Classification of special Anosov endomorphisms of infra-nil-manifolds

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Abstract. In this paper we give a classification of special endomorphisms of infra-nil-manifolds : Let $f : N/\Gamma \to N/\Gamma$ be a covering map of an infra-nil-manifold and denote by $A : N/\Gamma \to N/\Gamma$ the infra-nil-endomorphism which is homotopic to f. If f is a special TA-map, then A is a hyperbolic infra-nil-endomorphism and f is topologically conjugate to A.

1. Introduction

Finding a universal model for Anosov diffeomorphisms has been an important problem in dynamical systems. In this general context, Franks and Manning proved that every Anosov diffeomorphism of an infra-nil-manifold is topologically conjugate to a hyperbolic infra-nil-automorphism [**4, 5, 9, 10**]. Based on this result, Aoki and Hiraide has been studied the dynamics of covering maps of a torus [**2**]. The importance of infra-nil-manifolds comes from the following Conjecture 1.1 and Theorem 1.2 :

The first non-toral example of an Anosov diffeomorphism was constructed by S. Smale in [**13**]. He conjectured that, up to topologically conjugacy, the construction in Smale's example gives every possible Anosov diffeomorphism on a closed manifold.

Conjecture 1.1. *Every Anosov diffeomorphism of a closed manifold is topologically conjugate to a hyperbolic affine infra-nil-automorphism.*

Theorem 1.2 (Gromov [**6**]). *Every expanding map on a closed manifold is topologically conjugate to an expanding affine infra-nil-endomorphism.*

The conjecture has been open for many years (see [**3**] page 48). An interesting problem is to consider the conjecture for endomorphisms of a closed manifold. Our main theorem is a partial answer to the conjecture.

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In this paper we give a classification of special endomorphisms of infra-nil-manifolds. Infact, Aoki and Hiraide [**2**] in 1994 proposed two problems:

PROBLEM 1.3. Is every special Anosov differentiable map of a torus topologically conjugate to a hyperbolic toral endomorphism?

PROBLEM 1.4. Is every special topological Anosov covering map of an arbitrary closed topological manifold topologically conjugate to a hyperbolic infra-nil- endomorphism of an infra-nil-manifold ?

Aoki and Hiraide answered problem 1.3 partially as follows:

THEOREM 1.5 ([**2**] Theorem 6.8.1). *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a TA-covering map of an n-torus and denote by $A : \mathbb{T}^n \to \mathbb{T}^n$ the toral endomorphism homotopic to f. Then A is hyperbolic. Furthermore the inverse limit system of $(\mathbb{T}^n, f)$ is topologically conjugate to the inverse limit system of $(\mathbb{T}^n, A)$.*

THEOREM 1.6 ([**2**] Theorem 6.8.2). *Let f and A be as Theorem 1.5. Suppose f is special, then the following statements hold :*

1. *if f is a TA-homeomorphism, then A is a hyperbolic toral automorphism and f is topologically conjugate to A,*
2. *if f is a topological expanding map, then A is an expanding toral endomorphism and f is topologically conjugate to A,*
3. *if f is a strongly special TA-map, then A is a hyperbolic toral endomorphism and f is topologically conjugate to A.*

In [**14**], Sumi has altered the condition"strongly special" (part (3) of Theorem 1.6) to just "special" as follows:

THEOREM 1.7 ([**14**]). *Let f and A be as Theorem 1.5. If f is a special TA-map, then A is a hyperbolic toral endomorphism and f is topologically conjugate to A.*

In [**15**], Sumi generalized Theorem 1.5 and parts (1) and (2) of Theorem 1.6 for infra-nil-manifolds as follows:

THEOREM 1.8 ([**15**] Theorem 1). *Let $f : N/\Gamma \to N/\Gamma$ be a covering map of an infra-nil-manifold and denote as $A : N/\Gamma \to N/\Gamma$ the infra-nil-endomorphism homotopic to f. If f is a TA-map, then A is hyperbolic and the inverse limit system of $(N/\Gamma, f)$ is topologically conjugate to the inverse limit system of $(N/\Gamma, A)$.*

THEOREM 1.9 ([**15**] Theorem 2). *Let f and A be as in Theorem 1.8. Then the following statements hold:*

1. *if f is a TA-homeomorphism, then A is a hyperbolic infra-nil-automorphism and f is topologically conjugate to A,*
2. *if f is a topological expanding map, then A is an expanding infra-nil-endomorphism and f is topologically conjugate to A.*

In the paper, by using Theorem 1.7, we partially answer problem 1.4 of Aoki and Hiraide as follows:

THEOREM 1.10 (Main Theorem). *Let $f : N/\Gamma \to N/\Gamma$ be a covering map of an infra-nil-manifold and denote as $A : N/\Gamma \to N/\Gamma$ the infra-nil-endomorphism homotopic to f. If f is a special TA-map, then A is a hyperbolic infra-nil-endomorphism and f is topologically conjugate to A.*

COROLLARY 1.11. *If $f : N/\Gamma \to N/\Gamma$ is a special Anosov endomorphism of an infra-nil-manifold then it is conjugate to a hyperbolic infra-nil-endomorphism.*

2. Preliminaries

Let X and Y be compact metric spaces and let $f : X \to X$ and $g : Y \to Y$ be continuous surjections. Then f is said to be topologically conjugate to g if there exists a homeomorphism $\varphi : Y \to X$ such that $f \circ \varphi = \varphi \circ g$.

Let X be a compact metric space with metric d . For $f : X \to X$ a continuous surjection, we let

$$X_f = \{\tilde{x} = (x_i) : x_i \in X \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\},$$
$$\sigma_f((x_i)) = (f(x_i)).$$

The map $\sigma_f : X_f \to X_f$ is called the *shift map* determined by f. We call (X_f, σ_f) the *inverse limit* of (X, f). A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $e > 0$ (called an *expansive constant*) such that if x and y are any two distinct points of X then $d(f^i(x), f^i(y)) > e$ for some integer i. A continuous surjection $f : X \to X$ is called *c-expansive* if there is a constant $e > 0$ such that for $\tilde{x}, \tilde{y} \in X_f$ if $d(x_i, y_i) \le e$ for all $i \in \mathbb{Z}$ then $\tilde{x} = \tilde{y}$. In particular, if there is a constant $e > 0$ such that for $x, y \in X$ if $d(f^i(x), f^i(y)) \le e$ for all $i \in \mathbb{N}$ then $x = y$, we say that f is *positively expansive.* A sequence of points $\{x_i : a < i < b\}$ of X is called a δ-pseudo orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. Given $\epsilon > 0$ a δ-pseudo orbit of $\{x_i\}$ is called to be ϵ-traced by a point $x \in X$ if $d(f^i(x), x_i) < \epsilon$ for every $i \in (a, b-1)$. Here the symbols a and b are taken as $-\infty \le a < b \le \infty$ if f is bijective and as $-1 \le a < b \le \infty$ if f is not bijective. f has the *pseudo orbit tracing property* (abbrev. POTP) if for every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit of f can be ϵ-traced by some point of X . We say that a homeomorphism $f : X \to X$ is a topological Anosov map (abbrev. TA-map) if f is expansive and has POTP. Analogously, We say that a continuous surjection $f : X \to X$ is a topological Anosov map if f is c-expansive and has POTP, and say that f is a topological expanding map if f is positively expansive and open. We can check that every topological expanding map is a

TA-map (see [**2**] Remark 2.3.10).

A *Lie group* is a smooth manifold obeying the group properties and that satisfies the additional condition that the group operations are differentiable. Let N be a Lie group. A vector field X on N is said to be *invariant under left translations* if for each $g, h \in N$, $(dl_g)_h(X_h) = X_{gh}$, where $(dl_g)_h : T_hN \to T_{gh}N$ and $l_g : N \to N;\ x \mapsto gx$. A *Lie algebra* g is a vector space over some field F together with a binary operation $[\cdot,\cdot] : g \times g \to g$ called the *Lie bracket*, that satisfies:

(1) Bilinearity: $[ax+by, z] = a[x, z]+b[y, z],\ [z, ax+by] = a[z, x]+ b[z, y]\ \forall x, y, z \in g$,
(2) Alternativity: $[x, x] = 0\ \forall x \in g$,
(3) The Jacobi Identity: $[x, [y, z]]+[z, [x, y]]+[y, [z, x]] = 0\ \forall x, y, z \in g$

Let $Lie(N)$ be the set of all left-translation-invariant vector fields on N. It is a real vector space. Moreover, it is closed under Lie bracket. Thus $Lie(N)$ is a Lie subalgebra of the Lie algebra of all vector fields on N and is called the *Lie algebra* of N. A *nilpotent* Lie group is a Lie group which is connected and whose Lie algebra is a nilpotent Lie algebra. That is, its Lie algebra's central series eventually vanishes.

A group G is a *torsion group* if every element in G is of finite order. G is called *torsion free* if no element other than identity is of finite order. A *discrete* subgroup of a topological group G is a subgroup H such that there is an open cover of H in which every open subset contains exactly one element of H. In other words, the subspace topology of H in G is the discrete topology. A *uniform* subgroup H of G is a closed subgroup such that the quotient space G/H is compact.

