows from Lemma 2.3.

**Theorem 3.1.** A matrix field $\tau \in P^*_2(K;\mathbb{S})$ can be uniquely determined by the following degrees of freedom:

1. For each $\ell$ dimensional simplex $\triangle_\ell$ of $K$, $0 \leq \ell \leq n-1$, with $\ell$ linearly independent tangential vectors $t_1, \cdots, t_\ell$, and $n-\ell$ linearly independent normal vectors $\nu_1, \cdots, \nu_{n-\ell}$, the mean moments of degree at most $n-\ell$ over $\triangle_\ell$, of $t_i^T \nu_i, \nu_j^T \nu_j$, $i = 1, \cdots, \ell$, $i, j = 1, \cdots, n-\ell$, $\{C_{n+1-\ell + \ell(n-\ell)}^2\}$ degrees of freedom for each $\triangle_\ell$;
2. the average of $\tau$ over $K$, $n(n+1)/2$ degrees of freedom;
3. the values of moments $\int_K \tau : \theta d\mathbf{x}$, $\theta \in M_2(K)$, $\frac{(2n-1)!}{n(n+1)!} \frac{n(n+1)}{2} + \frac{n(n+1)}{2} - \frac{n(2n)!}{n!n!}$ degrees of freedom.

Proof. We assume that all degrees of freedom vanish and show that $\tau = 0$. Note that the mean moment becomes the value of $\tau$ for a 0 dimensional simplex $\triangle_0$, namely, a vertex, of $K$. The first set of degrees of freedom implies that $\tau \nu = 0$ on $\partial K$ while the second set of degrees of freedom shows $\div \tau = 0$. Then the third set of degrees of freedom proves that $\tau = 0$. Next we shall prove that the sum of these degrees of freedom is equal to the dimension of the space $P^*_2(K,\mathbb{S})$. In fact the sum of the first set of degrees of freedom is
\begin{align*}
\sum_{\ell=0}^{n-1} C_{n+1}^{\ell+1} \frac{(n-\ell)(n+\ell+1)}{2} C_n^\ell,
\end{align*}
we refer interested readers to [26, Theorem 2.1] for a detailed proof of the numbers of degrees of freedom in the first set. By the Chu-Vandermonde combinatorial identity [24], and its variant [24], see more details from [26],
\begin{align*}
\sum_{\ell=0}^{n-1} C_{n+1}^{\ell+1} \frac{(n-\ell)(n+\ell+1)}{2} C_n^\ell = \frac{(2n+1)!}{n!(n+1)!} \frac{n(n+1)}{2} - \frac{(2n-1)!}{n!(n-1)!} \frac{n(n+1)}{2}.
\end{align*}
Hence the desired result follows from (3.4), and the sum of the second and third sets of degrees of freedom.

Then we define
\begin{equation}
\Sigma^*_2 : = \{ \tau \in H(\div, \Omega; \mathbb{S}), \tau|_K \in P^*_2(K;\mathbb{S}) \text{ for any } K \in T_h \}.
\end{equation}

**Remark 3.1.** For $n = 2$, we recover the lowest order element in [10]; for $n = 3$ we obtain the lowest order element in [2], see also [9].
To define a family of first order mixed elements, we need a family of simplified lowest order mixed elements, which is defined by

\[ \hat{P}_2^*(K; S) := \{ \tau \in P_2^*(K; S), \text{div} \tau \in R(K) \}. \tag{3.10} \]

The dimension of \( \hat{P}_2^*(K; S) \) is

\[ \frac{(2n+1)!}{n!(n+1)!} \frac{n(n+1)}{2} - n \frac{(2n)!}{n!n!} + n(n+1) \frac{n(n+1)!}{2}. \]

A complete set of degrees of freedom for \( \hat{P}_2^*(K; S) \) is obtained by removing the \( \frac{n(n+1)!}{2} \) average values over \( K \) for \( P_2^*(K; S) \). Then we define

\[ \hat{\Sigma}_2 := \{ \tau \in H(\text{div}; \Omega; \tau|_K \in \hat{P}_2^*(K; S) \text{ for any } K \in \mathcal{T}_h \}. \tag{3.11} \]

**Remark 3.2.** For \( n = 2, 3 \), we recover the simplified lowest order elements in [10] and [7], respectively.

### 3.2. Higher order auxiliary mixed elements.

To define auxiliary mixed elements of order \( k > 2 \), we introduce the following divergence free space for element \( K \in \mathcal{T}_h \),

\[ \Sigma_{k+1\rightarrow k+n-1,DF}(K; S) := \{ \tau \in \Sigma_k(n-1) \setminus P_k(K; S), \text{div} \tau = 0 \}. \tag{3.12} \]

Since the dimension of the space \( P_{k+n-2}(K; \mathbb{R}) \setminus P_k(K; \mathbb{R}) \) is

\[ \frac{(k+2n-2)!}{n!(n+1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)}{2} - n \frac{(k+2n)!}{n!n!} + n(n+1) \frac{n(n+1)!}{2}, \]

the number of the divergence free constraints is

\[ n \left( \frac{(k+2n)!}{n!(n+1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)}{2} - n \frac{(k+2n-2)!}{n!(n+1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)!}{2} \right). \]

