Abstract – In addition to a brief review on narrowband direction of arrival (DOA) estimation, the main purpose of this article is to evaluate and compare the performance of narrowband DOA estimation methods. This paper presents the performance analysis of eight DOA estimation algorithms in three categories, minimum variance distortionless response (MVDR) and delay&sum of classical category, multiple signal classification (MUSIC), Root-MUSIC, estimation of signal parameters via rotational invariance technique (ESPRIT), and minimum norm of subspace-based category, and also deterministic maximum likelihood (DML) and stochastic maximum likelihood (SML) belong to ML category. We present the description and compare the performance, angular accuracy and resolution, of these algorithms for a uniform linear array (ULA), based on three metrics, the root mean square error (RMSE), normalized root mean square error (NRMSE) and the number of detected signal sources. Three cases, single, double and five source signals are simulated in MATLAB software for different signal to noise ratios (SNRs). According to the simulation results, Root-MUSIC, ESPRIT and MUSIC show higher accuracy and resolution than the other algorithms. It should be noted that MUSIC is more applicable because it can be used for different array geometries. Copyright © 2011 Praise Worthy Prize S.r.l. - All rights reserved.

Keywords: DOA Estimation, MVDR, MUSIC, Root-MUSIC, ESPRIT, Min-Norm, Delay & Sum, DML, SML, RMSE, NRMSE

A Comprehensive Performance Study of Narrowband DOA Estimation Algorithms

Shahriar Shirvani-Moghaddam¹, Sakineh Almasi-Monfared²

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>Number of sensors</td>
</tr>
<tr>
<td>M</td>
<td>Number of sources</td>
</tr>
<tr>
<td>x(t)</td>
<td>Input signal</td>
</tr>
<tr>
<td>S_m(t)</td>
<td>Narrowband source signals</td>
</tr>
<tr>
<td>η(t)</td>
<td>Additive white Gaussian noise (AWGN)</td>
</tr>
<tr>
<td>L</td>
<td>Number of trials</td>
</tr>
<tr>
<td>λ</td>
<td>Wavelength</td>
</tr>
<tr>
<td>d</td>
<td>Inter-element spacing</td>
</tr>
<tr>
<td>θ_m</td>
<td>Signal direction of arrival</td>
</tr>
<tr>
<td>a(θ_m)</td>
<td>Steering vector</td>
</tr>
<tr>
<td>A</td>
<td>Signal direction vectors</td>
</tr>
<tr>
<td>E[.]</td>
<td>Expectation value</td>
</tr>
<tr>
<td>σ_n²</td>
<td>Variance of noise</td>
</tr>
<tr>
<td>R</td>
<td>Covariance matrix</td>
</tr>
<tr>
<td>R_Ss</td>
<td>Correlation matrix of source signal</td>
</tr>
<tr>
<td>I</td>
<td>Identity matrix</td>
</tr>
<tr>
<td>ℋ</td>
<td>Hermitian operator</td>
</tr>
<tr>
<td>P(θ)</td>
<td>Signal power</td>
</tr>
<tr>
<td>(Ω)⁻¹</td>
<td>Inverse operator</td>
</tr>
<tr>
<td>P_MVDR(θ)</td>
<td>Pseudo-spectrum of MVDR</td>
</tr>
<tr>
<td>V_c</td>
<td>Signal Eigenvector</td>
</tr>
<tr>
<td>V_n</td>
<td>Noise Eigenvector</td>
</tr>
<tr>
<td>A</td>
<td>Diagonal matrix of the Eigenvectors</td>
</tr>
<tr>
<td>P_MUSIC</td>
<td>Normalized MUSIC angular spectrum</td>
</tr>
<tr>
<td>D(2)</td>
<td>Root-MUSIC polynomial</td>
</tr>
<tr>
<td>n_i(t)</td>
<td>Noise of i-th ESPRIT subarray</td>
</tr>
<tr>
<td>R_Nx</td>
<td>Correlation matrix of 1st ESPRIT subarray</td>
</tr>
<tr>
<td>R_Ny</td>
<td>Correlation matrix of 2nd ESPRIT subarray</td>
</tr>
<tr>
<td>V_x</td>
<td>Eigenvectors of 1st ESPRIT subarray</td>
</tr>
<tr>
<td>V_y</td>
<td>Eigenvectors of 2nd ESPRIT subarray</td>
</tr>
<tr>
<td>P_MN(θ)</td>
<td>Spectrum of Min-Norm</td>
</tr>
<tr>
<td>Tr[.]</td>
<td>Trace operator</td>
</tr>
<tr>
<td>argmax( )</td>
<td>Specific θ which maximize the function</td>
</tr>
<tr>
<td>Φ</td>
<td>M × M diagonal matrix</td>
</tr>
<tr>
<td>w</td>
<td>Array weight</td>
</tr>
<tr>
<td>e1</td>
<td>All zeros except the 1st element equal to 1</td>
</tr>
<tr>
<td>ω</td>
<td>Angular frequency</td>
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</tbody>
</table>