Let N be a Lie group and $Aut(N)$ be the set of all automorphisms of N. Let Γ be a uniform discrete subgroup of N such that for every $\overline{A} \in Aut(N)$, $\overline{A}(\Gamma) = \Gamma$. Then N/Γ is a closed manifold and $\overline{A}$ induces a diffeomorphism $A : N/\Gamma \to N/\Gamma,\ g\Gamma \mapsto \overline{A}(g)\Gamma$ for $g \in N$. If we want this diffeomorphism to be Anosov, $\overline{A}$ must be hyperbolic, i.e. the derivative $D_{e_N}\overline{A}$ at the identity e_N of N has no eigenvalues of modulus 1. It is known that this can happen only when N is nilpotent. So, if N is nilpotent then N/Γ is called a *nil-manifold*. Such a diffeomorphism $\overline{A}$ is called a *nil-automorphism* and is said to be *hyperbolic nil-automorphism* when $\overline{A}$ is hyperbolic.

Let X be a topological space and let G be a group. We say that G *acts* (continuously) on X if to $(g, x) \in G \times X$ there corresponds a point $g \cdot x$ in X and the following conditions are satisfied:

(1) $e \cdot x = x$ for $x \in X$ where e is the identity,
(2) $g \cdot (g' \cdot x) = gg' \cdot x$ for $x \in X$ and $g, g' \in G$,
(3) for each $g \in G$ a map $x \mapsto g \cdot x$ is a homeomorphism of X.

When G acts on X, for $x, y \in X$ letting

$$x \sim y \Leftrightarrow y = g \cdot x \text{ for some } g \in G$$

an equivalence relation $\sim$ in X is defined. Then the identifying space $X/\sim$, denoted as X/G, is called the orbit space by G of X. It follows that for $x \in X$, $[x] = \{g \cdot x : g \in G\}$ is the equivalence class.

An action of G on X is said to be *properly discontinuous* if for each $x \in X$ there exists a neighborhood $U(x)$ of x such that $U(x) \cap gU(x) = \emptyset$ for all $g \in G$ with $g \neq e_G$. Here $gU(x) = \{g \cdot y : y \in U(x)\}$.

Let $\pi : Y \to X$ be a covering map. A homeomorphism $\alpha : Y \to Y$ is called a *covering transformation* for π if $\pi \circ \alpha = \pi$ holds. We denote as $G(\pi)$ the set of all covering transformations for π. It is easy to see that $G(\pi)$ is a group, which is called the *covering transformation group* for π.

Now we give an extended definition of nil-manifolds. Let N be a connected simply connected nilpotent Lie group and C be a compact subgroup of $Aut(N)$. Consider the semidirect product $G = N \rtimes C$. An element of G is of the form (x, α) where x is the *translational part* and α is the *linear part*. G by multiplication $(x_1, \alpha_1)(x_2, \alpha_2) = (x_1\alpha_1(x_2), \alpha_1 \circ \alpha_2)$ is a group which acts on N by $(x, \alpha) \cdot y = x\alpha(y)$, for $x, y \in N, \alpha \in C$. Let Γ be a uniform discrete torsion free subgroup of G such that $\Gamma \cap G$, the subgroup of all pure translations, is isomorphic to a uniform subgroup of N, which denoted again by $\Gamma \cap G$, with finite index $[\Gamma, \Gamma \cap G]$. Note that $N/\Gamma \cap G$ is a nil-manifold. Γ acts on N by $\gamma \cdot x := \gamma(x) = (y, \alpha) \cdot x = y\alpha(x)$, for all $\gamma = (y, \alpha) \in \Gamma$ where $x, y \in N, \alpha \in C$. The quotient space of N under the action of Γ, N/Γ, is called an *infra-nil-manifold.* Since Γ is discrete, it acts properly discontinuously on N (see [**3**]). When $C = \{id_{Aut(N)}\}$ then the subgroup Γ of G is isomorphic to a subgroup of N denoted again by Γ and N/Γ is a nil-manifold (see [**13**]). Let $\overline{A} : N \to N$ be an automorphism of N such that by conjugating Γ by $\overline{A}$ in $Diff(N)$, $\overline{A} \circ \Gamma \circ \overline{A}^{-1} \subset \Gamma$. Then $\overline{A}$ projects to a linear covering map A of N/Γ . The map A is called an *infra-nil-endomorphism.* If the derivative $D_e\overline{A}$ at the identity e of N has no eigenvalues of modulus 1, we say A is *hyperbolic.* If A is hyperbolic, then A is a TA-covering map (See [**2**], Theorem 1.2.1).

THEOREM 2.1 ([**2**] Theorem 6.3.4). *If $\pi : Y \to X$ is the universal covering, then for each $b \in Y$*

1. *the map $\alpha \mapsto \alpha(b)$ is a bijection from $G(\pi)$ onto $\pi^{-1}(\pi(b))$,*
2. *the map $\psi_b : G(\pi) \to \pi_1(X, \pi(b))$ by $\alpha \mapsto [\pi \circ u_{\alpha(b)}]$ is an isomorphism where $u_{\alpha(b)}$ is a path from b to $\alpha(b)$.*

Furthermore, the action of $G(\pi)$ on Y is properly discontinuous and $Y/G(\pi)$ is homeomorphic to X .

THEOREM 2.2 ([**2**] Theorem 6.3.7). *Let G be a group and X a topological space. Suppose that G acts on X and the action is properly discontinuous. Then*

(1) *the natural projection $\pi : X \to X/G$ is a covering map,*
(2) *if X is simply connected, then the fundamental group $\pi_1(X/G)$ is isomorphic to G.*

COROLLARY 2.3. *Let N/Γ be an infra-nil-manifold and $\pi : N \to N/\Gamma$ be the natural projection. Then*

$$\Gamma \cong \pi_1(N/\Gamma) \cong G(\pi).$$

PROOF. Since Γ acts on N properly discontinuous, the natural projection $\pi : N \to N/\Gamma$ is a covering map. Since N is simply connected, by Theorem 2.2 we have $\Gamma \cong \pi_1(N/\Gamma)$.

On the other hand, since N is simply connected and Γ acts on N properly discontinuous the natural projection $\pi : N \to N/\Gamma$ is the universal covering map. So by Theorem 2.1 we have $\Gamma \cong G(\pi)$. □

Let M be a closed smooth manifold and let $C^1(M, M)$ be the set of all C^1 maps of M endowed with the C^1 topology. A map $f \in C^1(M, M)$ is called an *Anosov endomorphism* if f is a C^1 regular map and if there exist $C > 0$ and $0 < \lambda < 1$ such that for every $\tilde{x} = (x_i) \in M_f = \{\tilde{x} = (x_i) : x_i \in M \text{ and } f(x_i) = x_{i+1},\ i \in \mathbb{Z}\}$ there is a splitting

$$T_{x_i}M = E^s_{x_i} \oplus E^u_{x_i}, \quad i \in \mathbb{Z}$$

(we show this by $T_{\tilde{x}}M = \bigcup_i (E^s_{x_i} \oplus E^u_{x_i})$) so that for all $i \in \mathbb{Z}$:

(1) $D_{x_i}f(E^\sigma_{x_i}) = E^\sigma_{x_{i+1}}$ where $\sigma = s, u$,
(2) for all $n \geq 0$

$$\| D_{x_i}f^n(v) \| \leq C\lambda^n \| v \| \text{ if } v \in E^s_{x_i},$$
$$\| D_{x_i}f^n(v) \| \leq C\lambda^n \| v \| \text{ if } v \in E^u_{x_i}.$$

If, in particular, $T_{\tilde{x}}M = \bigcup_i E^u_{x_i}$ for all $\tilde{x} = (x_i) \in M_f$, then f is said to be *expanding differentiable map*, and if an Anosov endomorphism f is injective then f is called an *Anosov diffeomorphism.* We can check that every Anosov endomorphism is a TA-map, and that every expanding differentiable map is a topological expanding map (see [**2**] Theorem 1.2.1).

Let X and Y be metric spaces. A continuous surjection $f : X \to Y$ is called a *covering map* if for $y \in Y$ there exists an open neighborhood V_y of y in Y such that

$$f^{-1}(V_y) = \bigcup_i U_i \quad (i \neq i' \Rightarrow U_i \cap U'_i = \emptyset)$$

where each of U_i is open in X and $f_{|U_i} : U_i \to V_y$ is a homeomorphism. A covering map $f : X \to Y$ is especially called a *self-covering map* if

$X = Y$. We say that a continuous surjection $f : X \to Y$ is a local homeomorphism if for $x \in X$ there is an open neighborhood U_x, of x in X such that $f(U_x)$ is open in Y and $f_{|U_x} : U_x \to f(U_x)$ is a homeomorphism. It is clear that every covering map is a local homeomorphism. Conversely, if X is compact, then a local homeomorphism $f : X \to Y$ is a covering map (see [**2**] Theorem 2.1.1).

A map $f \in C^1(M, M)$ is said to be C^1-*structurally stable* if there is an open neighborhood $\mathcal{N}(f)$ of f in $C^1(M, M)$ such that $g \in \mathcal{N}(f)$ implies that f and g are topologically conjugate. Anosov [**1**] proved that every Anosov diffeomorphism is C^1-structurally stable, and Shub [**12**] showed the same result for expanding differentiable maps. However, Anosov endomorphisms which are not diffeomorphisms nor expanding do not be C^1-structurally stable ([**8**],[**11**]).

