Mathematical modeling of deflection of a beam: A finite element approach

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Abstract— introducing a suitable model for a structure to understand its behavior under different conditions of loading is very important. Mathematical modeling is the simulation of a physical structure or physical process by means of suitable analytical or numerical construct. One of suitable methods for finding deflection of a beam under different forms of loading is Finite Element Method (FEM). In this paper we find deflection of a beam using FEM based on Euler-Bernoulli and Timoshenko theory.

Keywords—Mathematical modeling; deflection of a beam; Finite element method, Euler-Bernoulli theory, Timoshenko theory.

I. INTRODUCTION

The finite element (FE) method was developed more by engineers using physical insight than by mathematicians using abstract method. In all application analyst seeks to calculate a filed quantity. For example in stress analysis it is the displacement filed or the stress filed. The FE method is a way of getting a numerical solution to a specific problem.

An unsophisticated description of finite element method is that it involves cutting a structure into several elements, describing the behavior of each element in a simple way, then reconnecting elements at “nodes” as if nodes were pins or drops of glue that holds elements together (Fig.1). This process results in a set of simultaneous algebraic equations [1].

A more sophisticated description of the FE method regards it as piecewise polynomial interpolation. That is, over an element, a filed quantity such as displacement is interpolated from values of the filed quantity at nodes. By connecting elements together, the filed quantity becomes interpolated over the entire structure in piecewise fashion, by as much polynomial expression as there are elements. Matrix symbolism for this set of equation is $KD = F$ , where D is vector of unknown (value of the filed quantity at nodes), F is a vector of known loads and K is a matrix of known constant. In stress analysis K is “stiffness matrix”. [1].

II. FORMULATION

A. Euler-Bernoulli beam

The Euler-Bernoulli equation for a beam is

$$\rho \frac{d^2 v}{dt^2} + \frac{d^2}{dx^2} (EI \frac{dv}{dx}) = q(x , t) \quad (1)$$

Where $v(x , t)$ is the transverse displacement of the beam, $\rho$ is mass density per volume, $EI$ is the beam rigidity, $q(x , t)$ is the externally applied pressure load, and $t$ and $x$ indicate time and spatial axis along the beam axis. We consider shape functions for spatial interpolation of the transverse deflection, $v$, in terms of nodal variables. To this end, we consider an element which has two nodes, one at each end, as shown in Fig1.

Figure 1. Two-nodes beam element

The deformation of a beam must have continuous slope as well as continuous deflection at any two neighboring beam elements. To satisfy this continuity requirement each node has deflection, $v_i$, and slope, $\theta_i$, as nodal variables. In this case, any two neighboring beam elements have common deflection and slope at the shared nodal point. This satisfies the continuity of both deflection and slop. The Euler-Bernoulli beam equation is based on the assumption that the plane normal to the neutral axis before deformation remains normal to the neutral axis after deformation (see Fig.2). This assumption denotes $\theta = \frac{dv}{dx}$ (i.e. slope is the first derivative of deflection in terms of $x$) [2].

Figure 2. Euler-Bernoulli beam

Because there are four nodal variables for the beam element, we assume a cubic polynomial function for $v(x)$
\( v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \) \hspace{1cm} (2)

After applying boundary condition, we find \( c_i \) in terms of \( v_i \) and \( \theta_i \), so we will have
\[
v(x) = H_1(x) v_1 + H_2(x) v_2 + H_3(x) v_3 + H_4(x) \theta_2 \hspace{1cm} (3)
\]

Where
\[
\begin{align*}
H_1(x) &= 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} \\
H_2(x) &= x - \frac{2x^2}{l} + \frac{x^3}{l^2} \\
H_3(x) &= \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \\
H_4(x) &= -\frac{x^2}{l} + \frac{x^3}{l^2}
\end{align*}
\]

The functions \( H_i(x) \) are called Hermitian shape functions and shown in Fig.3 [2].

The mass matrix is: \( M^e = \int_0^l \rho A \dot{\mathbf{H}}^T \dot{\mathbf{H}} \, dx \) (9)

\[
= \frac{\rho Al}{420} \begin{bmatrix} 185 & 221 & 54 & -134 \\ 221 & 41^2 & 131 & -201 \\ 54 & 131 & 186 & -224 \\ -134 & -201 & -224 & 41^2 \end{bmatrix}
\]

**B. Timoshenko beam**

The Timoshenko beam theory includes the effect of transverse shear deformation. As a result, a plane normal to the beam axis before deformation does not remain normal to the beam axis any longer after deformation. Figure 4 shows the deformation in contrast to that in Fig.2. While Galerkin's method was used to derive the finite element matrix equation for the Euler-Bernoulli beam equation, the energy method is used for the present formulation for the Timoshenko beam.