In addition, the dimension of the space \( \Sigma_{k+n-1}(K; S) \setminus P_k(K; S) \) is

\[ \left( \frac{(k+2n-1)!}{n!(k+n-1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)}{2} - n \frac{(k+2n)!}{n!n!} + n(n+1) \frac{n(n+1)!}{2} \right). \]

It follows that the dimension of the space \( \Sigma_{k+1\rightarrow k+n-1,DF}(K; S) \) is

\[ \left( \frac{(k+2n)!}{n!(k+n-1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)}{2} - n \frac{(k+2n-2)!}{n!(n+1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)!}{2} \right). \tag{3.13} \]

Define the following enriched \( P_k(K; S) \) space

\[ P_k^*(K; S) := P_k(K; S) + \Sigma_{k+1\rightarrow k+n-1,DF}(K; S). \tag{3.14} \]

It follows that the dimension of \( P_k^*(K; S) \) is equal to

\[ \text{the dimension of } P_k(K; S) + \text{ the dimension of } \Sigma_{k+1\rightarrow k+n-1,DF}(K; S) \]

\[ = \frac{(k+2n-1)!}{n!(k+n-1)!} \frac{n(n+1)}{2} - n \left( \frac{(k+2n)!}{n!(k+n-1)!} \frac{(n+k)!}{k!n!} \frac{n(n+1)!}{2} \right). \tag{3.15} \]

To present the degrees of freedom of \( P_k^*(K; S) \), we define

\[ M_k(K) := \{ \tau \in P_k^*(K; S), \text{div} \tau = 0 \text{ and } \tau \nu|_{\partial K} = 0 \}, \tag{3.16} \]

where \( \nu \) is the normal vector of \( \partial K \). For the space \( M_k(K) \), we have the following important result.
Lemma 3.2. The dimension of $M_k(K)$ is

\[(3.17) \quad \frac{(k + 2n - 3)!}{n!(k + n - 3)!} \frac{n(n + 1)}{2} + \frac{n(n + 1)}{2} - \frac{(k + 2n - 2)!}{n!(k + n - 2)!}.\]

Proof. The dimension of the space $\Sigma_{K,k+n-1,b}$ reads

\[(3.18) \quad \frac{(k + 2n - 3)!}{n!(k + n - 3)!} \frac{n(n + 1)}{2} \]

Since the dimension of $R(K)$ is $\frac{n(n+1)}{2}$, the dimension of $R^\perp(K)$ (with respect to $P_{k+n-2}(K;\mathbb{R}^n)$) is

\[(3.19) \quad \frac{n}{n!(k + n - 3)!} \frac{(k + 2n - 2)!}{2} - \frac{n(n + 1)}{2}.
\]

It follows from the definition of $P_k^*(K;\mathbb{S})$ and Lemma 2.2 that $M_k(K)$ contains all divergence free tensor-value functions of $\Sigma_{K,k+n-1,b}$. Then the desired result follows from Lemma 2.3.

Theorem 3.2. A matrix field $\tau \in P_k^*(K;\mathbb{S})$ can be uniquely determined by the following degrees of freedom:

1. For each $\ell$ dimensional simplex $\Delta_\ell$ of $K$, $0 \leq \ell \leq n - 1$, with $\ell$ linearly independent tangential vectors $t_1, \ldots, t_\ell$, and $n - \ell$ linearly independent normal vectors $v_1, \ldots, v_{n-\ell}$, the mean moments of degree at most $k + n - \ell - 2$ over $\Delta_\ell$, of $t_1^T \tau v_1, \ldots, t_\ell^T \tau v_\ell$, $l = 1, \ldots, \ell$, $i, j = 1, \ldots, n - \ell$, $(C_{n+1-\ell}^2 + (\ell + n-\ell)C_{k+n-2}^\ell)C_{k+n-2}^\ell$ degrees of freedom for each $\Delta_\ell$;
2. the values $\int_K \tau : \theta dx$ for any $\theta \in \epsilon(P_{k-1}(K;\mathbb{R}^n))$, $nC_{n+k-1}^n - \frac{n(n+1)}{2}$ degrees of freedom;
3. the values $\int_K \tau : \theta dx$ for any $\theta \in M_k(K)$, \(\frac{(k+2n-3)!}{2} \frac{n(n+1)}{2} - \frac{n(n+1)}{2} - \frac{n(k+2n-2)!}{n!(k+n-2)!}\) degrees of freedom.

Proof. We assume that all degrees of freedom vanish and show that $\tau = 0$. Note that the mean moment become the value of $\tau$ for a 0 dimensional simplex $\Delta_0$, namely, a vertex, of $K$. The first set of degrees of freedom implies that $\tau v = 0$ on $\partial K$ while the second set of degrees of freedom shows $\text{div} \tau = 0$. Then the third set of degrees of freedom proves that $\tau = 0$.