I. Introduction

Wireless direction finding is the procedure for determining signal source by observing signal direction of arrival (DOA). Its history came back to the beginning of the wireless communications. This technology is used in many fields such as radar [1], sonar, mobile communications [2], astronomy, seismology and etc. So DOA estimation using sensor array or direction finding has been an important subject in signal processing. In this way, DOA algorithms are needed that introduce high spatial resolution and low computational complexity.
There are different methods to estimate the DOA that are divided into three basic categories, classical, subspace-based, and maximum likelihood (ML)-based [3].

Classical DOA methods are essentially based on beam forming. The two classical techniques for DOA are the delay&sum [4] and the minimum variance distortionless response (MVDR) [5] methods. The basic idea behind the classical methods is to scan a beam through space and measure the power received from each direction. Directions from which the largest amount of power is received are taken to be the DOAs [3].

Among them, subspace-based algorithms also referred to as super-resolution techniques, offer a good tradeoff between resolution and computational complexity. The subspace methods include different versions of the multiple signal classification (MUSIC) such as Root-MUSIC, constrained MUSIC, beam-space MUSIC, uniform circular array (UCA)-Root-MUSIC, spatial smoothing (SS) MUSIC, modified MUSIC, Fourier domain Root-MUSIC, Fourier domain weighted least squares Root-MUSIC, interpolated Root-MUSIC [6]-[17]. In addition, the estimation of signal parameters via rotational invariance technique (ESPRIT) [17]-[18] and the minimum norm (Min-Norm) method [19] are two other ones.

The ML approach is applicable for both correlated and uncorrelated signals. However, the resulting ML estimator involves a multidimensional optimization procedure over the parameter space, which includes the polarization vector parameters in addition to the DOAs. In order to avoid an exhaustive search procedure of these parameters, a simulated annealing technique was used [20]. The ML-based methods for DOA estimation includes deterministic maximum likelihood (DML) [21],[22] and stochastic maximum likelihood(SML) [22].

The classical methods are conceptually simple but offer modest or poor performance while requiring a relatively large number of computations. Note that these algorithms are initially presented under the assumption that the signal sources are stationary in space and that the incoming signals are uncorrelated. The subspace methods perform well and have several computationally efficient variants. The ML method offers high performance but they are computationally expensive.

Nowadays, two-dimensional (2D) DOA estimation of multiple narrowband signals using sensor array techniques has played a major role in many applications. Therefore, many prominent algorithms have been extended to the 2D DOA estimation and also different array configurations such as L-shaped [23], two L-shaped [24],[25], V-shaped[26], Z-shaped [27],Y-shaped, parallel ULAs, displaced sensor array (DSA) and also ULA with two extra elements [28],[29] are proposed and developed for 2D DOA estimation. The main idea behind the great part of them is to do the 2D DOA estimation as two 1D DOA estimation. Hence, it is essential to evaluate and compare the performance of 1D DOA estimation algorithms in different cases. It means the flexible methods and efficient algorithms in 1D may be useful for 2D DOA estimation.

In this investigation, the more popular DOA estimation algorithms, MVDR and Delay&Sum methods of classical category, MUSIC, Root-MUSIC, ESPRIT, and Min-Norm methods of subspace-based category and finally DML and SML ones of ML-based category are illustrated with more details. In order to evaluate their performance and also to compare them, these methods are simulated in MATLAB software. All computer codes are run on a personal computer with an Intel Core2, 2.5 GHz CPU, and 3.21GB RAM. Also, following assumptions are considered:

1. Signal sources are narrowband and uncorrelated to each other;
2. Antenna array is an 8-element ULA considering \( \frac{d}{r} \) inter-element spacing;
3. The number of snapshots is 100.
4. Signal DOAs are \( \{0^\circ, 15^\circ, 30^\circ\} \) and \( \{0^\circ, 75^\circ, 85^\circ, 90^\circ\} \);
5. Signal to noise ratios are: [-10, 0, 10, 20, 30] dB;
6. Root mean square error (RMSE) and normalized root mean square error (NRMSE) criteria are extracted by averaging the simulations in 500 trials.

