The Realization of the Conditions for Optimal Matrices

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Abstract: In this paper we examine the characterizations of multihomogeneous or multigraded systems that can give optimal resultant matrices by determining a bijection with the permutations of \{1,\ldots, r\}. These characterizations are related to the study of the determinant of a resultant complex. Since multihomogeneous resultant has been studied in several areas of scientific and engineering applications such as geometric modeling, game theory, and computer vision, this study ultimately aim at providing some useful algebraic or computational tools that can be applied for deriving an economical solution related to resultant. As a matter of fact, there have been ongoing research on finding explicit determinantal formula for sparse resultant matrices leading to the construction of the smallest possible matrix, which is either optimal or close to optimal. The work is based on deriving related properties to the existence of determinantal formulae which have been further investigated and later utilized as computational tools. The main condition to this subject is the defect vectors. When all defects are zero, a determinantal formula of Sylvester type exists, and it has been verified by the incremental algorithm for constructing sparse resultant matrices. We consider systems with defect vectors between 0 and 2, and those with defect vectors not in this range. The construction of Sylvester type matrices for some examples is discussed.

Keywords: determinantal formulas, multigraded resultant, optimal matrices, Sylvester type formulas.

1. Introduction

In the last decade resultants have been used as a computational tool for elimination of variables and for the study of complexity of polynomial system solving. There have been ongoing researches on finding explicit determinantal formula for sparse resultant matrices leading to the construction of the smallest possible matrix, which is either optimal or close to optimal (see [1-14]). The work is based on deriving related properties to the existence of determinantal formulas which have been further investigated and later utilized as computational tools. Optimal matrices are known to exist only for certain classes of polynomial systems. In this paper we focus on the characterizations of multihomogeneous or multigraded systems that can give optimal resultant matrices by showing a bijection with the permutations of \{1,\ldots, r\}. These characteristics are related to the study of the determinant of a resultant complex.

The main idea of resultants is the transformation of the original nonlinear system of equations into a linear system of equations in order to reduce the problem of finding common zeros to linear algebra computations. The determinant of the corresponding matrix is called the resultant which is defined as an irreducible polynomial function on \(S(d_1,\ldots,d_r)^{\ast} \) [12]. Thus resultant provides the solvability condition of the given systems. The entries of resultant matrices are coefficients of the polynomials. If these coefficients are indeterminates, then the determinant of the resultant matrix is an irreducible polynomial in these indeterminates. If this determinant equals the resultant, then it is called determinantal formula. In [1], algorithmic methods for identifying and constructing determinantal formula for the sparse resultant are given. This work is based on [2], where the underlying resultant complex is made explicit so as to derive computational tools to produce the smallest such formula.

There is various type of determinantal formulas, Sylvester, Bezout or hybrid. In this paper, we only focus on Sylvester type of multihomogeneous resultant or multigraded resultant for the sparse resultant where the variable sets of multihomogeneous polynomials can be partitioned into \(r\) groups \(x_1,\ldots,x_t\). Hence, every polynomial is homogeneous in each group of \(l_k + 1\) variables in \(x_i\) where one of them is the homogenizing variables and of degree \(d_k\). If \(l_k = 1\) or \(d_k = 1\), for each \(k \in \{1,\ldots,r\}\), then the resultant is called multigraded resultant of type \((l_1,\ldots,l_r; d_1,\ldots,d_r)\) and defined as a unique (up to sign) irreducible polynomial \(R(f_1,\ldots,f_r)\) in \(Z[x_1,x_2,\ldots,x_t]^n\) which vanishes under a specialization if and only if the equations \(f_1=\cdots=f_r=0\) have a solution in projective space \(P^1 \times \cdots \times P^t\). [1-3]. For example, given a generic polynomial with two variable groups \(x, y\) and \(t\) of type \((1,1); 2,1)\); \(s\) and \(t\) are the homogenizing variables and the \(c_{ij}\) for \(i=1,2,3\) and \(j=1,\ldots,6\) are indeterminate coefficients:

\[
c_{ij}x^iy + c_{ij}x^jy + c_{ij}x^iy + c_{ij}x^jy + c_{ij}x^iy + c_{ij}x^jy + c_{ij}x^iy + c_{ij}x^jy + c_{ij}x^iy + c_{ij}x^jy .
\]  

(1)

Multihomogeneous resultant is defined as an irreducible polynomial in the coefficients of the polynomial which vanishes if and only if the polynomials have a common root in \(X\) [1]. Since multihomogeneous resultant has been studied in several areas of scientific and engineering applications such as geometric modeling, game theory, and computer vision, this study ultimately aim at providing some useful algebraic or computational tools that can be applied for deriving an economical solution related to resultant. The main condition to this subject is the defect vectors.

