Densely Co-Hopfian Modules

Sh. Asgari [1] and **A. Haghany**[2]
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, 84156-83111, IRAN.

ABSTRACT. The modules of the title are introduced as a proper generalization of quasi co-Hopfian modules, and they are characterized in several ways. The ring R is right nonsingular if and only if the densely co-Hopfian right R-modules and quasi co-Hopfian right R-modules coincide. Dense co-Hopficity is investigated for certain modules that have indecomposable decompositions complementing direct summands. For some classes of rings R, including rings with dense right socles, we determine when the injective envelope of an R-module is densely co-Hopfian.

Keywords: Goldie torsion theory, reduced rank, dense submodule, densely co-Hopfian, quasi co-Hopfian, weakly co-Hopfian.
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1 INTRODUCTION AND PRELIMINARIES

Throughout rings will have unity and modules will be unitary. Let M denote a right module over a ring R. Recall that M is called co-Hopfian if every injective R-endomorphism of M is surjective. Co-Hopfian modules are generalized in the following ways. In [8], a weakly co-Hopfian (wcH) module is defined by the property that all injective R-endomorphisms of the module are essential. While in [2], the module M is called quasi co-Hopfian (qcH) if $M/f(M)$ is singular whenever f is an injective R-endomorphism of M. These concepts are vastly investigated in [2], [3] and [8]. Clearly

$$\text{co-Hopfian} \Rightarrow \text{weakly co-Hopfian} \Rightarrow \text{quasi co-Hopfian}$$

and none of these implications can be reversed. In this paper we introduce and study a notion for modules called *densely co-Hopfian* (dcH). We say M_R is densely co-Hopfian if for all injective R-endomorphisms f of M, $f(M)$ is a *dense* submodule of M in the *Goldie torsion theory* on Mod-R, that is, $M/f(M)$ is Z_2-torsion. We show that Goldie torsion modules (i.e., Z_2-torsion modules), as well as quasi co-Hopfian modules, are densely co-Hopfian but not conversely. The dcH property is investigated for direct sums and in Theorem 2.13 we determine the dcH property for a certain module that has an indecomposable decomposition complementing direct summands. We then discuss when the dcH property transfers between a module and its injective envelope. We also consider modules with the property that all their submodules are dcH. Such modules will be called *completely co-Hopfian* (ccH), and they are characterized in Proposition 3.1. Finally in

[1]Corresponding author: sh_asgari@math.iut.ac.ir
[2] aghagh@cc.iut.ac.ir

Theorem 3.4 for three types of rings R, including rings with dense right socles, we prove that M_R is ccH if and only if the injective hull $E(M_R)$ of M_R is dcH.

We now fix our notation and state a few well known preliminary results that will be needed. Let M be a right R-module, A an R-submodule. Then $A \leq_e M$ will mean that A is an essential submodule of M. The singular submodule of M is denoted by $Z(M)$, and $Z_2(M)$ is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. The uniform dimension of $M/Z_2(M)$ is called the reduced rank of M, and we denote this by $\mathbf{r}(M)$. Recall that M is called singular if $M = Z(M)$, and nonsingular if $Z(M) = 0$. The module M is called *Goldie torsion* (or Z_2*-torsion*) if $Z_2(M) = M$. If M/A is Goldie torsion, then A is said to be a *dense* submodule of M, and this fact is denoted by $A \leq_d M$. Thus, for any module M_R, $Z_2(M) = \{x \in M : \text{ann}(x) \leq_d R_R\}$. As a hereditary torsion class, the class of Z_2-torsion modules is closed under submodules, factor modules, direct sums and extensions. These facts imply that $A \leq_d B$ and $B \leq_d C$ if and only if $A \leq_d C$. Moreover, the Goldie torsion theory is stable, that is, the Z_2-torsion class is closed under taking injective envelopes. Let us now write $Z(M/A) = A^*/A$ and $Z(M/A^*) = A^{**}/A$, and put on record several properties each of which is equivalent to A being dense in M. We omit the proofs.

Proposition 1.1 *Let $A \leq M$. The following are equivalent statements.*
(1) $A \leq_d M$.
(2) M/A^* *is singular.*
(3) $A^* \leq_e M$.
(4) $A^{**} = M$.
(5) $A + Z(M) \leq_e M$.
(6) $A + Z_2(M) \leq_e M$.
(7) $(A + Z_2(M))/Z_2(M) \leq_e M/Z_2(M)$.
(8) $A \oplus B \leq_e M$, *for some* Z_2*-torsion submodule* B *of* M.
(9) $A \cap B \neq 0$ *for every non-zero nonsingular submodule* B *of* M.
(10) *For every submodule* B *of* M, $A \cap B \leq Z_2(M)$ *implies that* $B \leq Z_2(M)$.
(11) *For all* $m \in M \backslash Z_2(M)$, *there exists* $r \in R$ *such that* $mr \in A \backslash Z_2(A)$.

