Multiproduct EPQ Model with Single Machine, Backordering, and Immediate Rework Process

Ata Allah Taleizadeh
Department of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran
E-mail: ata.taleizadeh@iust.ac.ir

Seyed Jafar Sadjadi
Department of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran
E-mail: sjsadjadi@iust.ac.ir

Seyed Taghi Akhavan Niaki*
Department of Industrial Engineering, Sharif University of Technology, Tehran, Iran
E-mail: Niaki@Sharif.edu
*Corresponding author

Abstract: Production systems with scrapped and rework items have recently become an interesting subject of research. While most attempts have been focused on finding the optimal production quantity in a simple production system, little work appears on a joint production environment. In this research, two joint production systems in a form of multiproduct single-machine with and without rework are studied where shortage is allowed and backordered. For each system, the optimal cycle length, the backordered and production quantities of each product are determined such that the cost function is minimized. Proof of the convexity of the involved
objective functions of each model is provided and numerical illustrations are given to demonstrate the applicability of the proposed models. Furthermore, the results obtained by solving the models with and without rework of defective items are compared. Sensitivity analysis and some managerial insights based on the numerical illustration are provided at the end.

[Received 12 January 2010; Revised 22 May 2010; Accepted 31 May 2010]

**Keywords:** Inventory Control; Multiproduct; Single-Machine; Rework; Limited Production Capacity, Backordering

**Biographical notes:** Ata Allah Taleizadeh is currently a Ph.D. student in Industrial Engineering at Iran University of Science and Technology. He received his B.Sc. and M.Sc. degrees both in Industrial Engineering. His research interest areas include inventory control, production planning, and integrated inventory–production–pricing models.

Seyed Jafar Sadjadi is an Associate Professor of Industrial Engineering at Iran University of Science and Technology. He has finished his BS and Master in Iran in the field on Industrial Engineering and did his PhD at the University of Waterloo, Canada in 1998. His research interest is Operations Research with Industrial Engineering and financial applications. Dr. Sadjadi is presently working as editor-in-chief of the International Journal of Industrial Engineering Computations, an indexed journal published by Growing Science.

Seyed Taghi Akhavan Niaki is a Professor of Industrial Engineering at Sharif University of Technology. His research interests are in the areas of Simulation Modeling and Analysis, Applied Statistics, Multivariate Quality Control and Operations Research. Before joining Sharif
University of Technology he worked as a systems engineer and quality control manager for Iranian Electric Meters Company. He received his Bachelor of Science in Industrial Engineering from Sharif University of Technology, his Master’s and his Ph.D. degrees both in Industrial Engineering from West Virginia University.

1 Introduction

The economic production quantity (EPQ) is a commonly used production model that has been studied extensively in the past few decades. One of the extension types of this model deals with the rework on imperfect quality products. Reworking imperfect products is known to be economically more attractive than to disposing of them. Pasandideh and Niaki (2008) extended a multiproduct EPQ model in which the orders may be delivered discretely in multiple pallet forms. Yoo et al. (2009) proposed a profit-maximizing EPQ model that incorporates both the imperfect production quality and two-way imperfect inspection. Jaber et al. (2008) introduced an EPQ model for items with imperfect quality subject to learning effects. They assumed imperfect quality items are withdrawn from inventory and sold at discount price. Jamal et al. (2004) developed an EPQ model to determine the optimum batch quantity in a single-stage system. They assumed rework is completed under two different operational policies, to minimize the total system cost, named immediate and N-Cycle rework processes. Cardenas-Barron (2009) introduced an EPQ model with a rework process at a single-stage manufacturing system with planned backorders. In this research, single-stage manufacturing system generates imperfect products, which are reworked in the same cycle. Goyal and Cardenas-Barron (2002) developed an EPQ model under a simple approach to determine economic production quantity for systems
that produce imperfect items. Huang (2004) extended the EPQ model in which imperfect products are allowed to be present into the produced lot sizes.

Chiu et al. (2004) considered the effects of random defective rate and imperfect rework process on EPQ model. Chang (2004) investigated the effects of imperfect products on the total inventory cost associated with an EPQ model. Hayek and Salameh (2001) developed an EPQ model by considering the imperfect products and rework items, assuming all the shortages are backordered and the percentage of defective products is a random variable. Goyal and Cardenas-Barron (2005) extended the EPQ model by considering an imperfect production system that produces defective products, supposing all the defective items that are randomly produced are reworked. Chiu et al. (2007) investigated an EPQ model that considers scrap, rework, and stochastic machine breakdowns to determine the optimal run time and production quantity. Sarkar et al. (2008) introduced an extended EPQ model dealing with the optimum production quantity in a multi-stage system in which rework is done under two different operational policies to minimize the total system cost. Liao et al. (2009) studied an integrated maintenance and production system with an EPQ model for an imperfect process involving a defective production system under increasing failure rate. Cardenas-Barron (2008) introduced a simple derivation to find the optimal production quantity of a system that includes a rework process. Choi et al. (2007) extended a production model in which demand is satisfied by recovering used products as well as new products. They assumed that a fixed proportion of the used products are collected from customers and later recovered for reuse. Taleizadeh et al. (2010a) introduced an EPQ model with scrapped items and limited production capacity as well as a multiproduct single-machine production system with stochastic scrapped production rate, partial backordering, and service level constraint (2010b). Furthermore, Taleizadeh et al. (2010c) studied a production quantity
model with random defective items, service level constraints, and repair failure in multiproduct single machine situations. Other related researches are: Cárdenas-Barrón (2007, 2009), Chandrasekaran et al. (2007), Liu et al. (2008), Mohan et al. (2008), and Parveen and Rao (2009).

2 Problem definition

The inventory control problem of this research involves imperfect production processes, in which due to process deterioration or some other factors, defective items of $n$ different types are generated at a rate $\beta_i; i=1,2,...,n$. The production rate of producing the $i^{th}$ item is $P_i$, assuming perfect items' quantities, i.e., $(1-\beta_i)P_i$, satisfy the corresponding demand, $d_i$ in each cycle. The demand, production, and proportion of defective rates of the products are considered constants. A cycle consists of production uptime ($T_i^{1}$ and $T_i^{3}$), reworking time ($T_i^{2}$), and production downtime ($T_i^{3}$ and $T_i^{4}$). During the reworking time, no defective items are produced and the production and rework tasks are performed utilizing shared resources. Since all of the products are manufactured on a single machine with a limited capacity, a unique cycle length for all items is considered, i.e, $T_1=T_2=\cdots=T_n=T$. All of the imperfect quality items are reworked and it is assumed that there are no scrapped items at the end of a rework process. Furthermore, both the production and rework processes are accomplished using the same resource, the same cost, and at the same speed where shortage is allowed in the form of backorder. Moreover, we assume that there is a real constant production capacity limitation on the single machine on which all products are produced and that the setup cost is nonzero.
3 Modeling

Jamal et al. (2004) developed an EPQ model to determine the optimum production quantity of an item where rework is performed in two different situations. The objective function of their model was to minimize the total inventory cost of the production system under consideration. In this paper, the model of Jamal et al. (2004) is extended and applied to a more realistic inventory control problem in which several items and several constraints are available. However, before developing the model, since the problem at hand is of a multiproduct type, for $i = 1, 2, \ldots, n$ the following notations are first defined for the parameters and the variables of the model.