The rest of this paper is organized as follows. In section II, the signal model and related formulae are presented. Section III is focused on DOA estimation methods and related formulations with more details. In this section some simulation results for each method are illustrated that show the spatial spectrum or estimated angular value and also the RMSE criterion for them. In section IV the performance of selected algorithms are compared based on NRMSE and the number of estimated signal sources. Finally, section V concludes the paper.

## II. Signal Model

Consider a ULA composed of \( N \) sensors, and it receives \( M \) narrowband source signals \( S_n(t) \) from desired users arriving at directions \( \theta_1, \theta_2, \ldots, \theta_m \). At a particular instant of time \( t = 1, 2, \ldots, K \) where \( K \) is the total number of snapshots taken, the desired users signal vector \( x(t) \) can be defined:

\[
x(t) = AS(t) + n(t)
\]

(1)

where \( A \) is the \( M \times N \) matrix of the desired users signal direction vectors and is given by:

\[
A(\theta) = [\alpha(\theta_1), \alpha(\theta_2), \ldots, \alpha(\theta_m)]
\]

(2)

where \( \alpha(\theta_m) \) is the \( N \times 1 \) array steering vector which represents the array response at direction \( \theta_m \) and is given by:

\[
a(\theta_m) = [\exp(j(\pi - 1)) \left(2\pi \left(\frac{d}{\lambda}\right) \sin(\theta_m)\right)]^T
\]

(3)

where \( \left[\cdot\right]^T \) is the transposition operator, \( d \) is the inter-
element spacing and $\lambda$ is the wavelength of the received signal.

In Eq. (1), $S(t)$ is the $M \times 1$ desired users’ source waveform vector defined as:

$$S(t) = [S_1(t), S_2(t), \ldots, S_M(t)]$$

and $n(t)$ is an additive white Gaussian noise (AWGN). The covariance matrix defined as:

$$R = E[XX^H] = AR_{SS}A^H + \sigma_n^2I$$

where $R_{SS} = E[SS^H]$ is an $M \times M$ desired users’ source waveform covariance matrix, $\sigma_n^2$ is the noise variance, and $I$ is an identity matrix of size $N \times N$.

III. DOA Estimation Methods

In this investigation, DOA estimation methods are categorized as follows:

1. classical
2. subspace-based
3. ML-based

In the following subsections, the details of each method are presented. Also, simulation results are illustrated that are extracted for three cases, single source, two and five sources. In addition, RMSE value is plotted for different DOAs of single source.

III.1. Classical Methods

The delay&sum method computes the DOA by measuring the signal power at each possible angle of arrival. In this method, the estimate of the angle of arrival is the direction of maximum power. In an ULA when the elements of the steering vectors have equal gains, the weight vector produces a sinc beam pattern that has large sidelobes. The largest sidelobe has a magnitude $13\,\text{dB}$ below that of the mainlobe. The limiting factor in the overall performance of this method is that it can steer its main beam but it has no control over its side lobes. So, despite the narrow main lobe width, the large side lobes allow unwanted power to enter into the computation of $P(\theta)$ for different angles of arrival and hence DOA resolution deteriorates.

The MVDR method is similar to the delay&sum technique in that it measures the power of the received signal in all possible directions. The power from the DOA, $\theta$, is measured by constraining the beamformer gain to be 1 in that direction and using the remaining degrees of freedom to minimize the contributions to the output power from signals coming from all other directions. This method has no drawbacks of delay&sum method, and provides higher resolution [3].

Fig. 1 shows the spatial spectrum of both delay&sum and MVDR algorithms in the case of three signal sources arriving at an 8-element ULA with $SNR = 30\,\text{dB}$. Due to higher angular accuracy and resolution of MVDR algorithm with respect to delay & sum, in the next parts of this paper, MVDR is considered. The peaks in the MVDR spectrum occur whenever the steering vector is orthogonal to the noise subspace, so the DOAs are estimated by detecting the peaks in the spectrum [30].