Next we shall prove that the sum of these degrees of freedom is equal to the dimension of the space $P_k^*(K;\mathbb{S})$. In fact, it follows from the Chu-Vandermonde combinatorial identity (2.13) and its variant (2.14) that the number of degrees in the first set is

\[(3.20) \quad \sum_{\ell=0}^{n-1} C_{n+1}^{\ell+1} \frac{(n-\ell)(n+\ell+1)}{2} C_{n+k-2} = \frac{n(n+1)}{2} C_{k+2n-1}^n - C_{k+2n-3}^n,\]

we refer interested readers to [26] Theorem 2.1 for a detailed proof of the numbers of degrees of freedom in the first set. The desired result follows from (3.15) and (3.17).

Then we define

\[(3.21) \quad \Sigma_{k,h}^* := \{\tau \in H(\text{div}, \Omega;\mathbb{S}), \tau|_K \in P_k^*(K;\mathbb{S}) \text{ for any } K \in T_h\}.\]

Remark 3.3. For $n = 2$, we recover the higher order elements in [10]; for $n = 3$ we obtain the higher order elements in [6].
4. A family of lower order mixed elements

4.1. Mixed methods. We propose to use the spaces $\Sigma_{k,h}$, with $2 \leq k \leq n$, defined in (2.1) to approximate $\Sigma$. In order to get a stable pair of spaces, we take the discrete displacement space as the full $C^{-1} P_{k-1}$ space

$$V_{k,h} := \{ v \in L^2(\Omega; \mathbb{R}^n), \ v|_K \in P_{k-1}(K; \mathbb{R}^n) \text{ for all } K \in \mathcal{T}_h \}. $$

Unfortunately, we cannot establish the stability of the pair of spaces $\Sigma_{k,h}$ and $V_{k,h}$. We propose to enrich $\Sigma_{k,h}$ by some $n-1$ dimensional simplex bubble function spaces. Given a $n-1$ dimensional simplex $F$ of $\mathcal{T}_h$, let $\omega_F := K^{-} \cup K^{+}$ denote the union of two elements that share $F$. Define

$$\mathbb{B}_F^1 := \left\{ \tau \in \Sigma_{2,h}, \tau = 0 \text{ on } \Omega \setminus \omega_F, \int_F \tau \nu \cdot pds = 0 \text{ for any } p \in (R(|\omega_F|_F))^{\perp}, \right. \\
the averages of $\tau$ over both $K^{-}$ and $K^{+}$ vanish, \\
the values of $\int_K \tau : \theta dx$ vanish for any $\theta \in M_2(K), K = K^{-}$ and $K^{+}$.

Here $\nu$ is the normal vector of $F$, and $(R(|\omega_F|_F))^{\perp}$ is the orthogonal complement of the restriction $R(|\omega_F|_F)$ on $F$ of $R(|\omega_F|)$ with respect to the $L^2$ inner product over $F$. We also need a subspace of $\mathbb{B}_F^1$ defined by

$$\mathbb{B}_F^2 := \{ \tau \in \mathbb{B}_F^1, \int_F \tau \nu \cdot pds = 0 \text{ for any } p \in P_0(F, \mathbb{R}^n) \}. $$

Hence we define the following enriched stress space

$$\Sigma_{k,h}^+ = \Sigma_{k,h} + \sum_F \mathbb{B}_F^1 \text{ for } 2 \leq k \leq n - 1;$$

and

$$\Sigma_{k,h}^+ = \Sigma_{k,h} + \sum_F \mathbb{B}_F^2 \text{ for } k = n.$$ 

**Lemma 4.1.** The space $\Sigma_{k,h}^+$ is a direct sum of the spaces $\Sigma_{k,h}$ and $\sum_F \mathbb{B}_F^1$ for $2 \leq k \leq n - 1$; is a direct sum of the spaces $\Sigma_{k,h}$ and $\sum_F \mathbb{B}_F^2$ for $k = n$.

**Proof.** We first prove the first part of the theorem. It suffices to show that, given $K \in \mathcal{T}_h$, assume the following degrees of freedom vanish for $\tau \in P_k(K, \mathbb{R}^n)$ with $2 \leq k \leq n - 1$, then $\tau \nu = 0$ on $\partial K$ where $\nu$ is the normal vector of $\partial K$.

- For each $\ell$ dimensional simplex $\Delta_\ell$ of $K$, $0 \leq \ell \leq n - 2$, with $\ell$ linearly independent tangential vectors $t_1, \cdots, t_\ell$, and $n - \ell$ linearly independent normal vectors $\nu_1, \cdots, \nu_{n-\ell}$, the mean moments of degree at most $n-\ell$ over $\Delta_\ell$, of $t_i^T \nu_i v^T \nu_j, l = 1, \cdots, \ell, i, j = 1, \cdots, n-\ell, (C^2_{n+1-\ell+l(n-\ell)})C_n^{\ell} = \frac{(n-\ell)(n+\ell+1)}{2} C_n^{\ell}$ degrees of freedom for each $\Delta_\ell$;

In fact, it follows from [20] Theorem 2.1] that such a set of degrees of freedom indicates the $\tau \nu = 0$ on $\partial K$.