$q_i$ The production lot size of the $i^{th}$ product in each cycle (a decision variable)

$T$ The cycle length (a decision variable)

$b_i$ The backordered quantity of the $i^{th}$ product in a cycle (a decision variable)

$d_i$ The demand rate of the $i^{th}$ product

$P_i$ The production (or rework) rate of the $i^{th}$ product

$\beta_i$ The proportion of defective items of the $i^{th}$ product

$C_i^s$ The setup cost of a production run for the $i^{th}$ product

$T_i^s$ The machine setup time to produce the $i^{th}$ product

$N$ Number of cycles per year

$TC (q,b)$ The total inventory costs per year

$I_i$ The maximum on-hand inventory of the $i^{th}$ product when a regular production process stops

$I_{Max}^i$ The maximum on-hand inventory of the $i^{th}$ product when the reworking ends
The production cost of the \( i^{th} \) product per item ($/item)

\( C_i^P \)

The inspection and rework cost of the \( i^{th} \) product per item ($/item)

\( C_i^R \)

The holding cost of the \( i^{th} \) product per item per unit time ($/item/unit time)

\( C_i^h \)

The shortage cost of the \( i^{th} \) product per item per unit time ($/item/unit time)

\( C_i^B \)

The multiproduct single machine (MP-SM) EPQ model with rework is first developed in section 3.1 and then a simpler model without rework is introduced in section 3.2.

3.1 MP-SM with rework

Figure (1) depicts a cycle for the inventory control problem under study. In order to model the problem, a single-product problem consisting of the \( i^{th} \) product is first developed, then, the model will extend to include several products.

be greater than or equal to the demand rate \( d_i \). As a result, we have:

\[
(1 - \beta_i) P_i - d_i \geq 0
\]  

(1)

The production cycle length (see Fig. 1) is the summation of the production uptimes, the rework time, and the production downtimes, i.e.,:

\[
T = \sum_{j=1}^{5} T_{ij}
\]  

(2)

The basic assumption in EPQ model with rework process is that \((1 - \beta_i) P_i \) must always where \( T_{i1} \) and \( T_{i5} \) are the production uptimes (the periods in which perfect and defective items are produced), \( T_{i2} \) is the reworking time, and the production downtimes are \( T_{i3} \) and \( T_{i4} \).
In this research, a part of the modeling procedure is adopted from Jamal et al. (2004) to model the problem. As noted before, since all products are manufactured on a single machine with a limited capacity, the cycle length for all products are equal ($T_1 = T_2 = \cdots = T_n = T$). Hence, based on Figure (1) we have:

\[ I_i = (1 - \beta_i)P_i - d_i \frac{Q_i}{P_i} - b_i \quad (3) \]

and

\[ I_i^{Max} = I_i + \beta_i (P_i - d_i) \frac{Q_i}{P_i} = (1 - \beta_i)P_i - d_i \frac{Q_i}{P_i} + \beta_i (P_i - d_i) \frac{Q_i}{P_i} - b_i \quad (4) \]

It is obvious from Figure (1) that:
\[
T_{i}^{1} = \frac{q_{i}}{P_{i}} - \frac{b_{i}}{(1 - \beta_{i})P_{i} - d_{i}}
\]

(5)

\[
T_{i}^{2} = \beta_{i} \frac{q_{i}}{P_{i}}
\]

(6)

\[
T_{i}^{3} = \left(1 - \frac{d_{i}}{P_{i}} - \beta_{i} \frac{d_{i}}{P_{i}} \right) q_{i} - \frac{b_{i}}{d_{i}}
\]

(7)

\[
T_{i}^{4} = \frac{b_{i}}{d_{i}}
\]

(8)

\[
T_{i}^{5} = \frac{b_{i}}{(1 - \beta_{i})P_{i} - d_{i}}
\]

(9)

Hence, using Equation (2) the cycle length for a single product problem is:

\[
T = \sum_{j=1}^{s} T_{j}^{i} = \frac{q_{i}}{d_{i}}
\]

(10)

Or,

\[
q_{i} = d_{i} T
\]

(11)

3.1.1 The objective function for a single product

The total cost of the system includes setup, processing, rework, shortage, and inventory carrying costs. Although the processing and rework costs are constants and do not affect the optimal solution, they are used in the objective function in order to determine the annual total cost. Defective items are produced in every batch and they are reworked within the same cycle. During the rework process, some processing and inventory holding costs are incurred for reworked quantities. Then, the total cost per year, \( TC(q, b) \), is obtained as:
Finally, the objective function of the joint production system (MP-SM with rework) using Equation (10) becomes:

\[
TC(T, b) = \sum_{i=1}^{n} C_i^s d_i + \sum_{i=1}^{n} C_i^r \beta_i q_i + \sum_{i=1}^{n} C_i^h \left[ \frac{I_i + I_i^{Max}}{2T} (T_i^1) + \frac{I_i + I_i^{Max}}{2T} (T_i^2) + \frac{I_i^{Max}}{2T} (T_i^3) \right] + \sum_{i=1}^{n} C_i^b b_i \left( T_i^4 + T_i^5 \right) / 2T
\]

(13)

3.1.2 The constraints for a single product

Since the production plus rework times of the \( i^{th} \) product is \( T_i^1 + T_i^2 + T_i^5 \) and the setup time is \( T_i^s \), the summation of the production, reworking and setup time (for all products) will be \( \sum_{i=1}^{n} (T_i^1 + T_i^2 + T_i^5) + \sum_{i=1}^{n} T_i^s \). It is clear that this summation must be smaller or equal to the cycle length (\( T \)). Hence, the capacity constraint of the model becomes:

\[
\sum_{i=1}^{n} (T_i^1 + T_i^2 + T_i^5) + \sum_{i=1}^{n} T_i^s \leq T
\]

(14)

Then, based on the Equations (3), (4), and (7) we have:

\[
\sum_{i=1}^{n} (1 + \beta) \frac{d_i}{P_i} T + \sum_{i=1}^{n} T_i^s \leq T
\]

(15)

3.1.3 The final model of MP-SM with rework

From Equations (3) to (11), (13) and (15), the final model of the joint production system is obtained as follows.
Min : \( TC(T,B) = \alpha_1 T^2 + \alpha_2 T + \frac{\alpha_3}{T} + \sum_{i=1}^{n} \alpha_{4i} \frac{b_i^2}{T} - \sum_{i=1}^{n} \alpha_{5i} b_i - \sum_{i=1}^{n} \alpha_{6i} T + \sum_{i=1}^{n} (C_i^p + \beta_i C_i^s) d_i \) \hspace{1cm} (16)

\[ s.t.: T \geq \frac{\sum_{i=1}^{n} T_i^i}{\left(1 - \sum_{i=1}^{n} (1 + \beta_i) \frac{d_i}{P_i}\right)} = T_{Lower} \] \hspace{1cm} (17)