![Fig. 1. The spatial spectrum of delay & sum and MVDR algorithms](image)

### III.1.1. MVDR Algorithm

In MVDR algorithm, the desired source DOA is a maximum likelihood estimate of the power arriving from one direction while all other sources are considered as interference. This method has better resolution than Bartlett and delay & sum. Its goal is to maximize the signal to interference ratio (SIR) while passing the signal of interest (SOI) undistorted in phase and amplitude. The pseudo-spectrum of the method is as shown in Eq. (6)[5]:

$$P_{MVDR}(\theta) = \frac{1}{\sigma^2(\theta)R_y^{-1}(\theta)}$$

The spatial spectrum of MVDR with a source at direction $0^\circ$ and two sources at DOAs: $[-2^\circ, 2^\circ]$ for $SNR = 30\,\text{dB}$ have been shown in Figs. 2(a), 2(b), respectively. The MVDR spectrum depicted in Fig. 2(b) shows that the DOA estimation algorithm has successfully resolved the sources located at $[-2^\circ, 2^\circ]$.
RMSE is the most applicable metric to evaluate the performance of DOA estimation algorithm that can be calculated as follows:

\[
RMSE = \sqrt{\frac{1}{M} \sum_{j=1}^{M} \sum_{i=1}^{I} (\hat{\theta}_j^i - \theta_j^i)^2}
\]  

(7)

where \( I \) is the number of simulation trials and the estimate of the \( i \)-th angle of arrival in the \( j \)-th trial is \( \hat{\theta}_j^i \).

The RMSE of MVDR versus SNR has been shown for different single sources at directions \([0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 90^\circ]\) in Fig. 3. As depicted in this figure, for low SNRs, the error at middle angles is less than \( 1^\circ \). For the sources farther away from the center of array, the error is increased. So, this algorithm is not able to detect angles larger than \( 75^\circ \).

\[
\begin{align*}
\text{Fig. 3. RMSE of MVDR versus SNR for different single sources at} \\
\text{DOAs}\{0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 90^\circ\}
\end{align*}
\]

III.2. Subspace-Based Methods

Subspace-based techniques produce the spatial spectrum by using Eigen-decomposition of the covariance matrix of input signals

III.2.1. MUSIC Algorithm

A common subspace-based DOA estimation algorithm is MUSIC. The Eigen decomposition of \( R \) can be expressed as:

\[
R = V_s A \Sigma V_s^H + \sigma_n^2 V_n \Sigma_n V_n^H
\]  

(8)

where \( V_s \), which consists of the \( M \) largest Eigen vectors of \( R \), is the signal subspace matrices, \( A \) is a diagonal matrix of the Eigen values corresponding to \( V_s \), and \( V_n \), constituted by the remaining Eigen vectors of \( R \), is referred to as the noise subspace matrices being orthonormal complementary to \( V_s \).

The normalized MUSIC angular spectrum is defined as [6]:

\[
P_{\text{MUSIC}}(\theta) = \frac{A A^H}{A^H V_n \Sigma_n V_n^H A}
\]  

(9)

The spatial spectrum of MUSIC with a source at direction \( \theta \) and two sources at directions \([-1^\circ, 1^\circ]\) for \( SNR = 30dB \) have been shown in Fig. 4(a), 4(b), respectively. By comparing Figs. 2 and Figs. 4, it is clear that the peaks of MUSIC spectrum are sharper than those in MVDR spectrum. It means that the MUSIC is more accurate and also offers higher resolution than MVDR.

The RMSE of MUSIC algorithm versus SNR has been shown in Fig. 5 for different single sources at directions \([0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 85^\circ]\).

\[
\begin{align*}
\text{Fig. 4. Spatial spectrum of MUSIC for SNR = 30dB considering} \\
\text{(a) a source at DOA:0^\circ, (b) two sources at DOAs:[-1^\circ, 1^\circ]}
\end{align*}
\]

\[
\begin{align*}
\text{Fig. 5. RMSE of MUSIC versus SNR for different single sources at} \\
\text{DOAs}\{0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 85^\circ\}
\end{align*}
\]

III.2.2. Root- MUSIC Algorithm

The Root-MUSIC algorithm proposed by Barabelis only applicable for ULAs. It has been shown that the Root-MUSIC algorithm provides improved resolution compared to the ordinary MUSIC method especially at low SNRs. As before, the MUSIC spectrum is defined as:

\[
P_{\text{MUSIC}}(\theta) = \frac{A A^H}{A^H V_n \Sigma_n V_n^H A} = \frac{A A^H}{A^H C A}
\]  