A notable property of dense submodules is that their inverse images under homomorphisms are again dense submodules. We shall also make use of the following well known facts, proofs of which are given for reader's convenience.

Proposition 1.2 (1) *The intersection of all dense submodules of* M *is the sum* $S(M)$ *of all nonsingular simple submodules of* M. *Consequently* $Z_2(M)S(R_R) = 0$.
(2) *The product of two dense right ideals is a dense right ideal.*

Proof. (1). Let $D(M)$ denote the intersection of all dense submodules of M. If D is a dense submodule and S is a nonsingular simple submodule of M, then $S \cap D \neq 0$. Hence $S \leq D$ and so $S(M) \leq D(M)$. On the other hand, if L is a complement to $Z_2(M)$ then

$L \leq_d M$. Thus $D(M)$ is nonsingular. However, $D(M) \leq \mathrm{Soc}(M)$ since every essential submodule is dense. Thus $D(M) \leq S(M)$.

Now if $m \in Z_2(M)$ then $\mathrm{ann}(m) \leq_d R_R$. However, $S(R_R)$ is the intersection of all dense right ideals of R, hence $S(R_R) \leq \mathrm{ann}(m)$. Thus $Z_2(M)S(R_R) = 0$.

(2) Let I and J be dense right ideals of the ring R. Then R/I and I/IJ are Z_2-torsion. Therefore from the isomorphism $[R/IJ]/[I/IJ] \cong R/I$ we conclude that R/IJ is Z_2-torsion, hence IJ is dense. □

2 DENSELY CO-HOPFIAN MODULES

Recall from §1 that M_R is called *densely co-Hopfian* (abbreviated dcH) if the image of any injective R-endomorphism of M is a dense submodule. The following result describes several equivalent conditions to dcH property. It reduces to some parts of [2, Theorem 4.1] when the base ring is right nonsingular.

Theorem 2.1 *The following statements are equivalent for an R-module M.*
(1) *M is dcH.*
(2) *M contains a dense submodule K which is dcH as an R-module and $f(K) \leq K$ for any injective endomorphism f of M.*
(3) *There exists a dense submodule K of M such that $f(K) \cap K \leq_d K$ whenever f is an injective endomorphism of M.*
(4) *If there exists an R-monomorphism $M \oplus N \to M$, then N is Z_2-torsion.*
(5) *For every dense submodule K of M and every injective endomorphism f of M, $f(K) \leq_d M$.*
(6) *For every non-Z_2-torsion submodule K of M and every injective endomorphism f of M, $f^{-1}(K)$ is non-Z_2-torsion.*
(7) *There exists a submodule K of M such that K and M/K are dcH and $f^{-1}(K) = K$ for any injective endomorphism f of M.*

Proof. Clearly (1) ⇒ (2) ⇒ (3) ⇒ (1).

(1) ⇒ (4). Let $f : M \oplus N \to M$ be a monomorphism and $\imath : M \to M \oplus N$ be the canonical injection. Then $f\imath$ is an injective endomorphism of M. Thus $f(M \oplus 0) \leq_d M$, hence $f(0 \oplus N)$ is Z_2-torsion by Proposition 1.1-(10), and so N is Z_2-torsion.

(4) ⇒ (1). Let f be an injective endomorphism of M. There exists a submodule $N \leq M$ such that $f(M) \oplus N \leq_e M$. By (4), N must be Z_2-torsion and consequently $f(M) \leq_d M$ by Proposition 1.1.

(1) ⇒ (5). Let f be an injective endomorphism of M. Then $f(M) \leq_d M$. On the other hand, $K \leq_d M$ implies that $f(K) \leq_d f(M)$. Thus $f(K) \leq_d M$.

(5) ⇒ (1). Apply (5) for $K = M$.

(1) ⇒ (6). Let f be an injective endomorphism of M and K be a non-Z_2-torsion submodule of M. By Proposition 1.1-(10), $f(M) \cap K$ is non-Z_2-torsion. Thus there exists $m \in M \setminus Z_2(M)$ such that $f(m) \in K$. Consequently $m \in f^{-1}(K)$ and $m \notin Z_2(M)$. Thus

$f^{-1}(K)$ is non-Z_2-torsion.

(6) $\Rightarrow$ (1). Let g be an injective endomorphism of M such that $g(M) \not\leq_d M$. There exists a non-Z_2-torsion submodule N such that $g(M) \oplus N \leq_e M$. Thus by (6), $g^{-1}(N)$ is non-Z_2-torsion. However, $g^{-1}(N) = g^{-1}(g(M) \cap N) = g^{-1}(0) = 0$, a contradiction.