The definition of defect vectors (\(\delta \in \mathbb{Z}^r\)) and critical vector (\(\rho \in \mathbb{N}^r\)) is as of Definition 1.2 in [1]. The fact that a determinantal formula of Sylvester type exists when all
defects are zero and it has been verified by the incremental algorithm for constructing sparse resultant matrices. Moreover, a determinantal vector $m$ exists if and only if $\delta_k \leq 2$ for all $k \in \{1, \ldots, r\}$ [2]. Therefore in this paper we consider systems with defect vectors between 0 and 2, and those with defect vectors not in this range in order to analyze the Sylvester-type formulae for some multihomogeneous polynomial systems which lead to the derivation of optimal matrices. The analysis is based on the output of testing data implemented in Maple program (mhomo). We start with the input of degree and variable groups. Then compute the defect vectors which tell us if and only if the system has any determinantal formula. We apply subroutine “minSyl” to find the vectors $m^k$ for all permutations $\pi: \{1, \ldots, r\} \rightarrow \{1, \ldots, r\}$, and the smallest possible Sylvester formulæ are among of these vectors.

It is interesting to compare dimension of resultant matrices that have been computed by the incremental algorithm in [8] and greedy algorithm in [16] for multigraded systems. The incremental algorithm implemented by [8] gives optimal matrix in the first round for a given permutation. More precisely, the first matrix constructed by the incremental algorithm has determinant equal to the sparse resultant of the system.

2. Degree of Resultants

This section focuses on the degree computation for resultants.

For sparse polynomial equations, mixed volume computation is an efficient way to count the degree of sparse resultant because it deals with a set of polynomials and standard Euclidean volume that provides a geometric way to bound the number of common zeros of a polynomial system. The proofs of this fact are as in [8, 9, 11, 15, and 16]. Hence it provides exact number of roots for sparse resultants. However, for generic polynomial equations of multigraded resultant, [3] gives the formula of resultants’ total degree,

\[
(l + 1) \left( \frac{1}{d_1} \cdots \frac{1}{d_r} \right).
\]  

(2)

Consider an unmixed multigraded of type $(1, 1; 2, 2)$ which is a sparse polynomial system. The mixed volume of this system is 12 counting by two-fold Newton polytopes [16], while formula (2) gives degree 24. The result is bigger because (2) assumed the system is generic regardless of the zero coefficients in the polynomial systems. Mixed volume computation considered a point set which is from the supports of the polynomial equations and applying Bernstein’s Theorem to produce optimal matrix. More precisely the theorem shows that the solution of the corresponding system can be predicted by mixed volume. In particular the sparse resultant is separately homogeneous in the coefficients of each polynomial with its degree equals the mixed volume of the other $n$ Newton polytopes. For instance as shown in [17] where the structure of the multivariate sparse resultant is actually tailored to the specific monomials that appear in the input polynomials.

In this paper, we verified the given polynomial systems has Sylvester type formulae with the smallest determinantal $m^5$ vectors which have relation with the value of defect vectors for sparse resultant matrices. Thus, leading to the construction of the smallest possible matrix, this is either optimal or close to optimal. The work is related to the study of the determinent of a resultant complex [2] and some analysis of algorithmic methods for identifying and constructing determinantal formulæ for the sparse resultant [1].

3. Determinantal Formulae

The existence of determinantal formula is when a matrix whose determinant equals the sparse resultant. There are two types of determinantal formula either Sylvester or Bézout. This study only focuses on the Sylvester type formulæ.