(10)

where \( C \) is:

\[
C = V_n \Sigma_n V_n^H
\]  

(11)
By writing the denominator as a double summation, one obtains:

\[
A^{H}V_{i}V_{j}^{H}A = \sum_{k=0}^{N-1} \sum_{p=0}^{N-1} C_{k,p} \exp(-j2\pi(p-k)d \sin \theta) \tag{12}
\]

\[
A^{H}V_{i}V_{j}^{H}A = \sum_{p-k=0}^{N-1} C_{p-k} \exp(-j2\pi(p-k)d \sin \theta) \tag{13}
\]

\(C_{k}\) is the sum of the \(l\)-th diagonal of the matrix \(C\). A polynomial \(D(z)\) can now be defined as follows:

\[
D(z) = \sum_{l=-N+1}^{N+1} C_{l} z^{-1} \tag{14}
\]

The polynomial \(D(z)\) is equivalent to \(A^{H}V_{i}V_{j}^{H}A\) evaluated on the unit circle. Because the MUSIC spectrum have \(r\) peaks, \(A^{H}V_{i}V_{j}^{H}A\) have \(r\) valleys and hence \(D(z)\) have \(r\) zeros on the unit circle. The rest of the zeros of \(D(z)\) will be away from the unit circle. It can be shown that if \(z = be^{jd}\) is a root of \(D(z)\), then:

\[
b e^{j\theta} = e^{j2\pi l \sin(\theta)} \Rightarrow b = 1 \tag{15}
\]

\[
\theta = \sin^{-1}(\frac{2\pi l}{N}), \quad i = 1, 2, \ldots, d \tag{16}
\]

In the absence of noise, \(D(z)\) will have roots that lie precisely on the unit circle, but considering the noise, the roots will be only close to the unit circle. The Root-MUSIC reduces the problem of DOA estimation to finding the roots of a \((2N+1)\)-th order polynomial[3].

Tables I and II show the estimated DOA values using Root-MUSIC for two cases, single and double sources.

Fig. 6 shows the RMSE of Root-MUSIC algorithm versus SNR for different single sources at DOAs: [0°, 15°, 30°, 45°, 60°, 75°, 90°].

III.2.3. ESPRIT Algorithm

ESPRIT is a computationally efficient and robust method of DOA estimation.

<table>
<thead>
<tr>
<th>Desired DOA (degree)</th>
<th>Estimated DOA-1 (degree)</th>
<th>Estimated DOA-2 (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0022</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>14.9939</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>29.9967</td>
<td>3.0207</td>
</tr>
<tr>
<td>60</td>
<td>60.0160</td>
<td>2.0221</td>
</tr>
<tr>
<td>75</td>
<td>74.9891</td>
<td>3.9893</td>
</tr>
<tr>
<td>85</td>
<td>85.0955</td>
<td>4.9759</td>
</tr>
<tr>
<td>90</td>
<td>-89.6253</td>
<td></td>
</tr>
</tbody>
</table>

It uses two identical arrays in the sense that array elements need to form matched pairs with an identical displacement vector, that is, the second element of each pair ought to be displaced by the same distance and in the same direction relative to the first element. However, this does not mean that one has to have two separate arrays. The array geometry should be such that the elements could be selected to have this property. For example, a ULA of four identical elements with inter-element spacing \(d\) may be thought of as two arrays of three matched pairs, one with first three elements and the second with last three elements such that the first and the second elements form a pair, the second and the third elements form another pair, and so on. The two arrays are displaced by the distance \(d\).

Let the signals induced on the \(l\)-th pair due to a narrowband source in direction \(\theta\) be denoted by \(x_l(t)\) and \(y_l(t)\). The phase difference between these two signals depends on the time taken by the plane wave arriving from the source considering the travel time between two elements. Assume that the two elements are separated by the displacement \(d\). Thus, it follows that:

\[
y_l(t) = x_l(t) e^{j2\pi d \cos \theta} \tag{17}
\]

Note that \(d\) is the magnitude of the displacement vector. This vector sets the reference direction and all angles are measured with reference to this vector. Let the array signals received by the two \(K\)-element arrays be denoted by \(x(t)\) and \(y(t)\). These are given by:

\[
x(t) = A s(t) + n_x(t) \tag{18}
\]

\[
y(t) = B s(t) + n_y(t) \tag{19}
\]
and:
\[
y(t) = A\Phi s(t) + n_y(t)
\]
where \( A \) is a \( K \times M \) matrix with its columns denoting the \( M \) steering vectors corresponding to \( M \) directional sources associated with the first subarray, \( \Phi \) is an \( M \times M \) diagonal matrix with its \( m \)-th diagonal element given by:
\[
\phi_{m,m} = e^{i2\pi d \cos \theta_m}
\]