Where;

\[ \alpha_1 = \sum_{i=1}^{n} \frac{C_i^h}{2} \left[ \frac{(1 + \beta_i)((1 - \beta_i)P_i - d_i))d_i^2 + \beta_i(P_i - d_i)d_i^2}{P_i} \right] > 0 \] \hspace{1cm} (18)

\[ \alpha_2 = \sum_{i=1}^{n} \frac{C_i^h}{2} \left[ \frac{3((1 - \beta_i)P_i - d_i)d_i^2 + (P_i - (1 + \beta_i)d_i)d_i}{P_i} \right] > 0 \] \hspace{1cm} (19)

\[ \alpha_3 = \sum_{i=1}^{n} C_i^s > 0 \] \hspace{1cm} (20)

\[ \alpha_{4i} = \frac{C_i^h}{2} \left[ \frac{(1 - \beta_i)(P_i - 2d_i)}{(1 - \beta_i)(P_i - d_i)d_i} \right] + \frac{C_i^h}{2} \left[ \frac{(1 - \beta_i)P_i}{(1 - \beta_i)(P_i - d_i)d_i} \right] > 0 \] \hspace{1cm} (21)

\[ \alpha_{5i} = \frac{C_i^h}{2} \left[ 1 + 2\beta_i \frac{d_i}{P_i} + \frac{(1 - \beta_i)(P_i - d_i)}{P_i} + \frac{\beta_i(P_i - d_i)}{P_i} \right] > 0 \] \hspace{1cm} (22)

\[ \alpha_{6i} = \frac{C_i^h}{2} \left[ \frac{(1 + \beta_i)d_i}{P_i} \right] > 0 \] \hspace{1cm} (23)

Note that since \( \sum_{i=1}^{n} (C_i^p + \beta_i C_i^s) d_i \) is constant, it has been removed from the objective function. In section four, a solution method is given for the developed model.

### 3.2 The MP-SM without rework

In this case, according to Figure (2) the objective function of the joint production system is obtained as follows.
Figure 2  On hand inventory of perfect quality items for MP-SM without rework

\[ I_i = \left( (1 - \beta_i) P_i - d_i \right) \frac{q_i}{P_i} - b_i \]

\[ (1 - \beta_i) P_i - d_i \]

\[ T_i^1 \]

\[ T_i^2 \]

\[ T_i^3 \]

\[ T_i^4 \]

\[ t \]

\[ i \]

\[ I_i \]

\[ \text{Production Cost} \]

\[ \frac{NC_i^P}{q_i} \]

\[ \text{Setup Cost} \]

\[ \frac{NC_i^S}{T_i^1} \]

\[ \text{Holding Cost} \]

\[ \frac{1}{2} \left( T_i^1 + T_i^2 \right) \]

\[ \text{Shortage Cost} \]

\[ \frac{NC_i^B b_i (T_i^3 + T_i^4)}{2} \]

\[ \sum_{i=1}^{n} C_i^P d_i + \sum_{i=1}^{n} \frac{I_i}{T_i^1} + \sum_{i=1}^{n} C_i^B \left[ \frac{I_i}{2T_i^1} (T_i^1 + T_i^2) \right] + \sum_{i=1}^{n} C_i^B b_i (T_i^3 + T_i^4) \]

\[ \text{Where;} \]

\[ T_i^1 = \frac{q_i}{P_i} - \frac{b_i}{(1 - \beta_i) P_i - d_i} \]

\[ T_i^2 = \frac{\left( 1 - \frac{d_i}{P_i} - \beta_i \frac{d_i}{P_i} \right) q_i}{d_i} - \frac{b_i}{d_i} \]

\[ T_i^3 = \frac{b_i}{d_i} \]

\[ \text{Where;} \]

\[ P_i \]

\[ b_i \]

\[ \beta_i \]

\[ q_i \]

\[ d_i \]
\[ T_{ij}^4 = \frac{b_i}{(1 - \beta_i)P_i - d_i} \]  \hspace{1cm} (28)

The capacity constraint in this case will be:

\[ \sum_{i=1}^{n} (T_{ij}^1 + T_{ij}^4) + \sum_{i=1}^{n} T_{ij}^s \leq T \]  \hspace{1cm} (29)

Hence, the final model of MP-SM EPQ without rework becomes:

\[
\begin{align*}
\text{Min} : & \quad TC(T, B) = \gamma_1 T + \frac{\gamma_2}{T} + \sum_{i=1}^{n} \gamma_3 b_i^2 T - \sum_{i=1}^{n} \gamma_4 b_i + \sum_{i=1}^{n} \gamma_5 s_i \\
\text{s.t.} : & \quad T \geq \frac{\sum_{i=1}^{n} T_{ij}^s}{1 - \sum_{i=1}^{n} \frac{d_i}{P_i}} = T_{Lower} \\
\end{align*}
\]  \hspace{1cm} (30)

Where;

\[ \gamma_1 = \sum_{i=1}^{n} C_i^h \left[ \frac{((1 - \beta_i)P_i - d_i)d_i^2}{P_i} \right] > 0 \]  \hspace{1cm} (32)

\[ \gamma_2 = \sum_{i=1}^{n} C_i^s > 0 \]  \hspace{1cm} (33)

\[ \gamma_3 = \left( \frac{C_i^h + C_i^s}{2} \right) \left[ \frac{2(1 - \beta_i)P_i}{((1 - \beta_i)P_i - d_i)d_i} \right] > 0 \]  \hspace{1cm} (34)

\[ \gamma_4 = \frac{C_i^h}{2} \left( 1 - (1 + \beta_i) \frac{d_i}{P_i} \right) + \frac{((1 - \beta_i)P_i - d_i)}{P_i} > 0 \]  \hspace{1cm} (35)

\[ \gamma_5 = C_i^p d_i + \frac{C_i^h}{2} \left( 1 - (1 + \beta_i) \frac{d_i}{P_i} \right) \frac{((1 - \beta_i)P_i - d_i)}{P_i^2} > 0 \]  \hspace{1cm} (36)

4 A solution method

The methods for solving both models of sections 3.1 and 3.2 are given in sections 4.1 and 4.2, respectively.
4.1 A solution method for MP-SM with rework

In order to derive an optimal solution for the final model, a proof of the convexity of the objective function is first provided. Then, a classical optimization technique using partial derivatives is utilized to derive the optimal solution.