3.1 Determinantal degree vectors ($m$)

Degree vectors $(m)$ is the main object of a linear map that yielding determinantal Sylvester type formulæ of multihomogeneous systems. The existence of vectors $m$ is by defect vectors. Weyman and Zelevinsky in [2] defined defect vectors as $\delta = l - \left\lceil \frac{l}{d} \right\rceil$.

In this section we present the degree vectors obtained by a Maple package named “mhomo.mpl” which was implemented by Dickenstein and Emiris [1]. In the algorithms they estimate the time computation for vector $m$, by giving bound for the range of $m$ to implement a computer search for determinantal degree vectors. The bound for $m$ is in the following lemma. We took some counter examples as input polynomials to examine more clearly how the sparse resultant matrices are constructed from the linear map and to observe the existence of determinantal degree vectors as well. The systems that we interested are multihomogeneous, either multigraded or non multigraded whose defect vectors between 0 and 2 and not in this range.

Lemma 3.1.1 For $m \in Z^\ast$ a determinantal and $P_i (m) \neq \phi$, for all $k \in \{1, \ldots, r\}$ implies

\[
\max \{-d_i, -l_i\} \leq m_k \leq d_i (n + 1) - 1 + \min \{d_i, l_i, 0\},
\]

with $P_i (m) = \{ p \in Z : m_k / d_i < p \leq m_k / d_i \}$ defined in [3]. As a consequent of this lemma, [1] also gives the following results.

Corollary 3.1.2. For $m \in Z^\ast$ a determinantal and $P_i (m) \neq \phi$, for some $k \in \{1, \ldots, r\}$, obtain

\[
0 \leq m_k \leq d_i (n + 1) - l_i - 1.
\]

Corollary 3.1.3. For $m \in Z^\ast$ a determinantal and $k \in \{1, \ldots, r\}$ implies

\[
\max \{-d_i, -l_i\} \leq m_k \leq d_i (n + 1) - 1 + \min \{d_i, l_i, 0\}.
\]

Theorem 3.1.4. A degree $m \in Z^\ast_{20}$ gives a Sylvester-type matrix if and only if there exists a permutation $\pi$ such that $m_j \geq m^\pi_j$ for $j = 1, \ldots, r$. The smallest Sylvester matrix is attained among the vectors $m^\ast$. From the above results, we observed that there are a finite number of vectors to be tested for enumeration of all possible determinantal $m$ not necessarily the one that
formulate Sylvester type formulae. For instance, type
(1,1,2,1), these bounds are computed by the routines in
"mhmomo.mpl" which was implemented by [1] where all \( m^n \)
including those that are pure Sylvester formula are being
searched algorithmically. The degree of the resultant in the
coefficients is \( \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] = 12 \) computed using (2). We shall
enumerate all 24 possible determinantal formulae in the
order of decreasing matrix dimension from 12 to 4, and make
the corresponding maps explicit. The assumption of \( P_1 \)'s set
is used only to bound the search so that no valid degree
vector is missed as marked NO_INT. Among of these degree
vectors, the smallest Sylvester-type matrix is attained as in
Theorem 3.1.4. The implementation of this example is
illustrated in the next section.

3.2 Sylvester-type formulae

Every Sylvester type formula, the matrix of the linear map
must be square and of the correct order (degree). For example, in the classical determinantal formulas of Sylvester
type for the resultant of two univariate polynomials is a
square matrix of order \( 2d \) [2]. This Sylvester matrix is the transformation matrix for the linear mapping
\[
\vartheta : S(d-1)^2 \rightarrow S(2d-1)
\]
defined by \((g_n, g_j) \mapsto f_n g_n + f_j g_j\). By linear map in the classical Sylvester type formula (3), its
determinant degree vector \( m = 2d - 1 \) and the upper bound
given by Corollary 3.1.2 is attained. Since \( R(f_n, f_j) = \det(\vartheta) \)
is an irreducible polynomial of degree \( 2d \) in the coefficients
of \( f_0 \) and \( f_1 \) whose entries are linear forms, the derivation of
the matrix is optimal.