\( s(t) \) denotes \( M \) source signals induced on a reference element, and \( n_x(t) \) and \( n_y(t) \), respectively, denote the noise induced on the elements of the two subarrays. Comparing the equations for \( x(t) \) and \( y(t) \), it follows that the steering vectors corresponding to \( M \) directional sources associated with the second subarray are given by \( A\Phi \).

Let \( V_x \) and \( V_y \) denote two \( K \times M \) matrices with their columns denoting the \( M \) Eigenvectors corresponding to the largest Eigenvalues of the two array correlation matrices \( R_x \) and \( R_y \), respectively. As these two sets of Eigenvectors span the same \( M \)-dimensional signal space, it follows that these two matrices \( V_x \) and \( V_y \) are related by a unique nonsingular transformation matrix \( \psi \), that is:
\[
V_x = \psi V_y
\]

Similarly, these matrices are related to steering vector matrices \( A \) and \( A\Phi \) by another unique nonsingular transformation matrix \( T \) as the same signal subspace is spanned by these steering vectors. Thus:
\[
V_x = AT
\]
and:
\[
V_y = A\Phi T
\]

Using Eq. (22), (23) and knowing that \( A \) is full rank:
\[
T y = \Phi
\]

According to this statement, the Eigenvalues of \( \psi \) are equal to the diagonal elements of \( \Phi \) and columns of \( T \) are Eigenvectors of \( \psi \).

This is the main relationship in the development of ESPRIT. It requires an estimate of \( \psi \) from the measurement of \( x(t) \) and \( y(t) \). An Eigen decomposition provides its eigenvalues, and by setting them to \( \Phi \) leads to the DOA estimates:
\[
\theta_m = \cos^{-1}\left(\frac{\text{Arg}(\lambda_m^m)}{2\pi d}\right), \quad m = 1, \ldots, M
\]

Note that the array geometry in Fig. 7 includes one sensor which belongs to both the first and the second subarrays [31],[32].

Fig. 7. An example of array geometry for ESPRIT [32]

Tables III and IV show the estimated DOA values by ESPRIT for different single and double sources, respectively. The RMSE of ESPRIT versus SNR have been shown in Fig. 8 for different single sources at directions [0°,15°,30°,60°,75°,85°].

**TABLE III**
The Estimated DOA Values by ESPRIT for Different Single Sources

<table>
<thead>
<tr>
<th>Desired DOA (degree)</th>
<th>Estimated DOA (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0017</td>
</tr>
<tr>
<td>15</td>
<td>15.0059</td>
</tr>
<tr>
<td>30</td>
<td>30.0186</td>
</tr>
<tr>
<td>60</td>
<td>60.0049</td>
</tr>
<tr>
<td>75</td>
<td>75.0513</td>
</tr>
<tr>
<td>85</td>
<td>84.9931</td>
</tr>
<tr>
<td>90</td>
<td>-88.8616</td>
</tr>
</tbody>
</table>

**TABLE IV**
The Estimated DOA Values by ESPRIT for Different Double Sources

<table>
<thead>
<tr>
<th>Desired DOAs (degree)</th>
<th>Estimated DOA-1 (degree)</th>
<th>Estimated DOA-2 (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.1</td>
<td>-1.0180</td>
<td>1.0522</td>
</tr>
<tr>
<td>-2.2</td>
<td>-2.0231</td>
<td>1.9360</td>
</tr>
<tr>
<td>-3.3</td>
<td>-2.9753</td>
<td>3.0104</td>
</tr>
<tr>
<td>-4.4</td>
<td>-3.9814</td>
<td>3.9663</td>
</tr>
<tr>
<td>-5.5</td>
<td>-4.9876</td>
<td>4.9692</td>
</tr>
</tbody>
</table>

Fig. 8. RMSE of ESPRIT versus SNR for different single sources at DOAs [0°,15°,30°,60°,75°,85°]
III.2.4. Min-Norm Algorithm

This algorithm is applicable for ULA and finds the DOA estimate by searching for peak locations in the spectrum. The power spectrum is given by:

\[ P_{MN}(\theta) = \frac{1}{|\mathbf{w}^T A|^2} \]  

(26)

where \( \mathbf{w} \) denotes the array weight such that it is of the minimum norm, has first element equal to unity and is contained in the noise subspace. The solution for this problem leads to the following expression for the spectrum:

\[ P_{MN}(\theta) = \frac{1}{|\mathbf{A}^T \mathbf{v}_n \mathbf{v}_n^T \mathbf{e}_1|^2} \]  

(27)

where the vector \( \mathbf{e}_1 \) contains all zeros except the first element equal to unity [19].