A map $f \in C^1(M, M)$ is said to be C^1-*inverse limit stable* if there is an open neighborhood $\mathcal{N}(f)$ of f in $C^1(M, M)$ such that $g \in \mathcal{N}(f)$ implies that the inverse limit (M_f, σ_f) of (M, f) and the inverse limit (M_g, σ_g) of (M, g) are topologically conjugate. Mané and Pugh [**8**] proved that every Anosov endomorphism is C^1-inverse limit stable. If the manifold M is an infra-nil-manifold, then this fact is a corollary of Theorem 1.8.

We define *special TA-maps* as follows. Let $f : X \to X$ be a continuous surjection of a compact metric space. Define the *stable* and *unstable* sets

$$W^s(x) = \{y \in X : \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0\},$$
$$W^u(\tilde{x}) = \{y_0 \in X : \exists \tilde{y} = (y_i) \in X_f \text{ s.t. } \lim_{i\to\infty} d(x_{-i}, y_{-i}) = 0\}.$$

for $x \in X$ and $\tilde{x} \in X_f$. A TA-map $f : X \to X$ is special if f satisfies the property that $W^u(\tilde{x}) = W^u(\tilde{y})$ for every $\tilde{x}, \tilde{y} \in X_f$ with $x_0 = y_0$. Every hyperbolic infra-nil-endomorphism is a special TA-covering map (See [**15**] Remark 3.13). By this and Theorem 1.10 We have the following corollary:

Corollary 2.4. *A TA-covering map of an infra-nil-manifold is special if and only if it is conjugate to a hyperbolic infra-nil-endomorphism.*

3. Some properties of infra-nil-manifolds

Let X be a topological space. We write $\Omega(X; x_0)$ the family of all closed paths from x_0 to x_0. Let $\Omega(X; x_0)/\sim$ be the identifying space with respect to the equivalence relation $\sim$ by homotopty. We write this set

$$\pi_1(X; x_0) = \Omega(X; x_0)/\sim .$$

The group $\pi_1(X;x_0)$ is called the *fundamental group* at a base point x_0 of X. If, in particular, $\pi_1(X;x_0)$ is a group consisting of the identity, then X is said to be *simply connected* with respect to a base point x_0.

Let x_0 and x_1 be points in X. If there exists a path w joining x_0 and x_1, then we can define a map

$$w_\sharp : \Omega(X;x_1) \to \Omega(X;x_0), \text{ by } w_\sharp(u) = (w \cdot u) \cdot \overline{w},$$

where $u \in \Omega(X;x_1)$, $(w \cdot u)$ is the concatenation of w and u and $\overline{w}$ is w in reverse direction. For $u, v \in \Omega(X;x_1)$ suppose $u \sim v$. Then $w_\sharp(u) \sim w_\sharp(v)$ and thus $w_\sharp$ induces a map

$$w_* : \pi_1(X;x_1) \to \pi_1(X;x_0), \text{ by } w_*([u]) = [w_\sharp(u)],$$

this map is an isomorphism (see [**2**] Lemma 6.1.4).

Remark 3.1. If X is a path connected space then we can remove the base point and write $\pi_1(X;x_0) = \pi_1(X)$.

Let X and Y be topological spaces and $f : X \to Y$ a continuous map. Take $x_0 \in X$ and let $y_0 = f(x_0)$. It is clear that $fu = f \circ u \in \Omega(X,y_0)$ for $u \in \Omega(X;x_0)$. Thus we can find a map

$$f_\sharp : \Omega(X;x_0) \to \Omega(Y;Y_0), \text{ by } f_\sharp(u) = fu,$$

where $u \in \Omega(X;x_0)$. If $u \sim v(F)$ for $u, v \in \Omega(X;x_0)$, then we have $fu \sim fv(f \circ F)$, from which the following map will be induced:

$$f_* : \pi_1(X;x_0) \to \pi_1(Y;y_0), \text{ by } f_*([u]) = [f_\sharp(u)] = [fu],$$

It is easy to check that f_* is a homomorphism. We say that f_* is a *homomorphism induced from a continuous map* $f : X \to Y$.

Let $f, g : X \to Y$ be homotopic and F a homotopy from f to g. Then for $x_0 \in X$ we can define a path $w \in \Omega(Y; f(x_0), g(x_0))$ by

$$w(t) = F(x_0, t) \quad t \in [0,1],$$

and the relation between homomorphisms $f_* : \pi_1(X;x_0) \to \pi_1(Y;f(x_0))$ and $g_* : \pi_1(X;x_0) \to \pi_1(Y;g(x_0))$ is: $g_* = \overline{w}_* \circ f_*$ (see [**2**] Lemma 6.1.9).

Let G be a group. We say that a path connected topological space X is of type $K(G,1)$ if $\pi_1(X) = G$ and $\pi_k(X) = 0$ for $k \neq 1$. $\pi_k(X)$ is the kth homotopy group of X.

Proposition 3.2 ([**2**] Proposition 6.7.8). *Let N be a topological space of type $K(G,1)$ and let M be a compact connected topological manifold. Let $x_0 \in M$ and $y_0 \in N$. Then, given a homomorphism $\varphi : \pi_1(M,x_0) \to \pi_1(N,y_0)$, there exists a continuous map $f : M \to N$ with $f(x_0) = y_0$ such that $f_* = \phi$. Conversely, if $f, g : M \to N$ are continuous maps with $f(x_0) = g(x_0) = y_0$ and if $f_*, g_* : \pi_1(M,x_0) \to \pi_1(N,y_0)$ satisfies $f_*(\alpha) = \rho g_*(\alpha)\rho^{-1}$, $\forall \alpha \in \pi_1(M,x_0)$ for some $\rho \in \pi_1(N,y_0)$, then f and g are homotopic.*

LEMMA 3.3 ([**2**] Remark 6.7.9). *Let $f, g : N/\Gamma \to N/\Gamma$ be continuous maps of an infra-nil-manifold and let $f(x_0) = g(x_0)$ for some $x_0 \in N/\Gamma$. Then f and g are homotopic if and only if $f_* = g_* : \pi_1(N/\Gamma, x_0) \to \pi_1(N/\Gamma, f(x_0))$. From this fact we have that if $f : N/\Gamma \to N/\Gamma$ is a continuous map, then there is a unique infra-nil-endomorphism $A : N/\Gamma \to N/\Gamma$ homotopic to f.*

LEMMA 3.4 ([**15**] Lemma 1.3). *Let $f : N/\Gamma \to N/\Gamma$ be a self-covering map of an infra-nil-manifold and $A : N/\Gamma \to N/\Gamma$ denote the infra-nil-endomorphism homotopic to f. If f is a TA-covering map, then A is hyperbolic.*

LEMMA 3.5 ([**15**] Lemma 1.5). *Let $f : N/\Gamma \to N/\Gamma$ be a self-covering map and let $\overline{f} : N \to N$ be a lift of f by the natural projection $\pi : N \to N/\Gamma$. If f is a TA-covering map then $\overline{f}$ has exactly one fixed point.*

For continuous maps f and g of N we define $D(f, g) = \sup\{D(f(x), g(x)) : x \in N\}$ where D denotes a left invariant, Γ-invariant Riemannian distance for N . Notice that $D(f, g)$ is not necessary finite.

Suppose that $f : N/\Gamma \to N/\Gamma$ is a TA-covering map. Let $A : N/\Gamma \to N/\Gamma$ be the infra-nil-endomorphism homotopic to f , and let $\overline{A} : N \to N$ be the automorphism which is a lift of A by the natural projection π . Since $D_e\overline{A}$ is hyperbolic by Lemma 3.4, the Lie algebra $Lie(N)$ of N splits into the direct sum $Lie(N) = E^s_e \oplus E^u_e$ of subspaces E^s_e and E^u_e such that $D_e\overline{A}(E^s_e) = E^s_e$, $D_e\overline{A}(E^u_e) = E^u_e$ and there are $c > 1, 0 < \lambda < 1$ so that for all $n \geq 0$

$$||D_e\overline{A}^n(v)|| \leq c\lambda^n||v|| \quad (v \in E^s_e),$$
$$||D_e\overline{A}^{-n}(v)|| \leq c\lambda^n||v|| \quad (v \in E^u_e),$$

where $||\cdot||$ is the Riemannian metric. Let $\overline{L}^\sigma(e) = \exp(E^\sigma_e)$ $(\sigma = s, u)$ and let $\overline{L}^\sigma(x) = x \cdot \overline{L}^\sigma(e)(\sigma = s, u)$ for $x \in N$. Since left translations are isometries under the metric D , it follows that for all $x \in N$

$$\overline{L}^s(x) = \{y \in N : D(\overline{A}^i(x), \overline{A}^i(y)) \to 0 \ (i \to \infty)\},$$
$$\overline{L}^u(x) = \{y \in N : D(\overline{A}^i(x), \overline{A}^i(y)) \to 0 \ (i \to -\infty)\}.$$

LEMMA 3.6 ([**7**] Lemma 3.2). *For $x, y \in N$, $L^s(X) \cap L^u(y)$ consists of exactly one point.*

For $x, y \in N$ denote as $\beta(x, y)$ the point in $L^s(X) \cap L^u(y)$.