For multivariate polynomial equations, we consider linear
mapping defined by [3],
\[
\varphi : S(m_1, \ldots, m_r) \rightarrow S(d_1 + m_1, \ldots, d_r + m_r),
\]
\((g_0, \ldots, g_r) \mapsto f_0 g_0 + \cdots + f_r g_r\). (4)

If the two spaces in (4) have the same dimension, then \( \varphi \)
is represented by a square matrix and \( R(f_0, \ldots, f_r) \) divides
\( \det(\varphi) \). Hence, \( R(f_0, \ldots, f_r) = \det(\varphi) \) is a Sylvester type
formula and can have several different formulas corresponding to different determinantal degree vector, \( m \).

To characterize the characteristics of polynomial systems
which are able to produce the smallest matrices we are
interested to look into the condition that determine the
smallest Sylvester matrices that is the defect vectors as
mentioned earlier.

Definition 3.2.1. [1-3] Fix natural numbers \( r \), and \((l_1, \ldots, l_r), \)
\((d_1, \ldots, d_r)\), define the defect vector \( \delta \in Z^r \) by
\[
\delta_k = l_k - \left\lfloor d_k l_k \right\rfloor, \text{ for all } k \in \{1, \ldots, r\}, \text{ where } l_k \geq \left\lfloor \frac{l_k}{d_k} \right\rfloor.
\]
Clearly the defect is always nonnegative vector.

Definition 3.2.2. [1] Given \( r \) and \((l_1, \ldots, l_r)\), \((d_1, \ldots, d_r)\) \( \in \mathbb{N}^r \),
define the critical degree vector \( \rho \in \mathbb{N}^r \) by \( \rho_j := (n+1)d_j - l_j - 1 \), for all \( k = 1, \ldots, r \).
The number \( l - \left\lfloor l/d_1 \right\rfloor \) can be thought of as the determinantal
complexity of Cayley’s formula. It is equal to zero only in
two extreme cases when \( l = 1 \) or \( d = 1 \). For \( l = 1 \) Cayley’s
formula becomes the Sylvester formula, and for \( d = 1 \) Cayley’s
formula expresses the resultant of a system of linear
forms as the determinant of their coefficient matrix
[11, p.429].

Proposition 3.2.3. For any \( k \in \{1, \ldots, r\} \), a pair \((l, d)\) of
positive integers, has defect, \( \delta_k = 0 \) if and only if
\( \min(l_k, d_k) = 1 \).

The proposition shows that only multigraded systems have
defect vectors zero. More precisely, since \( l_k \geq \left\lfloor \frac{l_k}{d_k} \right\rfloor \) for all \( k \) by Definition 3.2.1 and it is deduce to \( l_k = \left\lfloor l_k/d_k \right\rfloor \) if and
only if \( l_k = 1 \) or \( d_k = 1 \).

In this paper we consider systems with defect vectors
between 0 and 2, and those with defect vectors not in this
range. For the case of \( \delta_k \leq 2 \) for all \( k \), is a necessary and
sufficient condition for the existence of a determinantal
complex [2].

Definition 3.2.4. For each choice of permutation
\( \pi : \{1, \ldots, r\} \rightarrow \{1, \ldots, r\} \), there are three formulas for vector
\( m \) computation as follows:

1. \( m_\pi = (d_k - 1)l_k + d_k \sum_{j \neq (j \pi (k))} l_j \) when all defects are
zero, \( \delta_k = 0 \), for all \( k = 1, \ldots, r \) \[3\]
2. \( m_\pi = \left( 1 - \delta_k + \sum_{j \neq (j \pi (k))} l_k \right) \delta_k \), for \( 0 \leq \delta_k \leq 2 \) for
all \( k = 1, \ldots, r \) defines a determinantal complex [1].
3. \( m_\pi = \left( -\delta_k + \sum_{j \neq (j \pi (k))} l_k \right) - l_k \), for \( 0 \leq \delta_k \leq 1 \) for
all \( k = 1, \ldots, r \) defines a determinantal complex [1].