The spatial spectrum of Min-Norm with a source at direction \( \theta^0 \) and two sources at directions \([-1^\circ, 1^\circ]\) for \( SNR = 30dB \) have been shown in Figs. 9(a), 9(b), respectively. The RMSE of Min-Norm versus SNR has been shown in Fig. 10 for different single sources at directions \([0^\circ, 1.5^\circ, 30^\circ, 60^\circ, 75^\circ, 85^\circ]\).

\[ \text{Figs. 9. Spatial spectrum of Min-Norm for SNR= 30dB considering (a) a source at DOA:0^\circ, (b) two sources at DOAs: [-1^\circ, 1^\circ]} \]

\[ \text{Fig. 10. RMSE of Min-Norm versus SNR for different single sources at DOAs:0^\circ, 1.5^\circ, 30^\circ, 60^\circ, 75^\circ, 85^\circ}] \]

III.3. Maximum Likelihood-Based Methods

The ML method has such desirable theoretical properties, it is a commonly used method in estimation theory. For example, under fairly general conditions, the ML estimates converge to the true parameter values as the number of data values approaches infinity. Also, if there exists an estimate that achieves the Cramér-Rao bound, a lower bound on the minimum achievable variance for an unbiased estimate, then the estimate is the ML estimate [33]. DML and SML algorithms are two well-known versions of ML method.

III.3.1. DML Algorithm

Using the deterministic signal model, the DML approach is:

\[ \hat{\theta}_{DML} = \arg\max_{\theta} Tr(\mathbf{P}(\theta) R) \]  

(28)

Using the projection matrix onto the column space of \( \mathbf{A}(\theta) \):

\[ \mathbf{P}(\theta) = \mathbf{A}(\theta) \mathbf{A}(\theta)^H \]  

(29)

and the pseudo inverse of \( \mathbf{A}(\theta) \) [22]:

\[ \mathbf{A}(\theta)^H = (\mathbf{A}(\theta)^H \mathbf{A}(\theta) - \mathbf{I}) \]  

(30)

III.3.2. SML Algorithm

According to (Jaffer, 1988), the SML algorithm approach is:

\[ \hat{\theta}_{SML} = \arg\max_{\theta} (-\log |\mathbf{A}(\theta)\mathbf{P}(\theta)\mathbf{A}(\theta)^H + \hat{\sigma}^2(\theta)\mathbf{I}|) \]  

(31)

Using the signal projection:

\[ \mathbf{P}(\theta) = \mathbf{A}(\theta)^H \]  

(32)

together with the DOA dependent noise power estimation:

\[ \hat{\sigma}^2(\theta) = \frac{1}{N-M} Tr(\mathbf{P}(\theta) R) \]  

(33)

and the orthogonal projection matrix is:

\[ \mathbf{P}(\theta) = \mathbf{I} - \mathbf{P}(\theta) \]  

(34)

The computational effort of SML is much higher than DML, as the estimation of \( \hat{\sigma}^2(\theta) \) is already more expensive than the whole DML approach. Having a single snapshot and two targets, SML suffers numerical problems using double precision arithmetic. Some DOAs thus lead to negative logarithms. Nevertheless, the SML function is a bit noisy which makes the optimization expensive. As a side effect of the numerical noise, the
Increasing the SNR improves the performance of DML.

Tables V and VI show the estimated DOA by DML for single source and double sources, respectively. The RMSE of DML versus SNR is shown in Fig. 12 for different single sources at the following DOAs: $[0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 85^\circ]$.