For (1) $\Rightarrow$ (7), set $K = M$, and finally for (7) $\Rightarrow$ (1), define $\bar{f} : M/K \to M/K$ by $\bar{f}(m + K) = f(m) + K$. By hypothesis $f(M) + K \leq_d M$. On the other hand, $[f(M) + K]/f(M) \cong K/(f(M) \cap K) = K/f(K)$. By the assumption that K is dcH, the module $K/f(K)$ is Z_2-torsion and so $f(M) \leq_d f(M) + K$. It follows that $f(M) \leq_d M$. □

Corollary 2.2 *Let M be an R-module.*
(1) *If M is dcH then so is every direct summand of M.*
(2) *If $M/Z_2(M)$ is dcH then so is M. In particular, every module of finite reduced rank is dcH.*
(3) *Assume that I is a dense right ideal of R. If MI is dcH then so is M.*

Proof. (1). This follows from Theorem 2.1-(4).

(2). Apply Theorem 2.1-(7) for $K = Z_2(M)$.

(3). Clearly MI is a fully invariant submodule and $MI \leq_d M$. Thus (3) follows from Theorem 2.1-(2) for $K = MI$.

Remark 2.3 A module M is quasi co-Hopfian if $f(M)^* = M$ for every injective endomorphism f of M, and by Proposition 1.1, M is densely co-Hopfian if $f(M)^{**} = M$ for every injective endomorphism f of M.

Proposition 2.4 *Let M be a module.*
(1) *An infinite direct sum of copies of M is dcH if and only if M is Z_2-torsion.*
(2) *If $M = \sum_{i \in I} M_i$ such that $f(M_i) \cap M_i \leq_d M_i$ for any injective endomorphism f of M, then M is dcH.*

Proof. (1). Let $M^{(\Lambda)}$ be dcH. By Corollary 2.2-(1) we can assume that Λ is countable. Then $M/f(M)$ is Z_2-torsion where $f : M^{(\Lambda)} \to M^{(\Lambda)}$ is the shift map. However M is isomorphic to $M/f(M)$, hence M is Z_2-torsion. The converse is clear since every direct sum of Z_2-torsion modules is Z_2-torsion.

(2). Let f be an injective endomorphism of M and $N_i = f(M_i) \cap M_i$ for each $i \in I$. Define $\varphi : \bigoplus_{i \in I}(M_i/N_i) \to M/f(M)$ by $(m_i + N_i)_{i \in I} \mapsto (\Sigma_{j \in J}\, m_j) + f(M)$, where J is the largest subset of I such that $m_j \notin N_j$ for any $j \in J$. Then φ is an epimorphism and $\bigoplus_{i \in I}(M_i/N_i)$ is Z_2-torsion. Thus $M/f(M)$ is Z_2-torsion and so M is dcH. □

Corollary 2.5 *Let M be semisimple. Then M is dcH if and only if every nonsingular homogenous component of M is finitely generated.*

Proof. Let M be a semisimple dcH module and $H = S^{(\Lambda)}$ be a nonsingular homogenous component of M. By Corollary 2.2-(1), H is dcH and hence by Proposition 2.4-(1), H is finitely generated. The converse follows from Proposition 2.4-(2) as every homogenous component M_i of M is fully invariant and dcH. □

Proposition 2.6 *The following statements are equivalent for a module M.*
(1) *M is wcH.*
(2) *M is qcH and for any injective endomorphism f of M, $f(Z(M)) \leq_e Z(M)$.*
(3) *M is dcH and for any injective endomorphism f of M, $f(Z(M)) \leq_e Z(M)$.*
(4) *M is qcH and for any injective endomorphism f of M, $f(Z_2(M)) \leq_e Z_2(M)$.*
(5) *M is dcH and for any injective endomorphism f of M, $f(Z_2(M)) \leq_e Z_2(M)$.*

Proof. Clearly (1) ⇒ (2) ⇒ (3). For (3) ⇒ (1), by Proposition 1.1-(5), it is enough to show that $f(M) \leq_e f(M) + Z(M)$ for any injective endomorphism f of M. Let $K \leq f(M) + Z(M)$ and $K \cap f(M) = 0$. By hypothesis $K \cap Z(M) = 0$. Now let $x \in K$. There exist $y \in f(M)$ and $z \in Z(M)$ such that $x = y + z$, hence $xI = yI \leq K \cap f(M) = 0$ for some $I \leq_e R_R$. Thus $x \in K \cap Z(M) = 0$.