Let \( u \) and \( v \) be the axial and transverse displacements of a beam, respectively. Because of the transverse shear deformation, the slope of the beam \( \theta \) is different from \( dv/dx \). Instead, the slope equals \( (dv/dx) - \gamma \) where \( \gamma \) is the transverse shear strain. As a result, the displacement field in the Timoshenko beam can be written as
\[
\begin{align*}
\{u(x,y)\} &= -y \theta(x) \\
\{v(x)\} &= v
\end{align*}
\]

where the z-axis is located along the neutral axis of the beam and the beam is not subjected to an axial load such that the neutral axis does not have the axial strain. The axial and shear strains are
\[
\begin{align*}
\varepsilon &= -y \frac{d\theta}{dx} \\
\gamma &= -\theta + \frac{dv}{dx}
\end{align*}
\]

The element stiffness matrix can be obtained from the strain energy expression for an element. The strain energy for an element of length \( l \) is: \( U = \frac{b}{2} \int_0^l \int_{-h/2}^{h/2} \gamma \sigma \gamma \, dy \, dx \) (12)

in which the first term is the bending strain energy and the second term is the shear strain energy. Moreover, \( b \) and \( h \) are
the width and height of the beams respectively, and \( \mu \) is the correction factor for shear energy whose value is normally \( \frac{5}{6} \).

After taking integration with respect to \( y \) gives: (13)

\[
\int u = \frac{1}{2} \int \left( \frac{\partial v}{\partial x} \right)^2 + \frac{1}{2} \int \left( \frac{\partial \theta}{\partial x} \right)^2 \, dx
\]

where \( I \) and \( A \) are the moment of inertia and area of the beam cross-section. In order to derive the element stiffness matrix for the Timoshenko beam, the variables \( v \) and \( \theta \) need to be interpolated within each element. \( v \) and \( \theta \) are independent variables. That is, we can interpolate them independently using proper shape functions. This results in satisfaction of inter-element compatibility, i.e. continuity of both the transverse displacement \( v \) and slope \( \theta \) between two neighboring elements. We use the simple linear shape functions for both variables. That is,

\[
\begin{align*}
\begin{bmatrix}
   v_1 \\
   v_2
\end{bmatrix} &= \begin{bmatrix}
   H_1 & H_2
\end{bmatrix} \begin{bmatrix}
   v_1 \\
   v_2
\end{bmatrix}, \\
\begin{bmatrix}
   \theta_1 \\
   \theta_2
\end{bmatrix} &= \begin{bmatrix}
   H_1 & H_2
\end{bmatrix} \begin{bmatrix}
   \theta_1 \\
   \theta_2
\end{bmatrix}
\end{align*}
\]

where \( H_1 \) and \( H_2 \) are linear shape functions. Element stiffness matrix for the Timoshenko beam is:

\[
K_e = K_e^r + K_e^s
\]

where: (16)

\[
\begin{bmatrix}
   K_e^r & 0 \\
   0 & K_e^s
\end{bmatrix} = \begin{bmatrix}
   EI & 0 \\
   0 & GJ
\end{bmatrix}
\]

One thing to be noted here is that the bending stiffness term is obtained using the exact integration of the bending strain energy but the shear stiffness term, is obtained using the reduced integration technique [2].

III. PROBLEM

We suppose a cantilever with 10 element which is subjected to a point load at \( x=1 \) m.

In our project we want to find deflection beam based on Euler-Bernoulli and Timoshenko theory for different thickness and compare them with each other. We divided the beam to 10 elements with 11 nodes as shown in figure 6.

\[
\begin{array}{cccccccccc}
\text{Figure 6. Beam elements and nodes}
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{Figure 7. Degree of freedom of one element}
\end{array}
\]

A. Properties of the beam

Properties of the flexible cantilever and are shown in table 1.

<table>
<thead>
<tr>
<th>Physical parameter of problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Physical parameters</strong></td>
</tr>
<tr>
<td>Length(m)</td>
</tr>
<tr>
<td>Width(m)</td>
</tr>
<tr>
<td>Thickness(m)</td>
</tr>
<tr>
<td>Shear modul (GPa)</td>
</tr>
<tr>
<td>Young’s modulus(GPa)</td>
</tr>
<tr>
<td>Shear correction coefficient</td>
</tr>
</tbody>
</table>

IV. RESULTS

As we can see in the figures for thin beams the transverse shear deformation is negligible and the Euler-Bernoulli and Timoshenko beam theories give the same results.

\[
\begin{array}{cccc}
\text{Figure 8. Deflection of the cantiliver for } t=50 \text{ cm}
\end{array}
\]
V. CONCLUSION

In this project we find deflection of a beam by using finite element method. As we discussed the deflection depends on thickness and kind of theory we use. If L/t (i.e. the ratio of length to thickness) is small Timoshenko theory has more deflection than Euler-Bernoulli.

VI. REFERENCES