LEMMA 3.7 ([**7**] Lemma 3.2). *(1) For $L > 0$ and $\epsilon > 0$ there exists $J > 0$ such that for $x, y \in N$ if $D(\overline{A}^i(x), \overline{A}^i(y)) \leq L$ for all i with $|i| \leq J$, then $D(x, y) \leq \epsilon$.*
(2) For given $L > 0$, if $D(\overline{A}^i(x), \overline{A}^i(y)) \leq L$ for all $i \in \mathbb{Z}$, then $x = y$ $(x, y \in N)$.

LEMMA 3.8 ([**15**] Lemma 2.3). *Under the assumptions and notations as above, there is a unique map* $\overline{h} : N \to N$ *such that*

(1) $\overline{A} \circ \overline{h} = \overline{h} \circ \overline{f}$,
(2) $D(\overline{h}, id_N)$ *is finite,*

where $id_N : N \to N$ *is the identity map of* N. *Furthermore* $\overline{h}$ *is surjective and uniformly continuous under* D .

In addition, if f *is not an expanding map then* $\overline{h}$ *is a homeomorphism i.e.* $\overline{h}$ *is* D*-biuniformly continuous. (See* [**2**] *Proposition 8.4.2)*

LEMMA 3.9 ([**15**] Lemma 2.4). *For the semiconjugacy* $\overline{h}$ *of lemma 3.8, we have the following properties:*

(1) *There exists* $K > 0$ *such that* $D(\overline{h} \circ \gamma(x), \gamma \circ \overline{h}(x)) < K$ *for* $x \in \mathbb{N}$ *and* $\gamma \in \Gamma$.
(2) *For any* $\lambda > 0$, *there exists* $L \in \mathbb{N}$ *such that* $D(\overline{h} \circ \gamma(x), \gamma \circ \overline{h}(x)) < \lambda$ *for* $x \in N$ *and* $\gamma \in \overline{A}_*^L(\Gamma)$.
(3) *For* $x \in N$ *and* $\gamma \in \bigcap_{i=0}^{\infty} \overline{A}_*^i(\Gamma)$, *we have* $\overline{h} \circ \gamma(x) = \gamma \circ \overline{h}(x)$.
(4) *For* $x \in N$ *and* $\gamma \in \Gamma$, *we have* $\overline{h} \circ \gamma(x) \in \overline{L}^s(\gamma \circ \overline{h}(x))$.

REMARK 3.10. By part (2) of theorem 3.8, there is a $\delta_K > 0$ such that $D(\overline{h}(x), x) < \delta_K$ for $x \in N$, we have (see [**2**] page 270 (8.5))

$$\overline{W}^s(x) \subset U_{\delta_K}(\overline{L}^s(\overline{h}(x))) \quad \text{and} \quad \overline{W}^u(x; \mathbf{e}) \subset U_{\delta_K}(\overline{L}^u(\overline{h}(x))).$$

By lemma 3.5 if $f : N/\Gamma \to N/\Gamma$ is a TA-map and $\overline{f} : N \to N$ a lift of it, then there exists a unique fixed point say b such that $\overline{f}(b) = b$. For simplisity we can suppose that $b = e$. Indeed, we can choose a homeomorphism φ of N such that $\varphi(\pi(b)) = e$. Then $\varphi \circ f \circ \varphi^{-1}$ is a TA-covering map such that $\varphi \circ f \circ \varphi^{-1}(e) = e$.

LEMMA 3.11. *Let* $f : N/\Gamma \to N/\Gamma$ *be a continuous map of an infra-nil-manifold, and* $A : N/\Gamma \to N/\Gamma$ *be the unique infra-nil-endomorphism homotopic to* f, *then* $\overline{f}_* = \overline{A}_* : \Gamma \to \Gamma$.

PROOF. By corollary 2.3, $\overline{f}_*$ and $\overline{A}_*$ are two maps on Γ. For $[e] = \{x \in N : \gamma(x) = \gamma.x = e \text{ for some } \gamma \in \Gamma\}$, we have

$$f([e]) = f \circ \pi(e) = \pi \circ \overline{f}(e) = \pi(\overline{f}(e)) = \pi(e) = [e] = A([e]).$$

So according to lemma 3.3, $\overline{f}_* = \overline{A}_*$. □

Let $x \in N$, we define the stable set and unstable sets of x for f and A as follow (for more details see [**2**]):

$$\overline{W}^s(x) = \{y \in N : \lim_{i \to \infty} D(\overline{f}^i(x), \overline{f}^i(y)) = 0\},$$

$$\overline{W}^u(x, \mathbf{e}) = \{y \in N : \lim_{i \to -\infty} D(\overline{f}^i(x), \overline{f}^i(y)) = 0\},$$

Where $\mathbf{e} = (\ldots, e, e, e, \ldots)$.

Remark 3.12. By lemma 3.8, since $\overline{h}$ is D-uniformly continuous then $\overline{h}(\overline{W}^s(x)) = \overline{L}^s(\overline{h}(x))$ and $\overline{h}(\overline{W}^u(x;\mathbf{e})) = \overline{L}^u(\overline{h}(x))$.

Lemma 3.13. *The following statements hold:*

(1) $\gamma(\overline{W}^s(x)) = \overline{W}^s(\gamma(x))$ *for* $\gamma \in \Gamma$ *and* $x \in N$,
(2) $\gamma(\overline{W}^u(x;\boldsymbol{e})) = \overline{W}^u(\gamma(x);\boldsymbol{e})$ *for* $\gamma \in \Gamma$ *and* $x \in N$,

Proof. It is an easy corollary of lemma 6.6.11 of [**2**]. According to corollary 2.3, in the mentioned lemma put Γ instead of $G(\pi)$ and N instead of $\overline{X}$. □

Lemma 3.14. *The following statements hold:*

(1) $\gamma(\overline{L}^s(x)) = \overline{L}^s(\gamma(x))$ *for* $\gamma \in \Gamma$ *and* $x \in N$,
(2) $\gamma(\overline{L}^u(x)) = \overline{L}^u(\gamma(x))$ *for* $\gamma \in \Gamma$ *and* $x \in N$,

Proof. Proof is the same as in lemma 3.13. □

Lemma 3.15 ([**15**] Lemma 5.4). *Let* N/Γ *be an infra-nil-manifold. If* $f : N/\Gamma \to N/\Gamma$ *is a* TA*-covering map, then the nonwandering set* $\Omega(f)$ *coincides with the entire space* N/Γ.

Lemma 3.16 ([**2**] Lemma 8.6.2). *For* $\epsilon > 0$ *there is* $\delta > 0$ *such that if* $D(x,y) < \delta$, $x, y \in N$ *then* $\overline{W}^s(x) \subset U_\epsilon(\overline{W}^s(y))$ *and* $\overline{W}^u(x;\boldsymbol{e}) \subset U_\epsilon(\overline{W}^u(y;\boldsymbol{e}))$. *Where for a set* S, $U_\epsilon(S) = \{y \in N : D(y,S) < \epsilon\}$.

4. Proof of the main theorem

In this section , we give the proof in several lemmas and propositions, each of them based on the privious ones.

Sketch of proof. we suppose that $f : N/\Gamma \to N/\Gamma$ is a special TA-covering map of an infra-nil-manifold which is not injective or expanding, and $A : N/\Gamma \to N/\Gamma$ is the unique infra-nil-endomorphism homotopic to f. By lemma 3.8, there is a unique semiconjugacy $\overline{h} : N \to N$ between $\overline{f}$ and $\overline{A}$, such that by proposition 4.2.(3), $\overline{h}(\gamma(v)) = \gamma(\overline{h}(v))$, for each $\gamma \in \left(\overline{W}^u(e;\mathbf{e}) \rtimes \{id_{Aut(N)}\}\right) \cap \Gamma$ and $v \in \overline{W}^u(e;\mathbf{e})$. Through proposition 4.3 to proposition 4.13 we show that for all $\gamma \in \Gamma$ and $x \in N$, $\overline{h}(\gamma(x)) = \gamma(\overline{h}(x))$. Based on this result, $\overline{h}$ induces a homeomorphism $h : N/\Gamma \to N/\Gamma$ which is the conjugacy between f and A.

Lemma 4.1. *The following statements hold:*

(1) *Let* D *be the metric of* N *as above. for each* $x \in N$, $D(x^{-1}, e) = D(x, e)$.
(2) *If* $x \in \overline{W}^u(e;\boldsymbol{e})$, *then* $\overline{W}^u(x;\boldsymbol{e}) = \overline{W}^u(e;\boldsymbol{e})$.
(3) *If* $x \in \overline{L}^u(e)$, *then* $\overline{L}^u(x) = \overline{L}^u(e)$.