The last two satisfy the bounds of Corollary 3.1.3 and the
first formula is coincides to the second formula. Weyman
and Zelevinsky [2] proved that a vector \( m \in \mathbb{Z}^r \) is
determinantal if and only if \( K_2(m) = K_2(m) = 0 \). Also a
determinantal vector \( m \) exists if and only if \( \delta \leq 2 \) for all \( k \)
= 1, \ldots, r. We use these facts on some multihomogeneous
system of type \((l_1, \ldots, l_r; d_1, \ldots, d_r)\) to verify the existence of
Sylvester-type formulae. The results are presented in the next
section.

We are interested to compare the dimension of Sylvester
matrices constructed in present paper to those computed by
the greedy algorithm in our preliminary work [16] via
multipliers Maple package and incremental algorithm in [8] via
mixed volume computation. Table 1 explains the details.

Generally, using mixed volume the dimension of the matrix
is smaller for sparse polynomial equations. As a matter of
fact, almost all applications in the world are sparse.
Therefore mixed volume computation is more economical
because it deals with a set of polytopes of nonzero
coefficients of polynomials. Moreover it provides a
geometric way to bound the number of common zeros of a
polynomial system. In greedy algorithm, there is a
randomization step, so it might lead to matrices of slightly
smaller size.
In present paper, the algorithm in [1] which have implemented in mhomo Maple package, the smallest Sylvester matrices determine by the dimensions of $K_0(m)$ and $K_i(m)$. Sylvester-type formulae exist if dim$K_1$ equals dim$K_0$ that is the map $K_i(m) \rightarrow K_0(m)$ is surjective where by dim$K_1$ and dim$K_0$ correspond to the number of columns and rows respectively. The number of rows defines the degree of the extraneous factors in the determinant. In order to obtain the optimal matrices, dim$K_0$ must be reduced since extraneous factor leads to a superset of the common isolated roots. For Sylvester-type formula, the matrix must be square and of the correct order, so we are interested to further study whether dim$K_0$($m$) reduces if and only if dim$K_0(m)$ reduces so that the corresponding matrix is square. This is an open question raised in [1].

4. Implementation

We illustrated the implementation of Maple package called “mhomo.mpl” for some multihomogeneous systems:

Example 4.1. Set $l = (1,1), d = (2, 1)$. Let $\pi_l : \{1,2\} \rightarrow \{1,2\}$ and $\pi_d : \{1,2\} \rightarrow \{2,1\}$.

> l:=vector((1,1)); d:=vector((2,1));
> def0:=defects(l,d);
> perm.m-vec.dimK0.dimK1=|[1,2], 5 1, 12, 12
perm.m-vec.dimK0.dimK1=|[2,1], 3 2, 12, 12
list of minimal S-matrices: m-vector and
K1,K0-dims
[[5,1,12,12], [3,2,12,12]]
> summs:=allsums(l);
summs:=ARRAY([0,0,1,0,1,2,2,1])
> hasdeterm(l,d,perm, list, summs)
true
> allSyl:=findSyl(l,d);
sort(convert($\%$,list)), sort_fnc);
search of degree vecs from array[3,1] to
array[5,2]
First array[3,2]: dimK1=12, dimK0=12, dimK(-1)=0(should be 0).

#pure-Sylvester degree vectors = .4
tested 7, found 4 pure-Sylv formlae
[m,dimK1,dimK0]:
[[5,1,12,12], [3,2,12,12], [2,1,18,15], [5,2,24,18]]
> allDetVecs(l,d):
allmsrtd:=sort(convert($\%$,list), sort_fnc);

Tested 28 m-vectors: assuming Fk's nonempty.
Found 24 det'l m-vectors, listed with matrix dim:

\[
\begin{bmatrix}
3,0,4 & [2,2][1,1] \\
1,1,4 & [1,1][2,2] \\
4,0,5 & [12,NO_INT,2][1,1] \\
0,1,5 & [0,NO_INT,0][2,2] \\
\end{bmatrix}
\]

The Maple program (mhomo) finds the minimal Sylvester formulae by testing at most $r!$ vectors $m^n$. Here is a list of the $2 = 2!$ degree vectors $m^n$, permutation and the corresponding matrix dimension (dim$K_i \times$dim$K_j$):

\[m^n = (5, 1) \text{ permutation } (\pi_1 = (1, 2)) 12 \times 12\]

\[m^n = (3, 2) \text{ permutation } (\pi_2 = (2, 1)) 12 \times 12\]

