Fuzzy Isometries and Non-Existence of Fuzzy Contractive Maps on Fuzzy Metric Spaces

Gabjin Yun, Jeongwook Chang, and Seungsu Hwang

Abstract

In this paper, we first consider the disjoint union of two fuzzy metric spaces and non-existence of fuzzy contractive maps. We prove that there exists a natural fuzzy metric on the disjoint union of two fuzzy metric spaces. We also show that there is a fuzzy metric space which does not admit any fuzzy contractive map. Second, we introduce the notion of a fuzzy diameter for fuzzy metric spaces and obtain some properties on it and its relations to classical diameter for compact metric spaces. Moreover, using the notion of fuzzy diameter, we construct a fuzzy metric on the disjoint union of two fuzzy metric spaces. Finally, we consider the fuzzy product metric space of two fuzzy metric spaces and prove that there are fuzzy metrics on the product space such that the inclusions into the product space become fuzzy isometric embeddings. This shows that there is a family of fuzzy isometric embeddings from a fuzzy metric space into another fuzzy metric space.

Keywords: disjoint union, fuzzy contractive map, fuzzy diameter, fuzzy isometry, fuzzy metric space, fuzzy product metric space.

1. Introduction

Since the inception of fuzzy set theory [24], many authors introduced the notion of fuzzy metric spaces in different ways (cf. [2, 3, 16]). In particular, the notion of fuzzy metric space which is deeply related to the concept of probabilistic metric space given by Menger [18, 22] was introduced by Kramosil and Michalek in [15]. Later on, George and Veeramani [4] modified in a slight but appealing way the notion of fuzzy metric space of Kramosil and Michalek, and defined a Hausdorff and the first countable topology on the modified fuzzy metric space. This topology can be constructed on each fuzzy metric space in the sense of Kramosil and Michalek and it is metrizable [4, 7, 22]. Other recent contributions to the study of fuzzy metric spaces in the sense of [4] may be found in [5] and [8]. Related to fuzzy contractive maps between fuzzy metric spaces, many results on fixed point theory are known ([6, 9, 19, 20] and references therein). In particular, the authors introduced in [23] the notion of fuzzy dilation and minimal slope of a fuzzy mapping between fuzzy metric spaces and proved a new fixed point theorem. As an application of the fuzzy metric between fuzzy sets, Lee, Pedrycz and Sohn [17] have designed a similarity measure using a distance measure for fuzzy sets.

On the other hand, Gregori and Romaguera in [8] have introduced the concept of fuzzy isometry. Such a notion is natural in the sense that if a fuzzy metric on a space comes from a classical metric on a metric space, an isometry between classical metric spaces extends to an isometry between the fuzzy metric spaces. Gregori and Romaguera used this terminology to define a notion of completion of a fuzzy metric space. But they have not developed some of the properties related to fuzzy isometry itself. In fact, to our knowledge, few examples of fuzzy isometries or properties on fuzzy isometries have been published. In this paper, we construct several types of fuzzy metrics on the disjoint union of two fuzzy metric spaces so that the inclusions into the product space become fuzzy isometric embeddings. This shows that there is a family of fuzzy isometric embeddings from a fuzzy metric space into another fuzzy metric space.

The paper is organized as follows. In Section 2, we present definitions of fuzzy metric spaces in the sense of Kramosil and Michalek and in the sense of George and Veeramani, and describe some basic facts about fuzzy metric spaces as preliminaries. Then we construct a fuzzy metric on the disjoint union of two fuzzy metric spaces, and prove that there is a fuzzy metric space which does not admit fuzzy contractive maps.

In Section 3, we introduce the notion of fuzzy diameter of a fuzzy metric space and prove its relations with classical diameter for compact metric spaces. The diameter is one of the basic notions in compact metric spaces to investigate the topology of metric spaces. Moreover, using the notion of diameter, we show that there is another fuzzy metric on the disjoint union of two fuzzy metric spaces.

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In Section 4, we construct several types of fuzzy product metric spaces from fuzzy metric spaces having different continuous \( t \)-norms and prove the existence of a family of fuzzy isometric embeddings from a fuzzy metric space into another fuzzy product metric space.

2. Disjoint Union and Non-existence of Fuzzy Contractive Maps

We start with the definition of the continuous \( t \)-norm. The notion of triangular norms (briefly \( t \)-norm) was first introduced in the context of probabilistic metric space (cf. [22]) based on some ideas presented in [18]. Triangular norms have an important role in fuzzy metric spaces. There are well organized expository papers and a book on triangular norms including historical remarks [11]-[14].