Next we turn to the second part of the theorem. For this case, if the above set of degrees of freedom and the following set of degrees of freedom

- the average moment of degree zero of $\tau \nu$ for any $n-1$ dimensional simplex $\Delta_{n-1}$ of $K$ with the normal vector
vanish, we have \( \tau \nu = 0 \) on \( \partial K \), see [26] Theorem 2.1 for more details. This completes the proof.

It follows from the definition of \( V_{k,h} \) (\( P_{k-1} \) polynomials) and \( \Sigma_{k,h}^+ \) (enriched \( P_k \) polynomials) that

\[
\text{div} \Sigma_{k,h}^+ \subset V_{k,h}.
\]

This, in turn, leads to a strong divergence-free space:

\[
Z_h := \{ \tau_h \in \Sigma_{k,h}^+ \mid (\text{div} \; \tau_h, v) = 0 \; \text{for all} \; v \in V_{k,h} \}
\]

where \( Z_h \) is the divergence-free space defined in (4.6). This implies the above K-ellipticity condition (4.8). It remains to show the discrete B-B condition (4.9), in the following two lemmas.

**4.2. Stability analysis and error estimates.** The convergence of the finite element solution follows the stability and the standard approximation property. So we consider first the well-posedness of the discrete problem (4.7). By the standard theory, we only need to prove the following two conditions, based on their counterpart at the continuous level.

1. **K-ellipticity.** There exists a constant \( C > 0 \), independent of the meshsize \( h \) such that

\[
(A \tau, \tau) \geq C \| \tau \|^2_{H(\text{div})} \quad \text{for all} \; \tau \in Z_h,
\]

where \( Z_h \) is the divergence-free space defined in (4.6).

2. **Discrete B-B condition.** There exists a positive constant \( C > 0 \) independent of the mesh size \( h \), such that

\[
\inf_{0 \neq v \in V_{k,h}} \sup_{0 \neq \tau \in \Sigma_{k,h}^+} \frac{(\text{div} \; \tau, v)}{\| \tau \|_{H(\text{div})} \| v \|_0} \geq C.
\]

It follows from \( \text{div} \Sigma_{k,h}^+ \subset V_{k,h} \) that \( \text{div} \; \tau = 0 \) for any \( \tau \in Z_h \). This implies the above K-ellipticity condition (4.8). It remains to show the discrete B-B condition (4.9), in the following two lemmas.

For the analysis, we need a subspace \( \widetilde{\Sigma}_{k,h} := \Sigma_{k,h} \cap H^1(\Omega, S) \) of \( \Sigma_{k,h} \). For \( \tau \in \widetilde{\Sigma}_{k,h} \), the degrees of freedom on any element \( K \) are: for each \( \ell \) dimensional simplex \( \Delta_\ell \) of \( K \), \( 0 \leq \ell \leq n \), the mean moments of degree at most \( k - \ell - 1 \) over \( \Delta_\ell \) of \( \tau \). A standard argument is able to prove that these degrees of freedom are unisolvent.

**Lemma 4.2.** For any \( v_h \in V_{k,h} \), there is a \( \tau_h \in \widetilde{\Sigma}_{k,h} + \sum_{F} \mathbb{B}_F^1 \) with \( 2 \leq k \leq n - 1 \) such that, for all polynomial \( p \in R(K), K \in \mathcal{T}_h \),

\[
\int_K (\text{div} \; \tau_h - v_h) \cdot p \, dx = 0 \quad \text{and} \quad \| \tau_h \|_{H(\text{div})} \leq C \| v_h \|_0.
\]
Proof. Let $v_h \in V_{k,h}$. By the stability of the continuous formulation, cf. [10] for two dimensional case, there is a $\tau \in H^1(\Omega;\mathbb{S})$ such that,

$$\text{div } \tau = v_h \quad \text{and} \quad \|\tau\|_1 \leq C\|v_h\|_0.$$  

In this paper, we only consider the domain such that the above stability holds. We refer interested authors to [21] for the classical result which states it is true for Lipschitz domains in $\mathbb{R}^n$; see [20] for more refined results. First let $I_h$ be a Scott-Zhang [35] interpolation operator such that

$$\|\tau - I_h\tau\|_0 + \|\nabla I_h\tau\|_0 \leq Ch\|
abla \tau\|_0.$$  

These enriched bubble functions in $\sum_F B^1_F$ on the $n-1$ dimensional simplices $F$ allow for defining a correction $\delta_h \in \sum_F B^1_F$ such that

$$\int_F \delta_h \nu \cdot p ds = \int_F (\tau - I_h\tau) \nu \cdot p ds \quad \text{for any } p \in R(K)|_F.$$  

Finally we take

$$\tau_h = I_h\tau + \delta_h.$$  

We get a partial-divergence matching property of $\tau_h$: for any $p \in R(K)$, as the symmetric gradient $\epsilon(p) = 0$,

$$\int_K (\text{div } \tau_h - v_h) \cdot p dx = \int_K (\text{div } \tau_h - \text{div } \tau) \cdot p dx$$

$$= \int_{\partial K} (\tau_h - \tau) \nu \cdot p ds = 0.$$  

The stability estimate follows from (4.11) and the definition of the correction $\delta_h$. 