**Theorem (1):** The objective function \( TC(T, b) \) in (16) is convex.

**Proof:** Using Equation (37), since the Hessian matrix, obtained in Appendix (1), results in positive values for all nonzero \( b_i \) and \( T \), \( TC(T, b) = Z \) is convex.

\[
[T, b_1, b_2, \cdots, b_n] \times H \times b \geq 2 \alpha_i T^2 + \frac{2\alpha_2}{T} - 2 \sum_{i=1}^{n} \alpha_{5i} b_i T > 0
\]  \( (37) \)

To find the optimal production period length \( T \) and the optimal backorder quantities \( b_i \)'s, partial differentiations of \( Z \) with respect to \( T \) and \( b_i \) are obtained in Equations (38) and (39).

\[
\frac{\partial Z}{\partial T} = 2\alpha_i T^3 + \frac{\alpha_i}{T^2} \sum_{i=1}^{n} \alpha_{4i} b_i^2 - \frac{\alpha_{5i}}{T} = 0 \quad (38)
\]

\[
\Rightarrow 2\alpha_i T^3 - \frac{\alpha_{5i}}{T} = \left(\alpha_2 - \sum_{i=1}^{n} \alpha_{6i} b_i^2\right) T^2 = -\alpha_{5i} + \sum_{i=1}^{n} \alpha_{4i} b_i^2 = 0
\]

\[
\frac{\partial Z}{\partial b_i} = \frac{2\alpha_i b_i}{T} - \alpha_{5i} - \alpha_{6i} T = 0 \quad \Rightarrow 2\alpha_i b_i - \alpha_{5i} T - \alpha_{6i} T^2 = 0 \quad \Rightarrow b_i = \frac{\alpha_{5i} T + \alpha_{6i} T^2}{2\alpha_{4i}} \quad (39)
\]

Then the following solution procedure is used to solve and ensure the feasibility of the problem:
The solution procedure:

**Step1:** Check for feasibility:

If \((1 - \beta_i) P_i - d_i \geq 0\) and \(\sum_{i=1}^{n} (1 + \beta_i) \frac{d_i}{P_i} < 1\), go to step 2. Otherwise, the problem is infeasible.

**Step2:** Find a solution point:

Find the optimal solution using equations (38) and (39) and by an iterative approach. To do this, start with \(b_i = 0\) and insert \(b_i = 0\) into Equation (25). Then, the new real positive values of \(T\) are obtained. This iterative search will continue until the difference between two consecutive values of \(T\) is smaller than a given \(\Delta\).

**Step3:** Check the constraint:

Check the constraint based on the obtained value of \(T\). If \(T > T_{Lower}\) then \(T^* = T\). Otherwise, \(T^* = T_{Lower}\) and go to step 4.

**Step4:** Obtain the optimal solution:

Based on the obtained value of \(T^*\) and using \(q_j^* = d_j T^*\), \(b_j^*\) will be derived by Equation (39). Then calculate the objective function value and go to step 5.

**Step5:** Terminate the procedure.

4.2 A solution method for MP-SM without rework

Similar to the previous case, the objective function of the MP-SM without rework model is convex too. To find the optimal production period length \(T\) and the optimal backorder quantities \(b_j\)'s, partial differentiations of \(Z\) with respect to \(T\) and \(b_j\) are obtained as are given in Equations (40) and (41).
\[ \frac{\partial Z}{\partial T} = \gamma T T - \frac{\gamma_i b_i^2}{T^2} = \frac{\gamma_i + \sum_{i=1}^{n} b_i^2}{\gamma_i} = 0 \rightarrow T^2 = \frac{\gamma_i + \sum_{i=1}^{n} b_i^2}{\gamma_i} \tag{40} \]

\[ \frac{\partial Z}{\partial b_i} = \frac{2 \gamma_i b_i}{T} - \gamma_i = 0 \rightarrow b_i = \frac{\gamma_i T}{2 \gamma_i} \tag{41} \]

By substituting equation (41) in equation (40), we have:

\[ T = \sqrt{\frac{\gamma_i}{\gamma_i - \sum_{i=1}^{n} \frac{(\gamma_i)^2}{4 \gamma_i}}} \tag{42} \]

\[ b_i = \frac{\gamma_i}{2 \gamma_i} T \tag{43} \]

In this case, the following solution procedure is used to solve and ensure the feasibility of the problem:

**The solution procedure:**

**Step1:** Check for feasibility:

If \((1 - \beta_i) P_i - d_i \geq 0\) and \(\sum_{i=1}^{n} d_i P_i < 1\) and \(\gamma_i > \sum_{i=1}^{n} \frac{(\gamma_i)^2}{4 \gamma_i}\), go to step 2. Otherwise, the problem is infeasible.

**Step2:** Find a solution point:

Using equations (42) and (43) find the optimal solution.

**Step3:** Check the constraint:

Check the constraint based on the obtained value of \(T\). If \(T > T_{\text{Lower}}\) then \(T^* = T\). Otherwise, \(T^* = T_{\text{Lower}}\) and go to step 4.

**Step4:** Obtain the optimal solution:
Based on the obtained value of $T^*$ and using $q_i^* = d_i T^*$, $b_i^*$ will be obtained by Equation (43).

Then calculate the objective function value and go to step 5.

**Step5**: Terminate the procedure.

**5 Numerical Examples**

Consider two multiproduct EPQ problems consisting of ten products with breakdown and capacity constraint. In these examples the demand, production, and proportion of defective rates of each product are considered constant. Furthermore, the production rate of non-defective items is considered constant and is greater than the demand rate for each product. For each example, both the immediate rework and no rework situations are considered. Moreover, there are no scrapped or defective items during the rework process. The production and rework are accomplished using the same resource at the same speed and shortages are allowed as backorders.

The general and the specific data of these examples are given in Tables (1) and (2), respectively. Tables (3) and (4) show the best results for the two examples considering immediate rework using the first solution procedure. It should be noted that a value of $\Delta = 10^{-12}$ is assumed in the solution procedure.

Using the second solution procedure, Tables (5) and (6) show the best results for the two examples when rework is not performed.
Table 1  General Data for Example #1

<table>
<thead>
<tr>
<th>Product</th>
<th>$d_i$</th>
<th>$P_i$</th>
<th>$T_i^*$</th>
<th>$C_i^*$</th>
<th>$C_i^P$</th>
<th>$C_i^R$</th>
<th>$C_i^b$</th>
<th>$C_i^b$</th>
<th>$\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>5000</td>
<td>0.001</td>
<td>700</td>
<td>24</td>
<td>24</td>
<td>10</td>
<td>20</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>150</td>
<td>5500</td>
<td>0.002</td>
<td>650</td>
<td>22</td>
<td>22</td>
<td>9</td>
<td>18</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
<td>6000</td>
<td>0.003</td>
<td>600</td>
<td>20</td>
<td>20</td>
<td>8</td>
<td>16</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>250</td>
<td>6500</td>
<td>0.004</td>
<td>550</td>
<td>18</td>
<td>18</td>
<td>7</td>
<td>14</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>300</td>
<td>7000</td>
<td>0.005</td>
<td>500</td>
<td>16</td>
<td>16</td>
<td>6</td>
<td>12</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>350</td>
<td>7500</td>
<td>0.006</td>
<td>450</td>
<td>14</td>
<td>14</td>
<td>5</td>
<td>10</td>
<td>0.3</td>
</tr>
<tr>
<td>7</td>
<td>400</td>
<td>8000</td>
<td>0.007</td>
<td>400</td>
<td>12</td>
<td>12</td>
<td>4</td>
<td>8</td>
<td>0.35</td>
</tr>
<tr>
<td>8</td>
<td>450</td>
<td>8500</td>
<td>0.008</td>
<td>350</td>
<td>10</td>
<td>10</td>
<td>3</td>
<td>6</td>
<td>0.4</td>
</tr>
<tr>
<td>9</td>
<td>500</td>
<td>9000</td>
<td>0.009</td>
<td>300</td>
<td>8</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>0.45</td>
</tr>
<tr>
<td>10</td>
<td>550</td>
<td>9500</td>
<td>0.01</td>
<td>250</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 2  General Data for Example #2