Definition 2.1 (Schweizer and Sklar [22]): A binary operation \( * : [0,1] \times [0,1] \rightarrow [0,1] \) is a continuous \( t \)-norm if \(([0,1], *)\) is a commutative topological monoid with unit 1 such that \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) for \( a, b, c, d \in [0,1] \).

This implies that the binary operation \( * \) satisfies the following conditions:

(i) \( * \) is continuous
(ii) \( * \) is associative and commutative
(iii) \( a \ast 1 = a \) for every \( a \in [0,1] \)
(iv) \( a \ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) for \( a, b, c, d \in [0,1] \).

In particular, if \( * \) is a continuous \( t \)-norm, it follows from (iii) and (iv) that for every \( a \in [0,1] \),

\[
0 \ast a \leq 0 \ast 1 = 0 \quad \text{and} \quad 0 \ast a = a \ast 0 = 0 \quad \text{for any} \quad a \in [0,1].
\]

Kramosil and Michalek in [15] generalized the concept of probabilistic metric space given by Menger ([18], [22]) to the fuzzy framework as follows.

Definition 2.2 (Kramosil and Michalek [15]):
A 3-tuple \((X, M, *)\) is said to be a fuzzy metric space if \( X \) is an arbitrary set, \( * \) is a continuous \( t \)-norm, and \( M \) is a fuzzy set on \( X^2 \times [0,1) \) satisfying the following conditions:

(i) \( M(x, y, 0) = 0 \),
(ii) \( M(x, y, t) = 1 \) for all \( t > 0 \) if and only if \( x = y \),
(iii) \( M(x, y, t) = M(y, x, t) \),
(iv) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \),
(v) \( M(x, y) : [0, \infty) \rightarrow [0,1] \) is left-continuous, for all \( x, y, z \in X \) and \( t, s > 0 \).

\( M(x, y, t) \) can be thought of as the degree of nearness between \( x \) and \( y \) with respect to \( t \). We identify \( x = y \) with \( M(x, y, t) = 1 \) for all \( t > 0 \) and \( M(x, y, t) = 0 \) with \( \infty \).

In order to introduce a Hausdorff topology on fuzzy metric spaces, George and Veeramani in [4] modified in a slight but appealing way the notion of fuzzy metric space of Kramosil and Michalek.

Definition 2.3 (George and Veeramani [4]):
A 3-tuple \((X, M, *)\) is said to be a fuzzy metric space if \( X \) is a set, \( * \) is a continuous \( t \)-norm, and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \) satisfying the following conditions:

- for all \( x, y, z \in X \) and \( t, s > 0 \),
  (i) \( M(x, y, t) > 0 \),
  (ii) \( M(x, y, t) = 1 \) if and only if \( x = y \),
  (iii) \( M(x, y, t) = M(y, x, t) \),
  (iv) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \),
  (v) \( M(x, y, s : (0, \infty) \rightarrow [0,1]) \) is continuous.

In [4] and [5], George and Veeramani have shown that every fuzzy metric \((M, *)\) on a set \( X \) in the sense of George and Veeramani generates a topology \( \tau_M \). Thus, if \((X, M, *)\) is a fuzzy metric space in the sense of George and Veeramani, then we can consider the notion of compactness and boundedness of subsets of \( X \) with respect to the topology \( \tau_M \). It is known in [7] that a fuzzy metric space is compact if and only if it is precompact and complete in the sense of George and Veeramani ([4]). We will refer to these fuzzy metric spaces as KM fuzzy metric spaces and GV fuzzy metric spaces, respectively. From now on, a fuzzy metric space refers to a KM fuzzy metric space or a GV fuzzy metric space unless stated specifically.

Gregori and Romaguera in [8] has introduced the concept of fuzzy isometry. Such a notion is natural in the sense that if a fuzzy metric on a space comes from a classical metric on a metric space, an isometry between classical metric spaces extends to an isometry between fuzzy metric spaces.

Definition 2.4 (Gregori and Romaguera [8]):
Let \((X, M, *)\) and \((Y, N, \otimes)\) be two fuzzy metric spaces. A surjective map \( f : X \rightarrow Y \) is called an isometry if for each \( x_1, x_2 \in X \) and \( t > 0 \),

\[
N(f(x_1), f(x_2), t) = M(x_1, x_2, t).
\]

Note that if \( f : (X, M, *) \rightarrow (Y, N, \otimes) \) is an isometry, then \( f \) should be one-to-one. Two fuzzy metric spaces are called isometric if there exists an isometry \( f \) from \( X \) onto \( Y \).