Remark 4.1. A modification of the above proof applies for the case where $k = n$. In fact, the bubble functions in the spaces $\tilde{\Sigma}_{k,h}$ and $\sum_F^2 B^2_F$ on the $n-1$ dimensional simplices $F$ are able to control the constant subspace of $R(K)$ and its orthogonal complement, respectively.

We are in the position to show the well-posedness of the discrete problem.

Theorem 4.1. For the discrete problem (4.7), the K-ellipticity (4.8) and the discrete B-B condition (4.9) hold uniformly. Consequently, the discrete mixed problem (4.7) has a unique solution $(\sigma_h, u_h) \in \Sigma_{k,h}^+ \times V_{k,h}$.

Proof. The K-ellipticity immediately follows from the fact that $\text{div } \Sigma_{k,h}^+ \subset V_{k,h}$. To prove the discrete B-B condition (4.9), for any $v_h \in V_{k,h}$, it follows from Lemma 4.2 and Remark 4.1 that there exists a $\tau_1 \in \Sigma_{k,h}^+$ such that, for any polynomial $p \in R(K)$,

$$\int_K (\text{div } \tau_1 - v_h) \cdot p dx = 0 \quad \text{and} \quad \|\tau_1\|_{H(\text{div})} \leq C\|v_h\|_0.$$  

Then it follows from Lemma 2.3 that there is a $\tau_2 \in \Sigma_{k,h}$ such that $\tau_2|_K \in \Sigma_{K,k,b}$ and

$$\text{div } \tau_2 = v_h - \text{div } \tau_1, \|\tau_2\|_0 = \min\{\|\tau\|_0, \text{div } \tau = v_h - \text{div } \tau_1, \tau \in \Sigma_{K,k,b}\}$$  

Thus, we have $\tau = \tau_1 + \tau_2$, which proves the unique solvability of the discrete problem.
It follows from the definition of $\tau_2$ that $\|\text{div} \tau_2\|_0$ defines a norm for it. Then, a scaling argument proves
\begin{equation}
(4.16) \quad \|\tau_2\|_{H(\text{div})} \leq C\|\text{div} \tau_1 - v_h\|_0.
\end{equation}

Let $\tau = \tau_1 + \tau_2$. This implies that
\begin{equation}
(4.17) \quad \text{div} \tau = v_h \text{ and } ||\tau||_{H(\text{div})} \leq C\|v_h\|_0,
\end{equation}
this proves the discrete B-B condition \begin{equation}
(4.19).
\end{equation}

**Theorem 4.2.** Let $(\sigma, u) \in \Sigma \times V$ be the exact solution of problem \begin{equation}
\text{(4.11)}
\end{equation} and $(\tau_h, u_h) \in \Sigma_{k,h}^+ \times V_{k,h}$ the finite element solution of \begin{equation}
\text{(4.17)}.\end{equation} Then, for $2 \leq k \leq n$,
\begin{equation}
(4.18) \quad ||\sigma - \sigma_h||_{H(\text{div})} + ||u - u_h||_0 \leq Ch^k(||\sigma||_{k+1} + ||u||_k).
\end{equation}

**Proof.** The stability of the elements and the standard theory of mixed finite element methods \begin{equation}
\text{(14, 15)}\end{equation} give the following quasioptimal error estimate immediately
\begin{equation}
(4.19) \quad ||\sigma - \sigma_h||_{H(\text{div})} + ||u - u_h||_0 \leq C \inf_{\tau_h \in \Sigma_{k,h}^+, v_h \in V_{k,h}}(||\sigma - \tau_h||_{H(\text{div})} + ||u - v_h||_0).
\end{equation}

Let $P_h$ denote the local $L^2$ projection operator, or element-wise interpolation operator, from $V$ to $V_{k,h}$, satisfying the error estimate
\begin{equation}
(4.20) \quad ||v - P_h v||_0 \leq C h^k ||v||_k \text{ for any } v \in H^k(\Omega; \mathbb{R}^n).
\end{equation}

Choosing $\tau_h = I_h \sigma \in \Sigma_{k,h}$ where $I_h$ is defined in \begin{equation}
\text{(4.11)}\end{equation} as $I_h$ preserves symmetric $P_k$ functions locally,
\begin{equation}
(4.21) \quad ||\sigma - \tau_h||_0 + h||\sigma - \tau_h||_{H(\text{div})} \leq Ch^{k+1}||\sigma||_{k+1}.
\end{equation}