Proof. (1)

$$\begin{aligned} D(x^{-1},e) &= D(x^{-1}e, x^{-1}x) \\ (D \text{ is left invariant}) \quad &= D(e,x) \\ &= D(x,e) \end{aligned}$$

(2) Since $x \in \overline{W}^u(e;\mathbf{e})$, we have $D(\overline{f}^i(x), \overline{f}^i(e)) \to 0$ as $i \to -\infty$. Let $y \in \overline{W}^u(e;\mathbf{e})$, then $D(\overline{f}^i(y), \overline{f}^i(e)) \to 0$ as $i \to -\infty$. We have,

$$D(\overline{f}^i(y), \overline{f}^i(x)) < D(\overline{f}^i(y), \overline{f}^i(e)) + D(\overline{f}^i(e), \overline{f}^i(x)) \to 0 \text{ as } i \to -\infty.$$

So, $y \in \overline{W}^u(x;\mathbf{e})$, i.e. $\overline{W}^u(e;\mathbf{e}) \subset \overline{W}^u(x;\mathbf{e})$. Conversely, if $y \in \overline{W}^u(x;\mathbf{e})$ then $D(\overline{f}^i(y), \overline{f}^i(x)) \to 0$ as $i \to -\infty$, and

$$D(\overline{f}^i(y), \overline{f}^i(e)) < D(\overline{f}^i(y), \overline{f}^i(x)) + D(\overline{f}^i(x), \overline{f}^i(e)) \to 0 \text{ as } i \to -\infty.$$

So, $y \in \overline{W}^u(e;\mathbf{e})$, i.e. $\overline{W}^u(x;\mathbf{e}) \subset \overline{W}^u(e;\mathbf{e})$.
(3) Its proof is the same as part (1). □

For simplicity, let $\Gamma_{\overline{f}} = \big(\overline{W}^u(e;\mathbf{e}) \rtimes \{id_{Aut(N)}\}\big) \cap \Gamma$ and $\Gamma_{\overline{A}} = \big(\overline{L}^u(e) \rtimes \{id_{Aut(N)}\}\big) \cap \Gamma$.

Proposition 4.2. *The following statements hold:*

1. $\Gamma_{\overline{A}}$ *and* $\Gamma_{\overline{f}}$ *are subgroups of* Γ.
2. $\Gamma_{\overline{f}} \subset \Gamma_{\overline{A}}$.
3. $\overline{h}(\gamma(v)) = \gamma(\overline{h}(v))$, *for each* $\gamma \in \Gamma_{\overline{f}}$ *and* $v \in \overline{W}^u(e;\boldsymbol{e})$.
4. *If* $\overline{W}^u(\gamma_1(e);\boldsymbol{e}) = \overline{W}^u(\gamma_2(e);\boldsymbol{e})$, *for some* $\gamma_1,\gamma_2 \in \Gamma$, *then we have*

$$\gamma_1(\overline{h}(\gamma_1^{-1}(x)) = \gamma_2(\overline{h}(\gamma_2^{-1}(x)), \text{ for } x \in \overline{W}^u(\gamma_1(e);\boldsymbol{e}).$$

Proof. (1) Let $x, y \in \overline{L}^u(e)$, since $A^i(e) = e$, for all i, then by definition,

$$\begin{aligned} &\lim_{i\to-\infty} D(A^i(x),e) = \lim_{i\to-\infty} D(A^i(x),A^i(e)) = 0 \\ (4.1) \qquad &\lim_{i\to-\infty} D(A^i(y),e) = \lim_{i\to-\infty} D(A^i(y),A^i(e)) = 0. \end{aligned}$$

As D is left invariant we have

$$\begin{aligned} 0 \le \lim_{i\to-\infty} D(A^i(xy^{-1}),A^i(e)) &= \lim_{i\to-\infty} D(A^i(x)A^i(y^{-1}),e) \\ &= \lim_{i\to-\infty} D(A^i(x)A^{-i}(y),A^i(x)A^{-i}(x)) \\ (\text{D is left invariant}) \quad &= \lim_{i\to-\infty} D(A^{-i}(y),A^{-i}(x)) \\ &\le \lim_{i\to-\infty} D(A^{-i}(y),e) + D(e,A^{-i}(x)) \\ (\text{Lemma 4.1.(1) and (4.1)}) \quad &= \lim_{i\to-\infty} D(A^i(y),e) + D(A^i(x),e) = 0 \end{aligned}$$

Thus $xy^{-1} \in \overline{L}^u(e)$ and $\overline{L}^u(e)$ is a subgroup of N. So $\big(\overline{L}^u(e) \rtimes \{id_{Aut(N)}\}\big) \cap \Gamma$ is a subgroup of Γ.

For the second part, Let $\gamma_1, \gamma_2 \in \Gamma_{\overline{f}}$. Since Γ is a group we have $\gamma_1\gamma_2^{-1} \in \Gamma$. Now consider that $\gamma_1, \gamma_2 \in \big(\overline{W}^u(e;\mathbf{e}) \rtimes \{id_{Aut(N)}\}\big)$. There exist $x_1, x_2 \in \overline{W}^u(e;\mathbf{e})$ such that $\gamma_1 = (x_1, id_N)$ and $\gamma_2 = (x_2, id_N)$. Therefore,

$$\begin{aligned} x_1\overline{W}^u(e;\mathbf{e}) &= \gamma_1(\overline{W}^u(e;\mathbf{e})) \\ \text{(Lemma 3.13)} \quad &= \overline{W}^u(\gamma_1(e);\mathbf{e}) \\ &= \overline{W}^u(x_1;\mathbf{e}) \\ \text{(Lemma 4.1) (2)} \quad &= \overline{W}^u(e;\mathbf{e}). \end{aligned}$$

Similarly, $x_2\overline{W}^u(e;\mathbf{e}) = \overline{W}^u(e;\mathbf{e})$. So we have $x_1\overline{W}^u(e;\mathbf{e}) = x_2\overline{W}^u(e;\mathbf{e})$ and then $x_1x_2^{-1} \in \overline{W}^u(e;\mathbf{e})$. Finally,

$$\begin{aligned} \gamma_1\gamma_2^{-1} &= (x_1, id_N)(x_2, id_N)^{-1} \\ &= (x_1, id_N)(x_2^{-1}, id_N) \\ &= (x_1x_2^{-1}, id_N) \in \overline{W}^u(e;\mathbf{e}) \rtimes \{id_{Aut(N)}\}, \end{aligned}$$

and we have the result.

(2) Take $(x,\alpha) = \gamma \in \Gamma_{\overline{f}}$ $(x \in N, \alpha \in C)$, such that $\gamma \notin \Gamma_{\overline{A}}$. So, $x \notin \overline{L}^u(e)$ and for each $n \in \mathbb{Z}$, $n \neq 0$, $x^n \notin \overline{L}^u(e)$. By part (1), remark 3.10 and the fact that $\overline{h}(e) = e$, for all $n \in \mathbb{Z}$, we have $x^n \in \overline{W}^u(e;\mathbf{e}) \subset U_{\delta_K}(\overline{L}^u(e))$, which is impossible.

(3)Let $\gamma = (x, id_N)$, for some $x \in N$ and $v \in \overline{W}^u(e;\mathbf{e})$. We have

$$\begin{aligned} \gamma(v) &\in \gamma(\overline{W}^u(e;\mathbf{e})) \\ \text{(Lemma 3.13)} \quad &= \overline{W}^u(\gamma(e);\mathbf{e}) \\ &= \overline{W}^u(x;\mathbf{e}) \\ \text{(Lemma 4.1 (2))} \quad &= \overline{W}^u(e;\mathbf{e}), \end{aligned}$$

so,

$$\begin{aligned} \overline{h}(\gamma(v)) &\in \overline{h}(\overline{W}^u(e;\mathbf{e})) \\ \text{(Remark 3.12)} \quad &= \overline{L}^u(e). \end{aligned}$$

By part (2), $\gamma \in \Gamma_{\overline{A}}$, Thus

$$\begin{aligned} \gamma(\overline{h}(v)) &\in \gamma(\overline{h}(\overline{W}^u(e;\mathbf{e})) \\ \text{(Remark 3.12)} \quad &= \gamma(\overline{L}^u(e)) \\ \text{(Lemma 3.14)} \quad &= (\overline{L}^u(\gamma(e))) \\ &= \overline{L}^u(x) \\ \text{(Lemma 4.1 (3))} \quad &= \overline{L}^u(e). \end{aligned}$$

Again by Lemma 4.1 (3) and last part of the above relation, $\overline{L}^u(\gamma(\overline{h}(v))) = \overline{L}^u(e)$, and

$$\overline{h}(\gamma(v)) \in \overline{L}^u(e) = \overline{L}^u(\gamma(\overline{h}(v))).$$

On the other hand, by part (4) of lemma 3.9, $\overline{h}(\gamma(v)) \in \overline{L}^s(\gamma(\overline{h}(v)))$. Since $\overline{L}^u(\gamma(\overline{h}(v))) \cap \overline{L}^s(\gamma(\overline{h}(v))) = \{\gamma(\overline{h}(v))\}$ (see [**15**] Lemma 2.1), then $\overline{h}(\gamma(v)) = \gamma(\overline{h}(v))$.