Similarly, (1) ⇒ (4) ⇒ (5) ⇒ (1). □

Corollary 2.7 *Let M be a module such that $Z(M)$ or $Z_2(M)$ is wcH. Then M is dcH if and only if M is qcH if and only if M is wcH.*

Proposition 2.8 *Let R be a ring.*
(1) *The class of dcH R-modules coincides with the class of (weakly) co-Hopfian R-modules if and only if R is semisimple.*
(2) *The class of dcH R-modules coincides with the class of qcH R-modules if and only if R is right nonsingular.*

Proof. (1) follows from [2, Proposition 2.12].

(2) (⇒). Let M be an R-module. Then $Z_2(M)^{(\mathbb{N})}$ is Z_2-torsion and so it is dcH. Thus by hypothesis $Z_2(M)^{(\mathbb{N})}$ is qcH, hence $Z_2(M)$ is singular by [2, Lemma 3.3]. Consequently $Z_2(M) = Z(M)$, in particular, $Z_2(E) = Z(E)$ where $E = E(R_R)$. However E is extending and so $Z(E)$ is a direct summand of E. This implies that $Z(E) = 0$, hence $Z(R_R) = 0$.

(⇐). Since the notions of Z_2-torsion and singular are the same for a module over a right nonsingular ring, the properties dcH and qcH are equivalent. □

Example 2.9 We now construct examples of dcH modules which are neither Z_2-torsion nor qcH. Let R be a right Noetherian ring such that $Z(R_R) \neq 0$ and $Z_2(R_R) \neq R$. Then $Z_2(E) \neq Z(E)$ where $E = E(R_R)$ since $Z(R_R) \neq 0$; see the proof of Proposition 2.8-(2). Now by Corollary 2.2-(2) and [2, Proposition 2.2-(i) and Lemma 3.3], $R/Z_2(R_R) \oplus Z_2(E)^{(\mathbb{N})}$ is dcH which is neither Z_2-torsion nor qcH.

Theorem 2.10 *The following statements are equivalent for a ring R.*
(1) *Every (projective, free) R-module M is dcH.*
(2) *For every (projective, free) R-module M, $M/Z(M)$ is dcH.*
(3) *For every (projective, free) R-module M, $M/Z_2(M)$ is dcH.*
(4) *For every (projective, free) R-module M, $M/Z(M)$ is qcH.*
(5) *For every (projective, free) R-module M, $M/Z(M)$ is singular.*
(6) *Every (projective, free) R-module is Z_2-torsion.*
(7) *There exists a nilpotent dense right ideal in R.*
(8) $\mathrm{Rad}(R) \leq_d R_R$ *and* $\mathrm{Soc}(R_R) \leq_d R_R$.
(9) $Z(R_R) \leq_e R_R$.
(10) R_R *is Z_2-torsion.*

Proof. For (1) $\Rightarrow$ (6), let M be an R-module and Λ be an infinite set. By hypothesis $M^{(\Lambda)}$ is dcH, hence M is Z_2-torsion by Propositions 2.4-(1). The implication (6) $\Rightarrow$ (5) is clear since $Z(M/Z(M)) = Z_2(M)/Z(M)$. Clearly (5) $\Rightarrow$ (4) $\Rightarrow$ (2). Now let (2) hold. Then $L = (R/Z(R_R))^{(\mathbb{N})} \cong R^{(\mathbb{N})}/Z(R_R)^{(\mathbb{N})}$ is dcH. Since $L \oplus (R/Z(R_R)) \cong L$, Theorem 2.1-(4) implies that $R/Z(R_R)$ is Z_2-torsion, thus $Z(R_R) \leq_d R_R$ and so by Proposition 1.1-(5), $Z(R_R) \leq_e R_R$. This shows that (2) $\Rightarrow$ (9). Obviously (9) $\Rightarrow$ (10). If (10) holds then every R-module M is Z_2-torsion as $MZ_2(R_R) \leq Z_2(M)$, hence M is dcH. Thus (10) $\Rightarrow$ (1).