Let $v_h = P_h v$ and $\tau_h = I_h \sigma$ in \begin{equation}
\text{(4.19)}\end{equation}, by \begin{equation}
\text{(4.20)}\end{equation} and \begin{equation}
\text{(4.21)}, \end{equation} we obtain \begin{equation}
(4.18).\end{equation}

5. **First order mixed elements**

In order to get first order mixed elements, we propose to take the following discrete displacement space
\begin{equation}
(5.1) \quad V_{1,h} := \{v \in L^2(\Omega; \mathbb{R}^n), v|_K \in R(K) \text{ for any } K \in \mathcal{T}_h\}.
\end{equation}

To design the space for stress, we define
\begin{equation}
(5.2) \quad \Sigma_{1,h} := \{\tau \in H^1(\Omega; S), \tau|_K \in P_1(K, S) \text{ for any } K \in \mathcal{T}_h\}.
\end{equation}

Since the pair $(\Sigma_{1,h}, V_{1,h})$ is unstable, we propose to enrich $\Sigma_{1,h}$ by some $n-1$ dimensional simplex bubble function spaces. Given a $n-1$ dimensional simplex $F$ of $\mathcal{T}_h$, let $\omega_F := K^- \cup K^+$ denote the union of two elements that share $F$. Define
\begin{equation}
(5.3) \quad \hat{B}_F := \left\{\tau \in \Sigma_{2,h}^+, \tau = 0 \text{ on } \Omega\backslash\omega_F, \int_F \tau \nu \cdot p ds = 0 \text{ for any } p \in (R(\omega_F)|_F)\right\}^\perp,
\end{equation}
the values of $\int_K \tau : \theta dx$ vanish for any $\theta \in M_2(K), K = K^- \text{ and } K^+$.

This allows for defining the following enriched stress space
\begin{equation}
(5.4) \quad \hat{\Sigma}_{1,h}^+ = \Sigma_{1,h} + \sum_F \hat{B}_F.
\end{equation}
For this enriched space ˆΣ_{1,h}^+, the number of degrees of freedom on each simplex is 18 and 48 for n = 2, 3, respectively, which are the simplest conforming mixed elements so far. A similar argument of Lemma 4.1 shows that ˆΣ_{1,h}^+ is a direct sum of Σ_{1,h} and ∑_F ˆB_F.

The mixed finite element approximation of Problem (1.1) reads: Find (σ_h, u_h) ∈ ˆΣ_{1,h}^+ × V_{1,h} such that

\begin{align*}
(Aσ_h, τ) + (\text{div} τ, u_h) &= 0 \quad \text{for all } τ ∈ ˆΣ_{1,h}^+,
(div σ_h, v) &= (f, v) \quad \text{for all } v ∈ V_{1,h}.
\end{align*}

(5.5)

It follows from div ˆΣ_{1,h}^+ ⊂ V_{1,h} that div τ = 0 for any τ ∈ Z_h, which implies the above K-ellipticity condition (4.8). A similar proof of Lemma 4.2 shows the discrete inf–Sup condition (4.9). In particular, there exists an interpolation operator I_h : H^1(Ω, S) → ˆΣ_{1,h}^+ such that

\begin{align*}
∥τ − I_h τ∥_0 + h ∥\text{div}(τ − I_h τ)∥ &≤ h^k ∥τ∥_{k, k = 1, 2},
∥σ − σ_h∥_{H(div)} + ∥u − u_h∥_0 &≤ Ch(∥σ∥_1 + ∥u∥_1),
\end{align*}

(5.6)

and

\begin{align*}
\int_K \text{div}(τ − I_h τ) : pdx &= \int_{∂K} (τ − I_h τ)ν \cdot pds = 0 \quad \text{for any } p ∈ R(K)
\end{align*}

(5.7)

for any K ∈ T_h. A summary of these results leads to the error estimates in the following theorem.

**Theorem 5.1.** Let (σ, u) ∈ Σ × V be the exact solution of problem (1.1) and (τ_h, u_h) ∈ ˆΣ_{1,h}^+ × V_{1,h} the finite element solution of (5.5). Then,

\begin{align*}
∥σ − σ_h∥_{H(div)} + ∥u − u_h∥_0 &≤ Ch(∥σ∥_1 + ∥u∥_1),
(5.8)
\end{align*}

and

\begin{align*}
∥σ − σ_h∥_0 &≤ Ch^2∥σ∥_2.
(5.9)
\end{align*}

6. **The added face bubble functions in three dimensions**

Let F := Δ_2x1x2x3 be a face of element K := Δ_3x0x1x2x3, we construct the added face bubble functions. We have three face bubble functions of the Lagrange element of order 4:

\begin{align*}
ϕ_{i,F} = λ_1 λ_2 λ_3 (λ_i - \frac{1}{4}), i = 1, 2, 3.
\end{align*}

(6.1)

Note that ϕ_{i,F} vanish on face F' other than F of K.