Among these vectors which we find the smallest Sylvester matrix of row dimension 12, whereas the sparse resultant’s degree is 5, see Table 1. There are 4 pure Sylvester matrices, but only two is determinantal. This is measure by the matrix which has at least as many columns as rows. Moreover the smallest resultant matrix has size $4 \times 4$ as shown in dash line below, and corresponds to the degree vectors $(3, 0)$ or $(1,1)$ which satisfy $(3, 0) + (1, 1) = (4, 1)$ is the critical vector, (i.e. $\rho_1 = (3)2-1-1 = 4; \rho_2 = (3)1-1=1$). It is clear that for any determinantal $n^m$, the vector $(4, 1)$ is also determinantal yielding the same matrix dimension. For this type of polynomial system, the linear maps are,

\[\varphi_1 : S(3,0)^3 \rightarrow S(5,1); \quad \varphi_2 : S(1,1)^3 \rightarrow S(3,2) \quad (5)\]

defined by $g_1, g_2, g_3$ → $g_1 f_1 + g_2 f_2 + g_3 f_3$

follow (4). Note that all four vector spaces have dimension 12. The matrices which represent the $\varphi_1$ and $\varphi_2$ with respect to monomial bases for (1) respectively.

\[
\begin{bmatrix}
x^3 & x^3s & x^3s^2 & s^3 x^3 & x^3s & x^3s^2 & s^3 \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
x^3y & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\end{bmatrix}
\]
Example 4.2. Let $l = (2, 1), d = (2, 1)$. 
Let $\pi_1 : \{1, 2 \} \rightarrow \{1, 2 \}$ and $\pi_2 : \{1, 2 \} \rightarrow \{2, 1 \}$.

Let $N_s := \text{vector}(\{2, 1\}), D_s := \text{vector}(\{2, 1\})$.

> \text{minSyl}(N_s, D_s);
> \text{perm.m-vec.dim.K0.dim.K1} := [[1, 2], 61, 56, 60];
> \text{perm.m-vec.dim.K0.dim.K1} := [[2, 1], 43, 60, 72];

list of minimal S-matrices: m-vector and K1,K0-dims

[[6, 1, 60, 56], [4, 3, 72, 60]]

> \text{hasdeterm}(N_s, D_s, Ms, summs);

false

> allSyl := findSyl(K0, Ms);

false

> sort(\{\{list\}, sort\_fnc\});

search of degree vecs from array[4,1] to array[6,3]

First array[4,3]: dimK1=72, dimK0=60, dimK(-1)=0 (should be 0).

**pure-Sylvester degree vectors** = 5

tested 10, found 5 pure-Sylv formulae

[ms, dimK1, dimK0]:

[[6, 1, 60, 56], [4, 3, 72, 60], [6, 2, 120, 84], [5, 3, 120, 84], [6, 3, 180, 112]]

> allDetVecs(Ns, Ds);

allsmrd := sort(\{convert(%\{list\}, sort\_fnc\});

From: [-4 -3], to: 47, 31, start at: -4 -3

Tested 1820 m-vectors: assuming Pk's nonempty. Found 0 det's m-vectors, listed with matrix dim:

allsmrd := []

5. Conclusion

We have illustrated the determinantal formula of Sylvester-type for multihomogeneous system with defect vectors in a range between 0 and 2 and defect vectors not in this range. Emiris and Dickenstein [1] have implemented the algorithms
in Maple package (mhomo.mpl) for finding the vectors \( m \). Our implementation of these algorithms illustrates that for every multihomogeneous system with zero defect vectors, the Sylvester formulae are among these vectors \( m \). Otherwise, the system may or may not have any determinantal vectors \( m \). Even if these determinantal vectors exist, the vectors do not give a determinantal formula.

Further work, for the case when the determinantal Sylvester-type formulae cannot be derived, the question whether a hybrid determinantal formula can be obtained becomes the subject of our future research. We will focus on the formulation of Sylvester-Bézout construction.

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