Clearly (6) $\Rightarrow$ (3) and (3) $\Rightarrow$ (10) by setting $Z_2(M)$ instead of $Z(M)$ in the proof of (2) $\Rightarrow$ (9). In addition, if (6) holds the zero ideal is nilpotent and dense and so (6) $\Rightarrow$ (7). Now assume that (7) holds and let K be a nilpotent dense right ideal of R, say $K^n = 0$. If $n = 1$ then $K = 0$ is Z_2-torsion. If $n > 1$ then by Proposition 1.2-(2), K^{n-1} is a dense right ideal and so $KK^{n-1} = 0$ implies that K is Z_2-torsion. Therefore R contains a Z_2-torsion dense right ideal, hence R_R is Z_2-torsion. Thus every right ideal of R is dense and so (7) $\Rightarrow$ (8). Now let (8) hold and M be an R-module. Since $\mathrm{Rad}(R_R) \leq_d R_R$ and $M\mathrm{Rad}(R) \leq \mathrm{Rad}(M)$ we conclude that $\mathrm{Rad}(M) \leq_d M$. Thus $0 = \mathrm{Rad}(S) \leq_d S$ for every simple R-module S and so every simple R-module is Z_2-torsion (in fact it is singular), hence $\mathrm{Soc}(R_R) \leq Z_2(R_R)$. Since by hypothesis $\mathrm{Soc}(R_R) \leq_d R_R$ we conclude that $Z_2(R_R) \leq_d R_R$ and so $Z_2(R_R) = R_R$. Thus (8) $\Rightarrow$ (10). □

Let R be a right Artinian local ring that is not a division ring. Then $\mathrm{Rad}(R) \leq_e R_R$ and $\mathrm{Soc}(R_R) \leq_e R_R$ and so R satisfies all the conditions of Theorem 2.10. Moreover, $T = \begin{pmatrix} R & \mathrm{Rad}(R) \\ 0 & R \end{pmatrix}$ is right Artinian and clearly $\mathrm{Rad}(T) = \begin{pmatrix} \mathrm{Rad}(R) & \mathrm{Rad}(R) \\ 0 & \mathrm{Rad}(R) \end{pmatrix} \leq_e T_T$. Thus T also satisfies the conditions of Theorem 2.10 although T is not local. The next result, in particular, shows that for a right Artinian ring each of the statements of Theorem 2.10 holds if and only if $\mathrm{Rad}(R) \leq_e R_R$.

Proposition 2.11 *For a ring R, if $\mathrm{Soc}(R_R) \leq_e R_R$ then each of the statements in Theorem 2.10 is equivalent to any one of the following conditions.*
(*i*) $\mathrm{Rad}(R) \leq_e R_R$.
(*ii*) *For every R-module M, $M/\mathrm{Rad}(M)$ is singular.*

(iii) For every R-module M, $M/\mathrm{Rad}(M)$ is qcH.
(iv) Every simple R-module is singular.
(v) $\mathrm{Soc}(R_R)^2 = 0$.
(vi) $S(R_R)^2 = 0$.

Proof. By condition (6) of Theorem 2.10, every simple R-module is singular and so every maximal right ideal is essential. Thus $\mathrm{Soc}(R_R) \leq \mathrm{Rad}(R)$, and then the hypothesis $\mathrm{Soc}(R_R) \leq_e R_R$ implies that $\mathrm{Rad}(R) \leq_e R_R$. This shows that (6) $\Rightarrow$ (i). By [2, Proposition 2.13], $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$. Clearly $(ii) \Rightarrow (iv)$. The equality $Z(R_R)\mathrm{Soc}(R_R) = 0$ implies that $(iv) \Rightarrow (v)$. Obviously $(v) \Rightarrow (vi)$. Now let $S(R_R)^2 = 0$. Since $\mathrm{Soc}(R_R) \leq_e R_R$ and $S(R_R) \leq_d \mathrm{Soc}(R_R)$, we conclude that $S(R_R) \leq_d R_R$ and so $S(R_R)$ is a nilpotent dense right ideal of R. Thus $(vi) \Rightarrow$ (7) of Theorem 2.10. □

A module M is called *CS* or *extending* if every closed submodule of M is a direct summand of M. A ring for which every free module is CS is called Σ-extending [5, 12.21 and Corollary 11.4]. Over such rings, a dcH module is exactly a direct sum of a wcH module and a Z_2-torsion module, as the next result shows.

Proposition 2.12 *Let M be a homomorphic image of a CS module. M is dcH if and only if M is isomorphic to a direct sum of a nonsingular dcH module and a Z_2-torsion module.*

Proof. Assume that $M \cong L/K$ where L is a CS module and $K \leq L$. First we show that $Z_2(L/K)$ is a direct summand of L/K. Assume that $Z_2(L/K) = K'/K$. Then K' is a closed submodule of L; in fact, if $K' \leq_e N \leq L$ then N/K is Z_2-torsion since $(N/K)/(K'/K)$ and K'/K are Z_2-torsion. Thus $N/K \leq Z_2(L/K)$ and so $N = K'$. Therefore K' is a direct summand of L, say $L = K' \oplus K''$. Hence $L/K = K'/K \oplus (K'' + K)/K$ as desired. Now proposition is clear by Corollary 2.2-(1) and (2). □

Let M have an indecomposable decomposition that complements direct summands (see [1, § 12]). We call the direct sum of all isomorphic direct summands of such a decomposition, a homogeneous component of that decomposition.