Let t_{i,F}, i = 1, 2, 3, be unit tangential vectors of three edges of F. Let

\begin{align*}
T_{i,F} = t_{i,F} t_{i,F}^T, i = 1, 2, 3.
\end{align*}

(6.2)

Define T_{j,F}^+, j = 1, 2, 3 such that

\begin{align*}
T_{i,F} : T_{j,F}^+ = 0, T_{j,F}^+ : T_{l,F}^+ = δ_{jl}, i, j, l = 1, 2, 3.
\end{align*}

(6.3)

This allows the definition of the following space

\begin{align*}
Σ_{F,b} := \text{span}\{ϕ_{i,F} T_{j,F}^+, i, j = 1, 2, 3\}.
\end{align*}

(6.4)
On the face $F$, we have

\begin{equation}
(6.5) \quad x_i = \sum_{j=1}^{2} (x_{i,j} - x_{i,3}) \lambda_j + x_{i,3}, \ i = 1, 2, 3,
\end{equation}

where $x_i = (x_{i,0}, x_{i,1}, x_{i,2}, x_{i,3}), \ i = 0, 1, 2, 3$. We need a basis of the restriction of the rigid motion space on the face $F$:

\begin{equation}
(6.6) \quad v_{1,F} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ v_{2,F} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ v_{3,F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\end{equation}

and

\begin{equation}
(6.7) \quad v_{4,F} = \begin{pmatrix} (x_2 - x_{2,F}) \\ -(x_1 - x_{1,F}) \\ 0 \end{pmatrix}, \ v_{5,F} = \begin{pmatrix} 0 \\ (x_3 - x_{3,F}) \\ -(x_2 - x_{2,F}) \end{pmatrix}, \ v_{6,F} = \begin{pmatrix} (x_3 - x_{3,F}) \\ 0 \\ -(x_1 - x_{1,F}) \end{pmatrix}.
\end{equation}

Here $x_F = (x_{1,F}, x_{2,F}, x_{3,F})$ is the center of $F$. Define the basis $v_{i,F}^+, \ i = 1, 2, 3$, of the orthogonal complement space of the restriction of the rigid motion space on the face $F$ with respect to $P_1(F, \mathbb{R}^3)$, such that

\begin{equation}
(6.8) \quad \int_F v_{i,F}^+ \cdot v_{j,F} ds = 0, \ i = 1, 2, 3, \ j = 1, \ldots, 6.
\end{equation}

Then we define $\tau_{i,F}^* \in \Sigma_{F,b}, \ i = 1, \ldots, 6$ such that

\begin{equation}
(6.9) \quad \frac{1}{|F|} \int_F \tau_{i,F}^* \nu_F \cdot v_{j,F} ds = \delta_{i,j}, \ j = 1, \ldots, 6, \text{ and } \int_F \tau_{i,F}^* \nu_F \cdot v_{k,F}^+ ds = 0, \ k = 1, 2, 3.
\end{equation}

Finally, we take $\delta_{i,F} \in \Sigma_{K,A,b}$ such that $\text{div} \ \tau_{i,F} = \text{div}(\tau_{i,F}^* + \delta_{i,F}) \in P_1(K, \mathbb{R}^3)$. Then

\begin{equation}
(6.10) \quad \mathbb{B}^1_F = \text{span}\{\tau_{i,F}, \ i = 1, \ldots, 6\} \text{ and } \mathbb{B}^2_F = \text{span}\{\tau_{i,F}, \ i = 4, 5, 6\}.
\end{equation}

**Example** Let $F = \Delta x_1 x_2 x_3$ with $x_1 = (0, 0, 0)^T$, $x_2 = (1, 0, 0)^T$, and $x_3 = (0, 1, 0)^T$ and $\nu_F = (0, 0, 1)^T$. We have

\begin{equation}
(6.11) \quad T_{1,F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ T_{2,F} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ T_{3,F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{equation}

This implies that

\begin{equation}
(6.12) \quad T_{1,F}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T_{2,F}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ T_{3,F}^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{equation}

In addition, a basis of the restriction of the rigid motion space on the face $F$ reads

\begin{equation}
(6.13) \quad v_{1,F} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ v_{2,F} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ v_{3,F} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\end{equation}

and

\begin{equation}
(6.14) \quad v_{4,F} = \begin{pmatrix} (x_2 - \frac{1}{2}) \\ -(x_1 - \frac{1}{2}) \\ 0 \end{pmatrix}, \ v_{5,F} = \begin{pmatrix} 0 \\ 0 \\ -(x_2 - \frac{1}{2}) \end{pmatrix}, \ v_{6,F} = \begin{pmatrix} 0 \\ 0 \\ -(x_1 - \frac{1}{2}) \end{pmatrix}.
\end{equation}
Hence

\begin{equation}
\mathbf{v}^\perp_1 = \begin{pmatrix} (x_1 - \frac{1}{3}) \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}^\perp_2 = \begin{pmatrix} 0 \\ (x_2 - \frac{1}{3}) \\ 0 \end{pmatrix}, \quad \mathbf{v}^\perp_3 = \begin{pmatrix} (x_2 - \frac{1}{3}) \\ (x_1 - \frac{1}{3}) \\ 0 \end{pmatrix}.
\end{equation}