<table>
<thead>
<tr>
<th>Product</th>
<th>$d_i$</th>
<th>$P_i$</th>
<th>$T_i^*$</th>
<th>$C_i^*$</th>
<th>$C_i^P$</th>
<th>$C_i^R$</th>
<th>$C_i^b$</th>
<th>$C_i^b$</th>
<th>$\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>5000</td>
<td>0.001</td>
<td>700</td>
<td>24</td>
<td>24</td>
<td>10</td>
<td>20</td>
<td>0.05</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>5500</td>
<td>0.002</td>
<td>650</td>
<td>22</td>
<td>22</td>
<td>9</td>
<td>18</td>
<td>0.1</td>
</tr>
<tr>
<td>3</td>
<td>300</td>
<td>6000</td>
<td>0.003</td>
<td>600</td>
<td>20</td>
<td>20</td>
<td>8</td>
<td>16</td>
<td>0.15</td>
</tr>
<tr>
<td>4</td>
<td>400</td>
<td>6500</td>
<td>0.004</td>
<td>550</td>
<td>18</td>
<td>18</td>
<td>7</td>
<td>14</td>
<td>0.2</td>
</tr>
<tr>
<td>5</td>
<td>500</td>
<td>7000</td>
<td>0.005</td>
<td>500</td>
<td>16</td>
<td>16</td>
<td>6</td>
<td>12</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>600</td>
<td>7500</td>
<td>0.006</td>
<td>450</td>
<td>14</td>
<td>14</td>
<td>5</td>
<td>10</td>
<td>0.3</td>
</tr>
<tr>
<td>7</td>
<td>700</td>
<td>8000</td>
<td>0.007</td>
<td>400</td>
<td>12</td>
<td>12</td>
<td>4</td>
<td>8</td>
<td>0.35</td>
</tr>
<tr>
<td>8</td>
<td>800</td>
<td>8500</td>
<td>0.008</td>
<td>350</td>
<td>10</td>
<td>10</td>
<td>3</td>
<td>6</td>
<td>0.4</td>
</tr>
<tr>
<td>9</td>
<td>900</td>
<td>9000</td>
<td>0.009</td>
<td>300</td>
<td>8</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>0.45</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
<td>9500</td>
<td>0.01</td>
<td>250</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td>2</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 3  The Best Results for Example #1 with Immediate Rework

<table>
<thead>
<tr>
<th>Product</th>
<th>$T_{Lower}$</th>
<th>$T$</th>
<th>$T^*$</th>
<th>$q_i$</th>
<th>$b_i$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1246</td>
<td>0.7794</td>
<td>0.7794</td>
<td>77.9442</td>
<td>12.9534</td>
<td>63,610</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>116.9163</td>
<td>19.4380</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>155.8884</td>
<td>25.9523</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>194.8605</td>
<td>32.5130</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>233.8362</td>
<td>39.1414</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td>272.8047</td>
<td>45.7818</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td>311.7768</td>
<td>52.7732</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>350.7489</td>
<td>60.0117</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td>389.7210</td>
<td>68.1049</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td>428.6931</td>
<td>79.7102</td>
<td></td>
</tr>
</tbody>
</table>
### Table 4  The Best Results for Example #2 with Immediate Rework

<table>
<thead>
<tr>
<th>Product</th>
<th>$T_{Lower}$</th>
<th>$T$</th>
<th>$T^*$</th>
<th>$q_i$</th>
<th>$b_i$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.8931</td>
<td>0.6014</td>
<td>0.8931</td>
<td>89.3094</td>
<td>9.9875</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>178.6178</td>
<td>19.9476</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>267.9281</td>
<td>29.9332</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>357.2374</td>
<td>39.9866</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>446.5468</td>
<td>50.1512</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td>535.8562</td>
<td>60.4890</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td>625.1655</td>
<td>71.1199</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>714.4749</td>
<td>82.3392</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td>803.7843</td>
<td>95.09655</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td>893.936</td>
<td>114.3833</td>
<td></td>
</tr>
</tbody>
</table>

### Table 5  The Best Results for Example #1 without Rework

<table>
<thead>
<tr>
<th>Product</th>
<th>$T_{Lower}$</th>
<th>$T$</th>
<th>$T^*$</th>
<th>$q_i$</th>
<th>$b_i$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1294</td>
<td>7.2282</td>
<td>0.1294</td>
<td>12.9417</td>
<td>0.7356</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td>19.4125</td>
<td>1.0236</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td>25.8834</td>
<td>1.2616</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>32.3542</td>
<td>1.4511</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td>38.8251</td>
<td>1.5934</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td>45.2959</td>
<td>1.6893</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td>51.7668</td>
<td>1.7397</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td>58.2376</td>
<td>1.7452</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td>64.7085</td>
<td>1.7501</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td>71.1793</td>
<td>1.7589</td>
<td></td>
</tr>
</tbody>
</table>
A comparison study based on the results given in Tables (3)-(6) shows that lower optimum total costs are obtained for both examples in which reworking is allowed. Furthermore, the optimum cycle length obtained for the systems where rework is permitted is greater than those of the non-reworing systems.

### 6 A Sensitivity Analysis

The sensitivity analysis of this section is performed only for two previously mentioned examples of the immediate rework case. To study the effects of the parameter changes on the optimal result derived by the proposed method, the sensitivity analysis involves increasing or decreasing the parameters one at a time, by 20% and 40%. Tables (7) and (8) show the results of the sensitivity analysis for the first and second numerical example, respectively.

For the first numerical example, based on the results in Table (7), \( T^* = T = 0.7794 \), \( T_{lower} \) does not have any effect on the optimal value, and the capacity constraint is satisfied.
(T > T_{Lower}). Hence, when P_i increases by 20 and 40 percents, the changes in the values of T* and T are identical. This result is valid for a 20 percents decrease in the values of P_i as well. However, when P_i is decreased by 40 percents, T decreases from 0.7794 to 0.7540 and T_{Lower} increases from 0.1246 0.7998. Therefore, T < T_{Lower} and the optimal value of the cycle length becomes T^* = T_{Lower} = 0.7998. In this case, changes in the value of T* are considered with the situations of the three previous states (See Table 7). Since T* and T are the same, changing the values of C_i has the same effect on T* and T. Moreover, because C_i has not been used to calculate T_{Lower}, the changes in the values of C_i does not have any effect on T_{Lower}. By changing the values of \beta_i, the values of Z, T_{Lower}, and T are changed. Since the optimal value of the cycle length is equal to T, when \beta_i changes, the percentage of changes in the values of T* and T are identical. To study the effects of changes in T_i^*, because T_i^* is only used to calculate the values of T_{Lower}, when T_i^* is changed, the optimal value does not change. Since an increase of even 40 percents in the value of T_i^* does not cause T_{Lower} to become greater than T, T* and Z remain unchanged.