![Fig. 11. RMSE of DML and SML algorithms versus SNR for a single source at DOA $0^\circ$](image)

### TABLE V

<table>
<thead>
<tr>
<th>Desired DOA (degree)</th>
<th>Estimated DOA (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>75</td>
<td>75</td>
</tr>
<tr>
<td>85</td>
<td>84.9</td>
</tr>
<tr>
<td>90</td>
<td>89.6</td>
</tr>
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</table>

### TABLE VI

<table>
<thead>
<tr>
<th>Desired DOAs (degree)</th>
<th>Estimated DOA-1 (degree)</th>
<th>Estimated DOA-2 (degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1,1</td>
<td>-21.2</td>
<td>0</td>
</tr>
<tr>
<td>-6,6</td>
<td>3.4</td>
<td>28.4</td>
</tr>
<tr>
<td>-7,7</td>
<td>-6.5</td>
<td>6.1</td>
</tr>
<tr>
<td>-8,8</td>
<td>-8.3</td>
<td>8.3</td>
</tr>
<tr>
<td>-9,9</td>
<td>-9.4</td>
<td>9.3</td>
</tr>
<tr>
<td>-10,10</td>
<td>-10.1</td>
<td>10.1</td>
</tr>
</tbody>
</table>

![Fig. 12. RMSE of DML versus SNR for different single sources at DOAs: $[0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 85^\circ]$](image)

### IV. Comparison of DOA Algorithms Based on Numerical Results

As illustrated in the previous section, SNR and the angular distance between two adjacent signal sources have major impacts on the accuracy and resolution of DOA estimation algorithms. Therefore, in order to evaluate and compare the performance of different DOA estimation algorithms, RMSE criterion versus SNR for different angular difference is used.

Here, a new criterion, NRMSE, is proposed that show the performance of different algorithms especially in low SNRs and close sources. Maximum value of NRMSE is 1 when none of the sources is detected. For two sources, 50% of total RMSE belongs to first source and 50% for the second one. In this case, partial NRMSE is 0.5 when a source is not detected.

Figs. 13 show the NRMSE of DOA estimation algorithms versus angular distance between two sources for different SNRs. According to Figs. 13, the following results can be extracted:

1. DML shows better performance than the other algorithms in low SNR ($\sim 10$ dB).
2. Increasing the SNR improves the performance of different algorithms but it has neglected effect on DML performance.
3. Lower NRMSE belongs to Root-MUSIC, ESPRIT and MUSIC, respectively.

Another comparison is considered in the case of high number of sources which is greater than the half of the array elements. In this case, all algorithms are simulated in different SNRs. Table VII shows the number of detected sources in various SNRs. Root-MUSIC, ESPRIT and MUSIC offer higher number of detected sources in different SNRs.

### V. Conclusion

DOA estimation plays a major role in signal detection and adaptive antenna array beam forming in wireless communication applications. Efficient DOA estimation can improve the quality of transmitted and/or received signals and manage radio resources, effectively. The performance of DOA estimation depends on four parameters which are array geometry, DOA algorithm, the type of signal sources, correlated or uncorrelated, and signal bandwidth, narrowband or wideband. In this paper, we focused on DOA estimation algorithms in ULA geometry for narrowband uncorrelated sources. First, we presented and evaluated the performance of main algorithms of three well-known categories, classical, subspace-based, and ML-based methods. The performance of each algorithm was evaluated based on RMSE criterion in different DOAs. Also, the effect of SNR and the number of sources was investigated. In general, following results are extracted from numerical experiments:

- By increasing the SNR, RMSE will be decreased and both the resolution and angular accuracy will be...
increased.
- Increasing the number of sources is the reason to decrease the performance.
- Decreasing the angular difference between signal sources will decrease the performance.

![Comparison of different methods for SNR](image)

**TABLE VII**

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>SNR(dB)</th>
<th>Number of detected sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>MVDR</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>MUSIC</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Root-MUSIC</td>
<td>-10</td>
<td>4</td>
</tr>
<tr>
<td>ESPRIT</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Min-Norm</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>DML</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MVDR</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>MUSIC</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Root-MUSIC</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>ESPRIT</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Min-Norm</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>DML</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>MVDR</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>MUSIC</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Root-MUSIC</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>ESPRIT</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Min-Norm</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>DML</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>MVDR</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>MUSIC</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Root-MUSIC</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>ESPRIT</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Min-Norm</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>DML</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

We also compared these algorithms in various scenarios and SNRs. Simulation results showed that the Root-MUSIC, ESPRIT and MUSIC offer higher performance rather than MVDR, Min-Norm and ML algorithms, respectively. It should be noted that Root-MUSIC is an appropriate algorithm only for ULA and ESPRIT algorithm needs special array geometry, but the MUSIC algorithm can be used for all types of array geometries.

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**References**


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