(4) We have $x \in \overline{W}^u(\gamma_1(e);\mathbf{e}) = \gamma_1(\overline{W}^u(e;\mathbf{e}))$. Thus, $\gamma_1^{-1}(x) \in \overline{W}^u(e;\mathbf{e})$. Similarly, $\gamma_2^{-1}(x) \in \overline{W}^u(e;\mathbf{e})$. Now, by part (3),

$$\begin{aligned}\gamma_1\big(\overline{h}(\gamma_1^{-1}(x)\big) &= \overline{h}\big(\gamma_1(\gamma_1^{-1}(x))\big)\\ &= \overline{h}(x)\\ &= \overline{h}\big(\gamma_2(\gamma_2^{-1}(x))\big)\\ &= \gamma_2\big(\overline{h}(\gamma_2^{-1}(x)\big).\end{aligned}$$

□

According to part (4) of proposition 4.2, we can define a map $\overline{h}' : \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e}) \to \bigcup_{\gamma\in\Gamma} \overline{L}^u(\gamma(e))$, by

$$\overline{h}'(x) = \gamma(\overline{h}(\gamma^{-1}(x))) \quad x \in \overline{W}^u(\gamma(e);\mathbf{e})\ (\gamma \in \Gamma).$$

Next lemma shows some properties of $\overline{h}'$:

Proposition 4.3. *The following statements hold:*

(1) $\overline{A} \circ \overline{h}' = \overline{h}' \circ \overline{f}$ *on* $\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})$,
(2) $D(\overline{h}', id_{|\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})}) < \infty$,
(3) $\overline{h}'(\gamma(e)) = \gamma(e)$ *for* $\gamma \in \Gamma$,
(4) *if* $x \in \overline{W}^u(\gamma(e);\boldsymbol{e})$ $(\gamma \in \Gamma)$, *then* $\overline{h}'(x) \in \overline{L}^u(\gamma(e))$ *and* $\overline{h}'(x) \in \overline{L}^s(\overline{h}(x))$,
(5) *if* $y \in \overline{W}^s(x)$ *for* $x, y \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\boldsymbol{e})$, *then* $\overline{h}'(y) \in \overline{L}^s(\overline{h}'(x))$.

Proof. (1) Suppose that $x \in \overline{W}^u(\gamma(e);\mathbf{e})$, for some $\gamma \in \Gamma$. Notice that $\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e}) = \bigcup_{\gamma\in\Gamma_{\overline{f}}} \overline{W}^u(\gamma(e);\mathbf{e})$, for if $\gamma = (x,\alpha) \in \Gamma$ and $\gamma' = (x, id_N) \in \Gamma_{\overline{f}}$ $(x \in N,\ \alpha \in C)$, then

$$\gamma(e) = \gamma{\cdot}e = (x,\alpha){\cdot}e = x\alpha(e) = x = x\ id_N(e) = (x, id_N){\cdot}e = \gamma'{\cdot}e = \gamma'(e).$$

Hence, suppose that $\gamma \in \Gamma_{\overline{f}}$. Since $x \in \overline{W}^u(\gamma(e);\mathbf{e}) = \gamma\big(\overline{W}^u(e;\mathbf{e})\big)$, then

$$\gamma^{-1}(x) \in \overline{W}^u(e;\mathbf{e}). \tag{4.2}$$

By [**2**] page 205, we have $\overline{f}\big(\overline{W}^u(\gamma(e);\mathbf{e})\big) = \overline{W}^u(\overline{f}(\gamma(e));\mathbf{e})$. Here $\overline{f}(\gamma(e))$ means $\overline{f}_*(\gamma)\cdot e$ which by lemma 3.11 is equal to $\overline{A}_*(\gamma)\cdot e$ and $\overline{A}_*(\gamma) \in \Gamma$. Therefore,

$$\overline{f}(x) \in \overline{f}\big(\overline{W}^u(\gamma(e);\mathbf{e})\big) = \overline{W}^u(\overline{A}(\gamma(e));\mathbf{e}) \Rightarrow (\overline{A}_*(\gamma))^{-1}(\overline{f}(x)) \in \overline{W}^u(e;\mathbf{e}). \tag{4.3}$$

Thus we have

$$\begin{aligned}
\overline{A}\circ\overline{h}'(x) &= \overline{A}\big(\gamma(\overline{h}(\gamma^{-1}(x)))\big)\\
\text{((4.2) and proposition 4.2.(3))}\quad &= \overline{A}\big(\overline{h}(\gamma(\gamma^{-1}(x)))\big)\\
&= \overline{A}\circ\overline{h}(x)\\
\text{(lemma 3.8)}\quad &= \overline{h}\circ\overline{f}(x)\\
&= \overline{h}\Big((\overline{A}_*(\gamma))\big((\overline{A}_*(\gamma))^{-1}(\overline{f}(x))\big)\Big)\\
\text{((4.3) and proposition 4.2.(3))}\quad &= (\overline{A}_*(\gamma))\Big(\overline{h}\big((\overline{A}_*(\gamma))^{-1}(\overline{f}(x))\big)\Big)\\
&= \overline{h}'(\overline{f}(x))\\
&= \overline{h}'\circ\overline{f}(x).
\end{aligned}$$

(2) Let $x\in\overline{W}^u(\gamma(e);\mathbf{e})$, for some $\gamma\in\Gamma_{\overline{f}}$, and let $\delta_K>0$ be satisfying $D(\overline{h},id_N)<\delta_K$. Then

$$\begin{aligned}
D(\overline{h}'(x),x) &= D\big(\gamma(\overline{h}(\gamma^{-1}(x))),x\big)\\
\text{(lemma 4.1.(1))}\quad &= D\big(\overline{h}(\gamma^{-1}(x)),x\big)\\
\text{(lemma 4.1.(1))}\quad &= D\big(\overline{h}(\gamma^{-1}(x)),\gamma^{-1}(x)\big)\\
&< \delta_K.
\end{aligned}$$

(3) For any $\gamma\in\Gamma$, by definition we have

$$\overline{h}'\big(\gamma(e)\big)=\gamma(\overline{h}(\gamma^{-1}(\gamma(e))))=\gamma(\overline{h}(e))=\gamma(e).$$

(4) Let $x\in\overline{W}^u(\gamma(e);\mathbf{e})$, for some $\gamma\in\Gamma$. We have

$$\begin{aligned}
\overline{h}'(x)=\gamma(\overline{h}(\gamma^{-1}(x))) &\in \gamma(\overline{h}(\gamma^{-1}(\overline{W}^u(\gamma(e);\mathbf{e}))))\\
\text{(lemma 3.13)}\quad &= \gamma(\overline{h}(\overline{W}^u(e;\mathbf{e})))\\
\text{(proposition 4.2.(3))}\quad &= \overline{h}(\gamma(\overline{W}^u(e;\mathbf{e})))\\
\text{(lemma 3.13)}\quad &= \overline{h}(\overline{W}^u(\gamma(e);\mathbf{e})))\\
\text{(remark 3.12)}\quad &= \overline{L}^u(\gamma(e)),
\end{aligned}$$

and

$$\begin{aligned}
\overline{h}'(x) &\in \overline{L}^s(\overline{h}'(x))\\
&= \overline{L}^s\big(\gamma(\overline{h}(\gamma^{-1}(x)))\big)\\
\text{(proposition 4.2.(3))}\quad &= \overline{L}^s\big(\overline{h}(\gamma(\gamma^{-1}(x)))\big)\\
&= \overline{L}^s\big(\overline{h}(x)\big).
\end{aligned}$$

(5) By the second part of proof of (4), we have

$$\overline{L}^s(\overline{h}'(y))=\overline{L}^s\big(\overline{h}(y)\big)=\overline{h}(\overline{W}^s(y))=\overline{h}(\overline{W}^s(x))=\overline{L}^s\big(\overline{h}(x)\big)=\overline{L}^s(\overline{h}'(x)).$$

□

LEMMA 4.4 ([**15**] Lemma 7.6). *For each* $u, v \in N$, $\overline{W}^u(u; \boldsymbol{e}) \cap \overline{W}^s(v)$ *is the set of one point.*

According to the above lemma, define $\overline{\iota}(u,v) = \overline{W}^u(u; \mathbf{e}) \cap \overline{W}^s(v)$.

LEMMA 4.5. *For* $\epsilon > 0$*, there is* $\delta > 0$ *such that*

$$D(u,v) < \delta \Rightarrow max\{D(\overline{\iota}(u,v),u), D(\overline{\iota}(u,v),v)\} < \epsilon \quad (u,v \in N)$$

PROOF. Let $\epsilon > 0$ be given. Since $\overline{h}$ is D-biuniformly contiuous there exists $\epsilon' > 0$ such that

$$D(x,y) < \epsilon' \Rightarrow D(\overline{h}^{-1}(x), \overline{h}^{-1}(y)) < \epsilon \quad (x,y \in N).$$

By [**2**] theorem 6.6.5 or [**15**] lemma 7.2, since N is simply connected, $\overline{A}$ has local product structure (for definition and details, see [**2**]), and then for $\epsilon > 0$ there exists $\delta' > 0$ such that

$$D(u,v) < \delta' \Rightarrow max\{D(\beta(u,v),u), D(\beta(u,v),v)\} < \epsilon' \quad (u,v \in N)$$

Again since $\overline{h}$ is D-biuniformly continuous, there exists $\delta > 0$ such that

$$D(u,v) < \delta \Rightarrow D(\overline{h}(u), \overline{h}(v)) < \delta' \quad (u,v \in N).$$

We know that $\overline{h}\big(\overline{\iota}(u,v)\big) = \beta(\overline{h}(u), \overline{h}(v))$ therefore

$$\begin{aligned} D(u,v) < \delta \Rightarrow\ & D(\overline{h}(u), \overline{h}(v)) < \delta' \\ & \Rightarrow max\{D(\beta(\overline{h}(u), \overline{h}(v)), \overline{h}(u)), D(\beta(\overline{h}(u), \overline{h}(v)), \overline{h}(v))\} < \epsilon' \\ & \Rightarrow max\{D(\overline{h}(\overline{\iota}(u,v)), \overline{h}(u)), D(\overline{h}(\overline{\iota}(u,v)), \overline{h}(v)) < \epsilon' \\ & \Rightarrow max\{D(\overline{\iota}(u,v),u), D(\overline{\iota}(u,v),v)\} < \epsilon. \end{aligned}$$

□

PROPOSITION 4.6. $\overline{h}'$ *is* D*-uniformly continuous.*

PROOF. Suppose that the statement is false. So there is $\epsilon_0 > 0$, for all $\delta > 0$, there are $x, y \in \bigcup_{\gamma \in \Gamma} \overline{W}^u(\gamma(e); \mathbf{e})$ such that

$$D(x,y) < \delta \text{ and } D(\overline{h}'(x), \overline{h}'(y)) > \epsilon_0. \tag{4.4}$$

By definition of $\overline{L}^\sigma(x)$ $(x \in N,\ \sigma = s,u)$, for $w \in \overline{L}^s(v)$ there is $\epsilon_1 > 0$ such that

$$D(v,w) < \epsilon_0/2 \Rightarrow D(\overline{L}^u(v), \overline{L}^u(w)) > \epsilon_1. \tag{4.5}$$

Take $n > 0$ and $\delta_1 > 0$ such that $\epsilon^n \geq 2\delta_K$ and $\delta_1^n \leq 2\delta_K$.