In the second example, based on the results in Table (8), for a 40% increase on P_i, T_{Lower} = 0.1668 and T = 0.6141 which results in T^* = 0.6141. Therefore, T increases while T* decreases. Despite the fact that in the second example T < T_{Lower} which made T_{Lower} and T* interchangeable (see Table 8), a 40% increase in P_i results in T > T_{Lower} and then T and T* will be interchangeable. Such a result is obtained in a 20% increase in P_i as well. In cases where P_i decreases 20% or 40%, since \sum_{i=1}^{n} (1 + \beta_i) \frac{d_i}{P_i} becomes greater than one, the problem becomes
infeasible. Because changes in the values of $c_i^r$ do not have any effect on $T_{Lower}$ and since in this problem $T^* = T_{Lower}$, no changes occur in the values of $T^*$. For a 40% increase in $\beta$, however when $\sum_{i=1}^{n} d_i d_p (1 + \beta_i) / p_i$ becomes greater than one, the problem becomes infeasible. Nevertheless, with a 20% increase in $\beta$, $T^* = T_{Lower}$, both $T^*$ and $T_{Lower}$ will increase 306.69%. By a 40% decrease on $\beta$, $T_{Lower}$ decreases significantly (to 0.3561), while $T$ lowers to 0.5933. In this case, due to $T > T_{Lower}$, $T^*$ is equal to $T$ and there are no significant changes in the values of $T^*$. In cases where $T_i^*$ increases by 20% or 40%, because $T_i^*$ directly affects $T_{Lower}$ and $T^* = T_{Lower}$, changes in the values of $T^*$ and $T_{Lower}$ are identical. However, since $T_i^*$ is not used to calculate $T$, changes in the values of $T$ are not proportional to changes in $T_i^*$.

Table 7  Effects of Parameter Changes in Example #1

<table>
<thead>
<tr>
<th>% Changes in Parameters</th>
<th>% Changes in $T_{Lower}$</th>
<th>% Changes in $T$</th>
<th>% Changes in $T^*$</th>
<th>% Changes in $Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>-26.54</td>
<td>+1.67</td>
<td>+1.67</td>
<td>-0.25</td>
</tr>
<tr>
<td>+20</td>
<td>-17.4</td>
<td>+0.95</td>
<td>+0.95</td>
<td>-0.14</td>
</tr>
<tr>
<td>-20</td>
<td>+46.37</td>
<td>-1.31</td>
<td>-1.31</td>
<td>+0.2</td>
</tr>
<tr>
<td>-40</td>
<td>+541.91</td>
<td>-3.25</td>
<td>+2.62</td>
<td>+0.51</td>
</tr>
<tr>
<td>$c_i^r$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>0</td>
<td>+17.68</td>
<td>+17.68</td>
<td>+3.39</td>
</tr>
<tr>
<td>+20</td>
<td>0</td>
<td>+9.23</td>
<td>+9.23</td>
<td>+1.76</td>
</tr>
<tr>
<td>-20</td>
<td>0</td>
<td>-10.26</td>
<td>-10.26</td>
<td>-1.95</td>
</tr>
<tr>
<td>-40</td>
<td>0</td>
<td>-21.99</td>
<td>-21.99</td>
<td>-4.15</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>+13.84</td>
<td>+0.84</td>
<td>+0.84</td>
<td>+7.09</td>
</tr>
<tr>
<td>+20</td>
<td>+6.49</td>
<td>+0.41</td>
<td>+0.41</td>
<td>+3.54</td>
</tr>
<tr>
<td>-20</td>
<td>-5.68</td>
<td>-0.37</td>
<td>-0.37</td>
<td>-3.55</td>
</tr>
<tr>
<td>-40</td>
<td>-10.78</td>
<td>-0.72</td>
<td>-0.72</td>
<td>-7.09</td>
</tr>
<tr>
<td>$T_i^r$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>+40</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>+20</td>
<td>+20</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-20</td>
<td>-20</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-40</td>
<td>-40</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
A 20% decrease in $T_i^*$ has exactly the same effect as a 20% increase. Furthermore, when $T_i^*$ decreases 40%, $T_{\text{Lower}}$ becomes 0.5359 and $T$ remains untouched ($T = 0.6014$). Hence, $T$ becomes greater than $T_{\text{Lower}}$, and $T^*$ decreases from 0.8931 to 0.6014.

7 Managerial Insight

In the first example, according to the results in Table (7), while $Z$ and $T$ are slightly-sensitive to the changes in the values of $P_i$, $T_{\text{Lower}}$ is sensitive to them. Since $T^* = T$, $T^*$ is slightly-sensitive to the changes in the values of $P_i$. Generally, $Z$ and $T$ are slightly-sensitive and sensitive to the changes in the values of $C_i^*$, respectively, $T_{\text{Lower}}$ is insensitive to the changes in values of $C_i^*$, $T^*$ and $T$ are slightly-sensitive, and $T^*$ is sensitive to the changes in values of $\beta_i$.

Table 8  Effects of Parameter Changes in Example #2

<table>
<thead>
<tr>
<th>% Changes in Parameters</th>
<th>% Changes in $T_{\text{Lower}}$</th>
<th>$T$</th>
<th>$T^*$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>-81.32</td>
<td>+2.11</td>
<td>-31.24</td>
<td>-1.38</td>
</tr>
<tr>
<td>+20</td>
<td>-71.75</td>
<td>+1.19</td>
<td>-31.87</td>
<td>-1.27</td>
</tr>
<tr>
<td>-20</td>
<td>Infeasible</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>-40</td>
<td>Infeasible</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$C_i^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>0</td>
<td>+17.57</td>
<td>0</td>
<td>+2.03</td>
</tr>
<tr>
<td>+20</td>
<td>0</td>
<td>+9.18</td>
<td>0</td>
<td>+1</td>
</tr>
<tr>
<td>-20</td>
<td>0</td>
<td>-19.34</td>
<td>0</td>
<td>-1.56</td>
</tr>
<tr>
<td>-40</td>
<td>0</td>
<td>-21.89</td>
<td>0</td>
<td>-1.98</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>Infeasible</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>+20</td>
<td>+306.69</td>
<td>+0.74</td>
<td>+306.69</td>
<td>+41.91</td>
</tr>
<tr>
<td>-20</td>
<td>-42.99</td>
<td>-0.69</td>
<td>-33.12</td>
<td>-4.88</td>
</tr>
<tr>
<td>-40</td>
<td>-60.13</td>
<td>-1.34</td>
<td>-33.56</td>
<td>-8.65</td>
</tr>
<tr>
<td>$T_i^*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+40</td>
<td>+40</td>
<td>0</td>
<td>+40</td>
<td>+3.29</td>
</tr>
<tr>
<td>+20</td>
<td>+20</td>
<td>0</td>
<td>+20</td>
<td>+1.48</td>
</tr>
<tr>
<td>-20</td>
<td>-20</td>
<td>0</td>
<td>-20</td>
<td>-0.97</td>
</tr>
<tr>
<td>-40</td>
<td>-40</td>
<td>0</td>
<td>-32.66</td>
<td>-1.12</td>
</tr>
</tbody>
</table>
Furthermore, the results in Table (7) show that $T_{Lower}$ is very sensitive to the changes in the values of $T_i^s$.