Theorem 2.13 *Let M be a module such that $f(M)$ is a direct summand of M for every injective endomorphism f of M. Suppose M has an indecomposable decomposition that complements direct summands. Then M is dcH if and only if every non-Z_2-torsion homogeneous component of such a decomposition of M is a finite direct sum.*

Proof. ($\Rightarrow$). A non-Z_2-torsion homogenous component of an indecomposable decomposition of M is isomorphic to $N^{(\Lambda)}$, for some non-Z_2-torsion indecomposable submodule N of M. Since M is dcH so is $N^{(\Lambda)}$, thus by Proposition 2.4-(1), Λ is finite.

($\Leftarrow$). Let f be an injective endomorphism of M. By hypothesis $M = f(M) \bigoplus K$ for some submodule K. Assume that K is non-Z_2-torsion. By hypothesis there exists an indecomposable decomposition $M = \bigoplus_{\alpha\in A} M_\alpha$ that complements direct summands and so there exists such a decomposition for K by [1, Lemma 12.3], say $K = \bigoplus_{\beta\in B} K_\beta$. Then

$$M = \bigoplus_{\alpha\in A} M_\alpha \quad (*) \qquad ; \qquad M = \bigoplus_{\alpha\in A} f(M_\alpha) \bigoplus (\bigoplus_{\beta\in B} K_\beta) \quad (**)$$

By [1, Theorem 12.4], $(*)$ and $(**)$ are equivalent. Now, for a $\beta_1 \in B$ such that K_{β_1} is non-Z_2-torsion, $K_{\beta_1} \cong M_{\alpha_1}$ for some $\alpha_1 \in A$. By hypothesis the homogenous component of M corresponding to M_{α_1} has finitely many direct summands, say $M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}$. Then in $(**)$ there is a homogenous component with at least $n+1$ direct summands, i.e. $f(M_{\alpha_1}), f(M_{\alpha_2}), \ldots, f(M_{\alpha_n}), K_{\beta_1}$ which are all isomorphic to M_{α_1}. This contradicts the equivalence of $(*)$ and $(**)$. Therefore K is Z_2-torsion and so $f(M) \leq_d M$. □

An extending module M is called *continuous* if every submodule N of M which is isomorphic to a direct summand of M, is a direct summand of M. Every nonsingular injective module over a ring of finite reduced rank is a direct sum of indecomposable modules by [10, Theorem 1.2] and [11, Theorem 2.1]. Hence by [9, Exercise 37 of §6] and the well known fact that the endomorphism ring of an indecomposable continuous module is a local ring, a nonsingular continuous module over a ring of finite reduced rank has a decomposition into indecomposable continuous submodules such that the endomorphism ring of each direct summand is local. Since a continuous module has the finite exchange property, such a decomposition complements direct summands by [4, Theorem 14.22]. Moreover, a module M is called *Σ-quasi-injective* if every direct sum of copies of M is quasi-injective. By [6, Corollary 2.4], every Σ-CS module, hence every Σ-quasi-injective module has an indecomposable decomposition which complements direct summands. Bearing these facts in mind the following corollary easily follows from Theorem 2.13.

Corollary 2.14 *Let R be a ring.*

(1) *If R is of finite reduced rank and M is a continuous R-module, then M is dcH if and only if every nonsingular homogeneous component of a decomposition of M into indecomposable continuous submodules is a finite direct sum.*

(2) *If M is Σ-quasi-injective, then M is dcH if and only if every nonsingular homogeneous component of a decomposition of M into indecomposable Σ-quasi-injective submodules is a finite direct sum.*

Corollary 2.15 *A divisible abelian group M is dcH if and only if $\mathbf{r}(M) < \infty$.*

Proof. This is an immediate consequence of Corollary 2.14-(1) since a divisible abelian group is a direct sum of isomorphic copies of $\mathbb{Q}$ and of arbitrary Prüfer groups. □

3 COMPLETE CO-HOPFICITY AND INJECTIVE ENVELOPES

A natural question is whether the dense co-Hopficity passes to injective envelope. As the notions of dcH and qcH are the same for a nonsingular module, [2, Example 4.8] shows that in general the answer to this question is negative. By the additive property of reduced rank if M is of finite reduced rank, then so is $E(M)$ and hence by Corollary 2.2-(2), $E(M)$ is dcH. However, every submodule of a finite reduced rank module is dcH. Let us call a module all of whose submodules are dcH, a *completely co-Hopfian* module (ccH). Now it is natural to ask whether $E(M)$ is dcH if M is ccH. In the following we show that a quasi-injective dcH module is ccH. Moreover for some classes of rings we show that the answer to the latter question is affirmative.