By Lemma 3.16, there exists $\delta_2 > 0$ such that

$$D(v,w) < \delta_2 \Rightarrow \overline{W}^u(w, \mathbf{e}) \subset U_{\delta_1}\big(\overline{W}^u(v, \mathbf{e})\big). \tag{4.6}$$

Since $\overline{h}$ is continuous, take $\delta_3 > 0$ such that

$$D(u,v) < \delta_3 \Rightarrow D(\overline{h}(u), \overline{h}(v)) < \epsilon_0/2. \tag{4.7}$$

By lemma 4.5, there is $0 < \delta < \delta_2$ such that

$$D(x,y) < \delta \Rightarrow D(y, \overline{\iota}(y,x)) < \delta_3 \quad (x, y \in N). \tag{4.8}$$

Now consider $x, y \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e);\mathbf{e})$ as above. There exist $\gamma_x, \gamma_y \in \Gamma$ such that $x \in \overline{W}^u(\gamma_x(e),\mathbf{e})$ and $y \in \overline{W}^u(\gamma_y(e),\mathbf{e})$. We have

$$\begin{aligned}
D\big(\overline{h}'(x), \overline{h}'(\overline{\iota}(y,x))\big) &\geq D\big(\overline{h}'(x), \overline{h}'(y)\big) - D\big(\overline{h}'(y), \overline{h}'(\overline{\iota}(y,x))\big) \\
\text{(by (4.4))} \quad &\geq \epsilon_0 - D\big(\gamma_y(\overline{h}(\gamma_y^{-1}(y))), \gamma_y(\overline{h}(\gamma_y^{-1}(\overline{\iota}(y,x))))\big) \\
\text{(by proposition 4.2.(3))} \quad &= \epsilon_0 - D\big(\overline{h}(\gamma_y(\gamma_y^{-1}(y))), \overline{h}(\gamma_y(\gamma_y^{-1}(\overline{\iota}(y,x))))\big) \\
&= \epsilon_0 - D\big(\overline{h}(y), \overline{h}(\overline{\iota}(y,x))\big) \\
\text{(by (4.7) and (4.8))} \quad &> \epsilon_0 - \epsilon_0/2 = \epsilon_0/2.
\end{aligned} \tag{4.9}$$

By proposition 4.3.(4)

$$x \in \overline{W}^u(\gamma_x(e),\mathbf{e}) \Rightarrow \overline{h}'(x) \in \overline{L}^u(\gamma_x(e)),$$
$$\overline{\iota}(y,x) \in \overline{W}^u(\gamma_y(e),\mathbf{e}) \Rightarrow \overline{h}'(\overline{\iota}(y,x)) \in \overline{L}^u(\gamma_y(e)).$$

Thus by proposition 4.3.(5), (4.9) and (4.5) we have

$$D\big(\overline{L}^u(\gamma_x(e)), \overline{L}^u(\gamma_y(e))\big) > \epsilon_1.$$

Suppose $\gamma = \gamma_x^{-1}\gamma_y$. We have

$$\begin{aligned}
\gamma_x\gamma(e) = \gamma_y(e) \notin U_{\epsilon_1}\big(\overline{L}^u(\gamma_x(e))\big) &\Rightarrow \gamma(e) \notin U_{\epsilon_1}\big(\overline{L}^u(\gamma_x^{-1}\gamma_x(e))\big) = U_{\epsilon_1}\big(\overline{L}^u(e)\big) \\
&\Rightarrow \gamma^n(e) \notin U_{\epsilon_1^n}\big(\overline{L}^u(e)\big) \supset U_{2\delta_K}\big(\overline{L}^u(e)\big).
\end{aligned} \tag{4.10}$$

On the other hand,

$$\begin{aligned}
\overline{W}^u(\gamma_x\gamma(e);\mathbf{e}) &= \overline{W}^u(\gamma_y(e);\mathbf{e}) \\
(y \in \overline{W}^u(\gamma_y(e);\mathbf{e})) \quad &= \overline{W}^u(y;\mathbf{e}) \\
\text{(by (4.6))} \quad &\subset U_{\delta_1}\big(\overline{W}^u(x;\mathbf{e})\big) \\
(x \in \overline{W}^u(\gamma_x(e);\mathbf{e})) \quad &= U_{\delta_1}\big(\overline{W}^u(\gamma_x(e);\mathbf{e})\big).
\end{aligned}$$

Now we have

$$\begin{aligned}
\gamma_x^{-1}\big(\overline{W}^u(\gamma_x\gamma(e);\mathbf{e})\big) \subset U_{\delta_1}\big(\overline{W}^u(e;\mathbf{e})\big) &\Rightarrow \overline{W}^u(\gamma(e);\mathbf{e}) \subset U_{\delta_1}\big(\overline{W}^u(e;\mathbf{e})\big) \\
&\Rightarrow \overline{W}^u(\gamma^2(e);\mathbf{e}) \subset U_{\delta_1}\big(\overline{W}^u(\gamma(e);\mathbf{e})\big) \\
\text{(by induction)} \quad &\Rightarrow \overline{W}^u(\gamma^n(e);\mathbf{e}) \subset U_{\delta_1}\big(\overline{W}^u(\gamma^{n-1}(e);\mathbf{e})\big) \\
&\Rightarrow \overline{W}^u(\gamma^n(e);\mathbf{e}) \subset U_{\delta_1^n}\big(\overline{W}^u(e;\mathbf{e})\big) \subset U_{2\delta_K}\big(\overline{L}^u(e)\big) \\
&\Rightarrow \gamma^n(e) \in \overline{W}^u(\gamma^n(e);\mathbf{e}) \subset U_{2\delta_K}\big(\overline{L}^u(e)\big).
\end{aligned} \tag{4.11}$$

Finally, (4.10) and (4.11) make a contradiction. □

Let $\tilde{u} = (u_i) \in N_{\overline{f}}$ and for each $i \in \mathbb{Z}$, $\overline{f}_{u_i,u_{i+1}}$ be the lift of f by π such that $\overline{f}(u_i) = u_{i+1}$ and define

$$\overline{f}^i_{\tilde{u}} = \begin{cases} \overline{f}_{u_{i-1},u_i} \circ \ldots \circ \overline{f}_{u_0,u_1} & \text{for } i > 0, \\ (\overline{f}_{u_i,u_{i+1}})^{-1} \circ \ldots \circ (\overline{f}_{u_{-1},u_0})^{-1} & \text{for } i < 0, \\ id_N & \text{for } i = 0. \end{cases}$$

We define a map $\tau_{\tilde{u}} = \tau^f_{\tilde{u}} : N \to (N/\Gamma)_f$ by

$$\tau_{\tilde{u}}(x) = (\pi \circ \overline{f}^i_{\tilde{u}}(x))_{i=-\infty}^{\infty} \quad (x \in N).$$

Since $\overline{f}(e) = e$, then $\tau_{\mathbf{e}}(e) = \tau_{\tilde{e}}(e) = (\pi(e))_{i=-\infty}^{\infty}$.

Lemma 4.7 ([**2**] Lemma 6.6.8 (1)). *If $x \in X$ and $\tilde{u} \in N_f$ then $\pi(\overline{W}^u(x;\tilde{u})) = W^u(\tau_{\tilde{u}}(x))$.*

Let X be a compact metric set and $f : X \to X$ a continuous surjection. A point $x \in X$ is said to be a *nonwandering point* if for any neighborhood U of x there is an integer $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. The set $\Omega(f)$ of all nonwandering points is called the *nonwandering set*. Clearly $\Omega(f)$ is closed in X and invariant under f.

f is said to be *topologically transitive* (here X may be not necessarily compact), if there is $x_0 \in X$ such that the orbit $O^+(x_0) = \{f^i(x_0) : i \in \mathbb{Z}^{\geq 0}\}$ is dense in X. It is easy to check that if X is compact, a continuous surjection $f : X \to X$ is topologically transitive if and only if for any U, V nonempty open sets there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

A continuous surjection $f : X \to X$ of a metric space is *topologically mixing* if for nonempty open sets U, V there exists $N > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$. Topological mixing implies topological transitivity.