In the second example, the results in Table (8) show that while $Z$ and $T$ are slightly-sensitive to the changes in the values of $P_i$, $T^*$ and $T_{Lower}$ are sensitive and very sensitive to the changes in the values of $P_i$, respectively. Because $T^* = T_{Lower}$, while both $T^*$ and $T_{Lower}$ are insensitive to $C_i^s$, $Z$ and $T$ are slightly-sensitive and sensitive to the changes in values of $C_i^s$, respectively. In general, while $Z$ and $T_{Lower}$ are very sensitive, $T$ is slightly-sensitive to the changes in values of $\beta_i$; furthermore, since $T^* = T_{Lower}$ in the second example, $T^*$ is also very sensitive to the changes in values of $\beta_i$.

8 Conclusion

In this research, two EPQ models with production capacity limitation and backordered quantities were developed under the assumptions of immediate rework and no-rework situations. The primary aim of this research has been to determine the optimal period lengths, backorder, and production quantities. The objective functions of the proposed mathematical models were proved convex and two solution procedures were proposed. Then, two numerical examples with ten products were used to illustrate the implementation of the proposed methods. The results obtained by solving the two models were next compared and finally, sensitivity analyses have been performed to demonstrate the applicability of the proposed methodology and to provide some managerial insights for practitioners. Future researches may focus on the partial backordering strategies and multiproduct multi-constraint problems in an uncertain environment.
Furthermore, an unequal cycle length consideration for all products may reduce the total cost. This may be investigated in a future research.

9 Acknowledgement

The authors are thankful for the constructive comments of the anonymous reviewers that significantly improved the presentation of the paper.

References


Appendix 1: The Proof of the Convexity

Consider the objective function as

\[
\min Z = C_1 T^2 + C_3 T + \sum_{i=1}^{n} C_{4i} b_i^2 T - \sum_{i=1}^{n} C_{5i} b_i - \sum_{i=1}^{n} C_{6i} b_i T
\]

Differentiating the objective function with respects to \( T \) and \( b_i \) and setting the differentials equal to zero results in

\[
\frac{\partial Z}{\partial T} = 2C_1 T + C_3 - \frac{\sum_{i=1}^{n} C_{4i} b_i^2 T}{T} - \sum_{i=1}^{n} C_{6i} b_i = 0
\]

\[
\frac{\partial Z}{\partial b_i} = 2C_{4i} b_i T - C_{5i} - C_{6i} T = 0
\]

Then, the Hessian matrix \( H \) becomes
\[
H = \begin{bmatrix}
\frac{\partial^2 Z}{\partial T ^2} & \frac{\partial^2 Z}{\partial T \partial b_1} & \frac{\partial^2 Z}{\partial T \partial b_2} & \cdots & \frac{\partial^2 Z}{\partial T \partial b_n} \\
\frac{\partial^2 Z}{\partial b_1 \partial T} & \frac{\partial^2 Z}{\partial b_1 \partial b_1} & \frac{\partial^2 Z}{\partial b_1 \partial b_2} & \cdots & \frac{\partial^2 Z}{\partial b_1 \partial b_n} \\
\frac{\partial^2 Z}{\partial b_2 \partial b_1} & \frac{\partial^2 Z}{\partial b_2 \partial b_1} & \frac{\partial^2 Z}{\partial b_2 \partial b_2} & \cdots & \frac{\partial^2 Z}{\partial b_2 \partial b_n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 Z}{\partial b_n \partial T} & \frac{\partial^2 Z}{\partial b_n \partial b_1} & \frac{\partial^2 Z}{\partial b_n \partial b_2} & \cdots & \frac{\partial^2 Z}{\partial b_n \partial b_n}
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
2C_1 + \frac{2C_3 b_1^2}{T^2} + \frac{2 \sum_{i=1}^{n} C_{ai} b_i^2}{T^2} - \frac{2C_{ai} b_1}{T^2} - C_{a1} & - \frac{2C_{ai} b_2}{T^2} - C_{a2} & \cdots & - \frac{2C_{ai} b_n}{T^2} - C_{an} \\
- \frac{2C_{ai} b_1}{T^2} - C_{a1} & \frac{2C_{ai}}{T} & 0 & \cdots & 0 \\
- \frac{2C_{ai} b_2}{T^2} - C_{a2} & 0 & \frac{2C_{ai}}{T} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- \frac{2C_{ai} b_n}{T^2} - C_{an} & 0 & 0 & \cdots & \frac{2C_{ai}}{T}
\end{bmatrix}
\]

Hence,

\[
[T, b_1, b_2, \cdots, b_n] \times H \times [T, b_1, b_2, \cdots, b_n] = [2C_1 T + \frac{2C_3}{T^2} - \sum_{i=1}^{n} C_{ai} b_i T, - C_{a1} T, - C_{a2} T, \cdots, - C_{an} T] \times \begin{bmatrix} T \\ b_1 \\ \vdots \\ b_n \end{bmatrix}
\]

\[
= 2C_1 T^2 + \frac{2C_3}{T} - 2 \sum_{i=1}^{n} C_{ai} b_i T
\]

Substituting \( b_1 = \frac{C_{ai} T + C_{ai} T^2}{2C_{ai}} \) we will have

\[
[T, b_1, b_2, \cdots, b_n] \times H \times [T, b_1, b_2, \cdots, b_n] = 2 \left( C_1 T^2 + \frac{C_3}{T} - \sum_{i=1}^{n} C_{ai} \frac{C_{ai} T^2 + C_{ai} T^3}{2C_{ai}} \right)
\]

We need to prove that \( C_1 T^2 + \frac{C_3}{T} > \sum_{i=1}^{n} C_{ai} \frac{C_{ai} T^2 + C_{ai} T^3}{2C_{ai}} \). Depending on the value of \( T \) two cases may happen.
First case: $0 < T < 1$

In this case since $\sum_{i=1}^{n} \frac{C_{n}T_{i}^2 + C_{n}T_{i}^3}{2C_{n}}$ is greater than $\sum_{i=1}^{n} \frac{C_{n}T_{i}^2 + C_{n}T_{i}^3}{2C_{n}}$, we need to prove $C_{i}T_{i}^2 + \frac{C_{i}}{T_{i}} > \sum_{i=1}^{n} \frac{C_{n}T_{i}^2 + C_{n}T_{i}^3}{2C_{n}}$. To do this, we will show that $C_{i} \geq \sum_{i=1}^{n} \frac{C_{n} + C_{n}}{2C_{n}}$.