7. Numerical Test

We compute a 2D pure displacement problem on the unit square \( \Omega = [0, 1]^2 \) with a homogeneous boundary condition that \( u \equiv 0 \) on \( \partial \Omega \). In the computation, we let the compliance tensor in (1.1)

\[ A\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + n\lambda} \text{tr}(\sigma) \delta \right), \quad n = 2, \]

where \( \delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), and \( \mu = 1/2 \) and \( \lambda = 1 \) are the Lamé constants. Let the exact solution be

\begin{equation}
\mathbf{u} = \begin{pmatrix} e^{x-y}x(1-x)y(1-y) \\ \sin(\pi x) \sin(\pi y) \end{pmatrix}.
\end{equation}

The true stress function \( \sigma \) and the load function \( f \) are defined by the equations in (1.1), for the given solution \( u \).

In the computation, the level one grid consists of two right triangles, obtained by cutting the unit square with a north-east line. Each grid is refined into a half-sized grid uniformly, to get a higher level grid. In all the computation, the discrete systems of equations are solved by Matlab backslash solver.

We use the bubble enriched \( P_2 \) symmetric stress finite element with \( P_1 \) discontinuous displacement finite element, \( k = 2 \) in (4.1) and in (4.5), and \( k = 2 \) in (2.1). That is, 3 \( P_3 \) bubbles are enriched each edge. In Table 7.1, the errors and the convergence order in various norms are listed for the true solution (7.1). The optimal order of convergence is observed for both displacement and stress, see Table 7.1 as shown in the theorem.

| \( ||u - u_h||_0 \) | \( h^n \) | \( ||\epsilon_h||_0 \) | \( h^3 \) | \( ||\text{div} \epsilon_h||_0 \) | \( h^3 \) |
|---|---|---|---|---|
| 1 | 0.27452 | 0.0 | 1.24637 | 0.0 | 6.97007772 | 0.0 |
| 2 | 0.07432 | 1.9 | 0.18054 | 2.8 | 2.13781130 | 1.7 |
| 3 | 0.01959 | 1.9 | 0.02429 | 2.9 | 0.57734125 | 1.9 |
| 4 | 0.00497 | 2.0 | 0.00314 | 2.9 | 0.14709450 | 2.0 |
| 5 | 0.00125 | 2.0 | 0.00040 | 3.0 | 0.03694721 | 2.0 |

As a comparison, we also test the Arnold–Winther element from [10], which has a same degree of freedom as ours, 21, on each element. But the displacement in that element is approximated by the rigid-motion space only, instead of the full \( P_1 \) space, i.e., 3 dof vs 6 dof on each triangle. The total degrees of freedom for the stress for the new element are \( 3|\mathcal{V}| + 3|\mathcal{E}| + 3|\mathcal{K}| \), where \( |\mathcal{V}| \), \( |\mathcal{E}| \), and \( |\mathcal{K}| \) are the numbers of vertices, edges and elements of \( T_h \), respectively, while those for the Arnold–Winther element are \( 3|\mathcal{V}| + 4|\mathcal{E}| \). Since the three bubble functions on each element can be easily condensed, these two elements almost have the same complexity for solving.
The errors and the orders of convergence are listed in Table 7.2. Because the new element uses the full $P_1$ displacement space, the order of convergence is one higher than that of the Arnold–Winther element. Also as the new element includes the full $P_2$ stress space, the order of convergence of stress is one order higher, see the data in Tables 7.1 and 7.2.

Table 7.2. The errors, $\epsilon_h = \sigma - \sigma_h$, and the order of convergence, by the Arnold-Winther 21/3 element [10], for (7.1).

<table>
<thead>
<tr>
<th></th>
<th>$| u - u_h |_H^1$</th>
<th>$h^n$</th>
<th>$| \epsilon_h |_0$</th>
<th>$h^n$</th>
<th>$| \text{div} \epsilon_h |_0$</th>
<th>$h^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.30554</td>
<td>0.0</td>
<td>1.58058</td>
<td>0.0</td>
<td>10.31991249</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.22589</td>
<td>0.4</td>
<td>0.89927</td>
<td>0.8</td>
<td>6.81340378</td>
<td>0.6</td>
</tr>
<tr>
<td>3</td>
<td>0.10922</td>
<td>1.0</td>
<td>0.25584</td>
<td>1.8</td>
<td>3.61633797</td>
<td>0.9</td>
</tr>
<tr>
<td>4</td>
<td>0.05354</td>
<td>1.0</td>
<td>0.06633</td>
<td>1.9</td>
<td>1.83690959</td>
<td>1.0</td>
</tr>
<tr>
<td>5</td>
<td>0.02066</td>
<td>1.0</td>
<td>0.01674</td>
<td>2.0</td>
<td>0.92212628</td>
<td>1.0</td>
</tr>
</tbody>
</table>

References


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