Proposition 3.1 *The following statements are equivalent for a module M.*
(1) *M is ccH.*
(2) *Every dense submodule of M is dcH.*
(3) *Every non-dense submodule of M is dcH.*
(4) *$X^{(\mathbb{N})}$ cannot be embedded in M, for any non-Z_2-torsion module X.*

Proof. Clearly (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3).

(2) $\Rightarrow$ (1). This follows by Corollary 2.2-(1), since every submodule is a direct summand of an essential submodule.

(3) $\Rightarrow$ (1). Let K be a submodule of M which is not dcH. There exists an injective endomorphism f of K such that $f(K) \not\leq_d K$. Thus $f(K) \not\leq_d M$, however $f(K) \cong K$ implies that $f(K)$ is not dcH which contradicts (3).

(1) $\Rightarrow$ (4). This follows by Proposition 2.4-(1).

(4) $\Rightarrow$ (1). Let K be a submodule of M. If K is not dcH then by Theorem 2.1-(4), there exists a non-Z_2-torsion module X such that $K \bigoplus X$ can be embedded in K. This implies that $X^{\mathbb{N}}$ can be embedded in M which contradicts (4). □

Proposition 3.2 *The following statements are equivalent if M is quasi-injective.*
(1) *M is dcH.*
(2) *M is ccH.*
(3) *$E(M)$ is dcH.*

Proof. (1) $\Rightarrow$ (2). It suffices to show that every essential submodule of M is dcH. Let K be an essential submodule of M and g be an injective endomorphism of K. As M is quasi-injective, there exists an endomorphism f of M such that $f\mid_K = g$. The essentiality of K implies that f is an injective endomorphism of M, hence $f(M) \leq_d M$. Clearly $f(K) \leq_d f(M)$, thus $f(K) \leq_d M$ and so $f(K) \leq_d K$.

(2) $\Rightarrow$ (3). This follows from Theorem 2.1-(2) for $K = M$.

(3) $\Rightarrow$ (1). By applying (1) $\Rightarrow$ (2) to the module $E(M)$ we conclude that $E(M)$ is ccH, hence M is dcH. □

The following lemma is useful. Recall that a module M is called *compressible* (resp. *retractable*) if there exists a monomorphism (resp. non-zero homomorphism) $f : M \to N$ for any non-zero submodule N of M.

Lemma 3.3 *Let R be a ring of finite reduced rank which is semiprime or has ACC on two-sided ideals. Then every non-zero nonsingular right R-module M contains an essential submodule $L = \bigoplus_{i \in I} U_i$ where each U_i is a uniform compressible right R-module.*

Proof. First we show that every non-zero submodule N of M has a uniform compressible submodule. Since R has finite reduced rank, the additive property of reduced rank implies that the reduced rank of every cyclic submodule of N is finite. However N is nonsingular, hence the uniform dimension of every cyclic submodule of N is finite. This implies that N contains a cyclic uniform submodule U. By [7, Exercise 3W], there exists a non-zero right ideal I of R such that I can be embedded in U. As any non-zero right ideal of a semiprime ring is retractable, if R is semiprime, I is a retractable nonsingular uniform R-module. Thus I is compressible and so N contains a uniform compressible submodule isomorphic to I. Now if R has ACC on two-sided ideals, there exists an associated prime P of U. Then $P = \mathrm{ann}_R(V)$ for a non-zero submodule V in U. Clearly $Z(V_R) = 0$ implies that $Z(V_{R/P}) = 0$, hence there exists a non-zero right ideal A of R/P such that A can be embedded in $V_{R/P}$. Therefore A is a retractable nonsingular uniform R/P-module. Hence A is compressible as an R/P-module and so A is also compressible as an R-module. Consequently, N contains a uniform compressible submodule isomorphic to A. Now let $L = \bigoplus_{i \in I} U_i$ be a maximal direct sum of uniform compressible submodules of M. By what we have shown above, L must be essential in M. □

Theorem 3.4 *Let R be a ring for which $\mathrm{Soc}(R_R) \leq_d R_R$ or let R be a ring of finite reduced rank which is either semiprime or has ACC on two-sided ideals. Then M is a ccH R-module if and only if $E(M)$ is dcH.*

Proof. By Proposition 3.2, it suffices to show that if M is ccH then $E(M)$ is dcH. Assume that $\mathrm{Soc}(R_R) \leq_d R_R$. Clearly $E(M)\mathrm{Soc}(R_R) \leq \mathrm{Soc}(E(M)) = \mathrm{Soc}(M)$. Since M is ccH, $E(M)\mathrm{Soc}(R_R)$ is dcH. Thus by Corollary 2.2-(3) we conclude that $E(M)$ is dcH.