Substituting for $C_{i}, C_{n}, C_{n}, C_{n}$ in the right hand sight of the current inequality results in:

$$\sum_{i=1}^{n} \left[ (1 - \beta_{i}^2) P_{i}^2 - P_{i}d_{i} + \beta_{i} P_{i}^2 - \beta_{i} P_{i}d_{i} \right] > \sum_{i=1}^{n} \left[ (1 - \beta_{i}^2) P_{i}^2 - (1 - \beta_{i}^2) \beta_{i} P_{i}d_{i} - (1 + \beta_{i}) P_{i} - (1 + \beta_{i}) \beta_{i} d_{i} + \frac{C_{i}^2}{(1 - \beta_{i}) P_{i} - d_{i} + C_{i}^2 (1 - \beta_{i}) P_{i}} \right]$$

$$\Rightarrow \sum_{i=1}^{n} \left[ (1 - \beta_{i}^2) P_{i}^2 - (1 - \beta_{i}^2) \beta_{i} P_{i}d_{i} - (1 + \beta_{i}) P_{i}d_{i} + (1 + \beta_{i}) \beta_{i} d_{i} - \beta_{i} d_{i} - \beta_{i} d_{i} - \beta_{i} d_{i} - \beta_{i} d_{i} \right]$$

$$\Rightarrow \sum_{i=1}^{n} \left[ (1 - \beta_{i}^2) P_{i}^2 - (1 - \beta_{i}^2) \beta_{i} P_{i}d_{i} - (1 + \beta_{i}) P_{i}d_{i} - (1 + \beta_{i}) \beta_{i} d_{i} - \beta_{i} d_{i} \right]$$

$$\Rightarrow \sum_{i=1}^{n} \left[ (1 - \beta_{i}^2) P_{i}^2 - (1 - \beta_{i}^2) \beta_{i} P_{i}d_{i} - (1 + \beta_{i}) P_{i}d_{i} - (1 + \beta_{i}) \beta_{i} d_{i} - \beta_{i} d_{i} \right]$$

It is clear that

$$\sum_{i=1}^{n} \left[ (1 - \beta_{i}^2) P_{i}^2 - (1 - \beta_{i}^2) \beta_{i} P_{i}d_{i} - (1 + \beta_{i}) P_{i}d_{i} - (1 + \beta_{i}) \beta_{i} d_{i} - \beta_{i} d_{i} \right]$$

In the denominator of the last term of the current inequality, $(1 - \beta_{i}) P_{i}$ being the production rate during $T_{i}$ is greater than $d_{i}$ i.e., $(1 - \beta_{i}) P_{i} - d_{i} > 0$ and that $C_{i}^2 (1 - \beta_{i}) P_{i}$ is a relatively big

30
positive. Hence,

\[ \sum_{i=1}^{n} (\beta_i P_i^2 - \beta_i P_i d_i) > \sum_{i=1}^{n} (-\beta_i^3 P_i d_i - \beta_i^2 d_i^2) \]

Therefore, we need to show that

\[ \sum_{i=1}^{n} (P_i^2 - P_i d_i) > \sum_{i=1}^{n} (-\beta_i^2 P_i d_i - \beta d_i^2) \]

The right hand side of the above inequality is negative and the left hand side is positive (because 

\[ ((1-\beta_i P_i - d_i) > 0 \rightarrow P_i > d_i \rightarrow P_i^2 > P_i d_i) \]. Hence, the theorem is proved.

**Second Case:** \( T \geq 1 \)

In this case since \( \sum_{j=1}^{s} C_a \frac{C_a T^3 + C_a T^3}{2C_i} \) is greater than \( \sum_{j=1}^{s} C_a \frac{C_a T^3 + C_a T^3}{2C_i} \), if we show that

\[ C_i T^2 + \frac{C_i}{T} \geq \sum_{j=1}^{s} C_a \frac{C_a T^3 + C_a T^3}{2C_i} \quad \text{or} \quad C_i T^2 \geq \sum_{j=1}^{s} C_a \left( \frac{C_a + C_a}{2C_i} \right)^3 \]

the theorem holds. In other words we need to show that \( C_i \geq \sum_{j=1}^{s} C_a \frac{(C_a + C_a)}{2C_i} T \). Similar to the first case, substituting \( C_i, C_a, C_i, \) \( C_a \) and doing simplification, we have

\[ \sum_{i=1}^{n} (P_i^2 - \beta_i^2 P_i^2 - P_i d_i + \beta_i P_i^2 - \beta_i P_i d_i) \]

\[ \frac{\sum_{i=1}^{n} (P_i^2 - \beta_i^2 P_i^2 - \beta_i^2 P_i d_i - \beta_i d_i^2 - P_i d_i - \beta_i^2 d_i^2)}{C_i ((1-\beta_i P_i - d_i)) + C_i (1-\beta_i) P_i^3} T \]

Since \( \frac{T}{C_i ((1-\beta_i P_i - d_i)) + C_i (1-\beta_i) P_i^3} \leq 1 \), (The elements of the denominator \( ((1-\beta_i P_i - d_i) > 0 \),

\( C_i (1-\beta_i) P_i^3 \) is a relatively big positive, and hence \( T \) cannot be greater than the denominator) therefore,
\[ \sum_{i=1}^{n} \left( P_i^2 - \beta_i^2 P_i^2 - P_i d_i + \beta_i P_i^2 - \beta_i P d_i \right) > \sum_{i=1}^{n} \left( P_i^2 - \beta_i^2 P_i^2 - \beta_i^3 P d_i - \beta_i d_i^2 - P d_i - \beta_i^3 d_i^2 \right) \]
\[ > \frac{\sum_{i=1}^{n} \left( P_i^2 - \beta_i^2 P_i^2 - \beta_i^3 P d_i - \beta_i d_i^2 - P d_i - \beta_i^3 d_i^2 \right)}{C_i \left( (1 - \beta_i) P_i - d_i \right) + C_i \left( 1 - \beta_i \right) P_i} \]

Thus, similar to the first case, we need to show that
\[ \sum_{i=1}^{n} \left( P_i^2 - \beta_i^3 P_i^2 - P_i d_i + \beta_i P_i^2 - \beta_i P d_i \right) > \sum_{i=1}^{n} \left( P_i^2 - \beta_i^3 P_i^2 - \beta_i^3 P d_i - \beta_i d_i^2 - P d_i - \beta_i^3 d_i^2 \right) \]

Alternatively, based on what we showed in the first case, we need to show that
\[ \sum_{i=1}^{n} \left( P_i^2 - P d_i \right) \sum_{i=1}^{n} \left( - \beta_i^2 P_i d_i - d_i^2 - \beta_i d_i^2 \right) \]

Which has been shown to hold.