For continuing, we need next theorem for which proof one can see [**2**] Theorem 3.4.4.

Theorem 4.8 (Topological decomposition theorem). *Let $f : X \to X$ be a continuous surjection of a compact metric space. If $f : X \to X$ is a TA-map, then the following properties hold:*

(1) *(Spectral decomposition theorem due to Smale) The nonwandering set, $\Omega(f)$, contains a finite sequence B_i $(1 \leq i \leq l)$ of f-invariant closed subsets such that*
   (i) *$\Omega(f) = \bigcup_{i=1}^{l} B_i$ (disjoint union),*
   (ii) *$f_{|B_i} : B_i \to B_i$ is topologically transitive.*
   Such the subsets B_i are called basic sets.
(2) *(Decomposition theorem due to Bowen) For B a basic set there exist $a > 0$ and a finite sequence C_i $(0 \leq i \leq a-1)$ of closed subsets such that*

(i) $C_i \cap C_j = \emptyset$ $(i \neq j)$, $f(C_i) = C_{i+1}$ *and* $f^a(C_i) = C_i$,
(ii) $B = \bigcup_{i=0}^{a-1} C_i$,
(iii) $f^a_{|C_i} : C_i \to C_i$ *is topologically mixing,*
Such the subsets C_i *are called elementary sets.*

LEMMA 4.9 ([**15**] Lemma 5.4). $\Omega(f) = N/\Gamma$.

LEMMA 4.10. N/Γ *is indeed an elementary set.*

PROOF. By lemma 3.5, let $\overline{f} : N \to N$ be the lift of f such that $\overline{f}(e) = e$. By the commuting diagram:

$$\begin{array}{ccc} N & \xrightarrow{\overline{f}} & N \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ N/\Gamma & \xrightarrow[f]{} & N/\Gamma \end{array}$$

we have,

$$f([e]) = f(\pi(e)) = \pi(\overline{f}(e)) = \pi(e) = [e].$$

Therefore, $[e]$ is a fixed point of f.

By lemma 4.9, $\Omega(f) = N/\Gamma$. As N is connected and π is a continuous surjection then N/Γ is connected. In the proof of part (1) of spectral decomposition theorem, they prove that basic sets are close and open. Hence by connectedness of $\Omega(f) = N/\Gamma$, it consists of only one basic set, say B. On the other hand, by part (2) of spectral decomposition theorem, $N/\Gamma = B$ is the union of elementary sets. There is an elementary set, say C, such that $[e] \in C$. Since elementary sets are disjoint, by condition $f(C_i) = C_{i+1}$, $N/\Gamma = B$ consists of only one elementary set. $\square$

LEMMA 4.11 ([**2**] Remark 5.3.2 (2)). *Let* $f : X \to X$ *be a* TA*-map of a compact metric space and let* C *be an elementary set of* f. *If* $\tilde{u} = (u_i) \in N_f$ *and* $x_i \in C$ *for all* $i \in \mathbb{Z}$ *then* $W^u(\tilde{x}) \cap C$ *is dense in* C.

LEMMA 4.12. $\bigcup_{\gamma \in \Gamma} \overline{W}^u(\gamma(e); \mathbf{e})$ *is dense in* N.

PROOF. By lemma 3.13 and lemma 4.7 we have

$$\bigcup_{\gamma \in \Gamma} \overline{W}^u(\gamma(e); \mathbf{e}) = \bigcup_{\gamma \in \Gamma} \gamma(\overline{W}^u(e; \mathbf{e})) = \pi^{-1}(W^u(\tau_{\mathbf{e}}(e))). \tag{4.12}$$

We have $\tau_{\mathbf{e}}(e) = \big(\pi(e)\big)_{i=-\infty}^{\infty} \in (N/\Gamma)_f$. On the other hand, Since by lemma 4.10, $\Omega(f) = N/\Gamma$ is an elementary set, say C, and for $\big(\pi(e)\big)_{i=-\infty}^{\infty}$ we have $\pi(e) \in N/\Gamma = C$ for all $i \in \mathbb{Z}$, by lemma 4.11 we have

$$W^u(\tau_{\mathbf{e}}(e)) = W^u(\tau_{\mathbf{e}}(e)) \cap (N/\Gamma) = W^u(\tau_{\mathbf{e}}(e)) \cap C$$

is dense in $C = N/\Gamma$. By relation (4.12), we have the desired result. $\square$

By lemma 4.6, $\overline{h}'$ is extended to a continuous map $\tilde{h} : N \to N$. From proposition 4.3 (1), (2) and (3), and lemma 3.8, we have $\overline{h} = \tilde{h}$ and $\overline{h}(\gamma(e)) = \gamma(e)$ for all $\gamma \in \Gamma$.

Proposition 4.13. *For all $\gamma \in \Gamma$ and $x \in N$, $\overline{h}(\gamma(x)) = \gamma(\overline{h}(x))$.*

Proof. According to lemma 3.9.(4), we have

$$\overline{h}(\gamma(x)) \in \overline{L}^s(\gamma(\overline{h}(x))). \tag{4.13}$$

Suppose that $x \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e); \mathbf{e})$. Then there is $\gamma_x \in \Gamma$ such that $x \in \overline{W}^u(\gamma_x(e); \mathbf{e})$. For each $\gamma \in \Gamma$ we have

$$\gamma(x) \in \overline{h}\big(\overline{W}^u(\gamma_x(e); \mathbf{e})\big) = \overline{W}^u\big(\gamma(\gamma_x(e)); \mathbf{e}\big).$$

Thus

$$\begin{aligned} \overline{h}(\gamma(x)) &\in \overline{h}\bigg(\overline{W}^u\big(\gamma(\gamma_x(e)); \mathbf{e}\big)\bigg) \\ \text{(by Remark 3.12)} \quad &= \overline{L}^u\big(\overline{h}(\gamma(\gamma_x(e)))\big) \\ &= \overline{L}^u\big(\gamma(\gamma_x(e))\big). \end{aligned} \tag{4.14}$$

On the other hand,

$$\begin{aligned} \gamma(\overline{h}(x)) &\in \gamma\big(\overline{h}(\overline{W}^u(\gamma_x(e); \mathbf{e}))\big) \\ \text{(by Remark 3.12)} \quad &= \gamma\big(\overline{L}^u(\overline{h}(\gamma_x(e)))\big) \\ &= \gamma\big(\overline{L}^u(\gamma_x(e))\big) \\ \text{(by Lemma 3.14)} \quad &= \overline{L}^u(\gamma(\gamma_x(e))). \end{aligned} \tag{4.15}$$

By (4.15), we have $\overline{L}^u\big(\gamma(\gamma_x(e))\big) = \overline{L}^u\big(\gamma(\overline{h}(x))\big)$. Therefore, by (4.14) we have

$$\overline{h}(\gamma(x)) \in \overline{L}^u\big(\gamma(\overline{h}(x))\big). \tag{4.16}$$

By (4.13) and (4.16) we have

$$\overline{h}(\gamma(x)) \in \overline{L}^u\big(\gamma(\overline{h}(x))\big) \cap \overline{L}^s\big(\gamma(\overline{h}(x))\big) = \{\gamma(\overline{h}(x))\}. \tag{4.17}$$

Thus for each $x \in \bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e); \mathbf{e})$ we have $\overline{h}(\gamma(x)) = \gamma(\overline{h}(x))$. Since $\overline{h}$ is continuous and $\bigcup_{\gamma\in\Gamma} \overline{W}^u(\gamma(e); \mathbf{e})$ is dense in N, we have the desired result. □

According to proposition 4.13, $\overline{h}$ induces a homeomorphism $h : N/\Gamma \to N/\Gamma$ such that $h \circ \pi = \pi \circ \overline{h}$. i.e. the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{\ \overline{h}\ } & N \\ \pi\big\downarrow & & \big\downarrow\pi \\ N/\Gamma & \xrightarrow[\ h\ ]{} & N/\Gamma \end{array}$$

h is the conjugacy between f and A. For if $x \in N/\Gamma$ then there is $y \in N$ such that $x = \pi(y)$ and

$$\begin{aligned} h \circ f(x) &= h \circ f(\pi(y)) = h(f \circ \pi(y)) = h(\pi \circ \overline{f}(y)) \\ &= h \circ \pi(\overline{f}(y)) = \pi \circ \overline{h}(\overline{f}(y)) = \pi(\overline{h} \circ \overline{f}(y)) \\ &= \pi(\overline{A} \circ \overline{h}(y)) = \pi \circ \overline{A}(\overline{h}(y)) = A \circ \pi(\overline{h}(y)) \\ &= A(\pi \circ \overline{h}(y)) = A(h \circ \pi(y)) = A \circ h(\pi(y)) \\ &= A \circ h(x). \end{aligned}$$

So the Main Theorem is proved.

Proof of Corrollary 1.11. As mentioned in section 2, every endomorphism of a compact metric space is a covering map. Every Anosov endomorphism is a TA-map (see [**2**] Theorem 1.2.1). It is easy to show that if f is a diffeomorphism or an expanding map then it is special. By Theorem 1.9, it is conjugate to a hyperbolic infra-nil-automorphism or an expanding infra-nil-endomorphism, respectively, which are hyperbolic infra-nil-endomorphisms. In Theorem 1.10, we prove the case that f is not injective or expanding. So in this case f is conjugate to a hyperbolic infra-nil-endomorphism too.

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