Now assume that R is a ring of finite reduced rank which is semiprime or has ACC on two-sided ideals. Since $E(M)$ is extending, $E(M) = Z_2(E(M)) \oplus E'$ for some submodule E'. Hence by Corollary 2.2-(2), it is enough to show that E' is dcH. Clearly, we can assume that $E' \neq 0$. By Lemma 3.3, $E' \cap M$ contains an essential submodule $L = \bigoplus_{i \in I} U_i$ where each U_i is a uniform compressible right R-module. However $E' \cap M$ is essential in E', hence $E' = E(\bigoplus_{i \in I} U_i)$. Because R is of finite reduced rank, $E' = \bigoplus_{i \in I} E(U_i)$ by [11, Theorem 2.1-(5)]. Since $E(U_i)$ is indecomposable injective, if we show that every homogenous component of this decomposition is a finite direct sum, then by Corollary 2.14-(1), E' is dcH. Assume that U_i and U_j are uniform compressible right modules with $E(U_i)$ and $E(U_j)$ two direct summands of a homogenous component of E' (i.e. $E(U_i) \cong E(U_j)$), hence there exist monomorphisms $f_{(i,j)} : U_i \to U_j$ and $f_{(j,i)} : U_j \to U_i$. On the other hand, every direct summand of L is a dcH R-module by our assumption, especially so is $\bigoplus_{k\in K} U_k$ which corresponds to a homogenous component of E'. However by monomorphisms $f_{(i,j)}$ there is a shift map on $\bigoplus_{k\in K} U_k$ which is a monomorphism if K is infinite. Moreover its image is not dense, which is a contradiction. Hence every homogeneous component of E' is a finite direct sum, as desired. □

Corollary 3.5 *Let R be a ring.*
(1) *If* $\mathrm{Soc}(R_R) \leq_d R_R$*, then M is ccH if and only if* $\mathrm{Soc}(M)$ *is dcH.*
(2) *If R is right Artinian, then an R-module M is ccH if and only if* $\mathbf{r}(M) < \infty$.

Proof. (1). By Theorem 3.4, it is enough to show that if $\mathrm{Soc}(M)$ is dcH then $E(M)$ is dcH. However, similar to the proof of Theorem 3.4, from $E(M)\mathrm{Soc}(R_R) \leq \mathrm{Soc}(E(M)) = \mathrm{Soc}(M)$ the result follows.

(2). It is enough to show that if an R-module M is ccH then $\mathbf{r}(M) < \infty$. Since R is right Artinian, $\mathrm{Soc}(M) \leq_e M$ and so $\mathrm{Soc}(M) \leq_d M$. Thus by the additive property of reduced rank $\mathbf{r}(M) = \mathbf{r}(\mathrm{Soc}(M))$. Moreover, there are only finitely many simple right R-modules up to isomorphism. Thus the result follows by (1) and Corollary 2.5. □

Corollary 3.6 *Let R be a ring for which* $\mathrm{Soc}(R_R) \leq_d R_R$ *or let R be a ring of finite reduced rank which is either semiprime or has ACC on two-sided ideals. The following statements are equivalent.*
(1) *In Mod-R, { ccH modules } = { modules of finite reduced rank }.*
(2) *Up to isomorphisms, there are only finitely many nonsingular indecomposable injective R-modules.*

Proof. Assume that $\mathrm{Soc}(R_R) \leq_d R_R$. First note that there is a one to one correspondence between non-isomorphic nonsingular indecomposable injective R-modules and non-isomorphic nonsingular simple R-modules. For (1) ⇒ (2), let $\{S_\lambda : \lambda \in \Lambda\}$ be any nonempty set of non-isomorphic nonsingular simple R-modules. Then by Corollary 2.5, $\bigoplus_{\lambda\in\Lambda} S_\lambda$ is ccH, hence it is of finite reduced rank by (1). Thus Λ must be finite and so (2) holds. The implication (2) ⇒ (1) follows by Corollaries 2.5 and 3.5-(1).

Now assume that R is a ring of finite reduced rank which is semiprime or has ACC on two-sided ideals. For (1) ⇒ (2), let $\{M_\lambda : \lambda \in \Lambda\}$ be any set of non-isomorphic nonsingular indecomposable injective right R-modules. Then $\bigoplus_{\lambda\in\Lambda} M_\lambda$ is injective and by Corollary 2.14-(1), it is dcH and so by (1) it is of finite reduced rank. Thus Λ must be finite and so (2) holds. The implication (2) ⇒ (1) follows by Theorem 3.4 and Corollary 2.14-(1). □

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