For a fuzzy metric space \((X, M, *)\) and
\[ x \in X, t > 0, \text{ define } M_0(x, t) \text{ by } \]
\[ M_0(x, t) = \inf \{M(x, x', t) : x' \in X\}. \tag{2.1} \]

Note that since \( M(x, y, t) = M(y, x, t) \), we have \( \inf \{M(x', x, t) : x' \in X\} = M_0(x, t) \).

**Lemma 2.5:** Let \((X, M_0, *)\) be a fuzzy metric space. \(M_0(x, y, t) : (0, \infty) \rightarrow [0, 1]\) is nondecreasing. Furthermore, if \((X, M_0, *)\) is a compact \(t\)-fuzzy metric space, then for \(x \in X\) and for each \(t > 0\), there exists a point \(x_0 \in X\) such that \(M_0(x, x_0, t) = M_0(x, t)\).

**Proof:** The first statement follows from the fact that \(M(x, y, t) : (0, \infty) \rightarrow [0, 1]\) is nondecreasing for each \(x, y \in X\) in [6]. The second fact follows from [21].

**Example 2.6:** Let \((X, d)\) be a compact metric space. Define \(M_d(x, y, t) = \frac{t}{t + d(x, y)}\) for \(x, y \in X\) and \(t > 0\), and let \(a \ast b = ab\) for \(a, b \in [0, 1]\). Then it is well-known that \((X, M_d, *)\) is a fuzzy metric space, which is called the standard fuzzy metric space induced by the metric \(d\). It is easy to see that for any \(x \in X\), since \(d\) is continuous on \(X \times X\) and \(X\) is compact, there exists a point \(x' \in X\) such that \(d(x, x') = \sup \{d(x, y) : y \in X\}\). Thus
\[ M_0(x, t) = \frac{t}{t + d(x, y)} \geq \frac{t}{t + \text{diam}(X)}, \]
where \(\text{diam}(X)\) denotes the diameter of the metric space \(X\) defined by \(\text{diam}(X) = \sup \{d(x, y) : x, y \in X\}\).

Furthermore there exists a point \(x_0 \in X\) such that
\[ M_0(x_0, t) = \frac{t}{t + \text{diam}(X)} \]
for any \(t > 0\).

To introduce a metric on the disjoint union of two fuzzy metric spaces, we first define a continuous \(t\)-norm on the disjoint union. However, we do not obtain a continuous \(t\)-norm on the disjoint union in a natural way by joining the two given continuous \(t\)-norms appropriately because of associativity.

**Definition 2.7:** Let \(\ast\) and \(\otimes\) be two continuous \(t\)-norms on \([0, 1]\). We say that \(\ast\) is weaker than \(\otimes\), denoted by \(\ast \prec \otimes\), if for any \(a, b \in [0, 1]\), \(a \ast b \leq a \otimes b\).

**Example 2.8:** Define, for \(a, b \in [0, 1]\),
\[ a \ast b = \max\{0, a + b - 1\}, \quad a \otimes b = ab \]
and \(a \land b = \min\{a, b\}\). Then \(\ast, \otimes, \land\) are all continuous \(t\)-norms. It is easy to see \(\ast\) is weaker than \(\ominus\) and \(\otimes\) is weaker than \(\land\), i.e., \(\ast \prec \ominus \prec \land\). In fact, it is obvious that \(a + b - 1 \leq ab \leq \min\{a, b\}\).

For two fuzzy metric spaces \((X, M_0, \ast)\) and \((Y, N_0, \otimes)\), let \(Z = X \coprod Y\) be the disjoint union. Define \(U : Z^2 \times (0, \infty) \rightarrow [0, 1]\) by
\[ U(x, y, t) = \begin{cases} M(x, y, t), & x, y \in X \\ N(x, y, t), & x, y \in Y \end{cases} \tag{2.2} \]
For \(x \in X, y \in Y\), define \(U(x, y, t) = U(x, y, t)\).

The following theorem shows that \((Z = X \coprod Y, U, \ast)\) is in fact a fuzzy metric space if \(\ast\) is weaker than \(\ominus\) and \(X\) and \(Y\) are compact with respect to the topologies generated by the metrics \((M_0, \ast)\) and \((N_0, \otimes)\), respectively. In the aspect of metric spaces, this type of theorem is meaningful. One basic and easy method of investigating properties on compact metric spaces is to embed a given metric space into another bigger metric space. In fact, any metric space can be embedded naturally into the disjoint union of the metric space and another metric space if the disjoint union of two metric spaces has a natural metric.

**Theorem 2.9:** Let \((X, M, \ast)\) and \((Y, N, \otimes)\) be two compact fuzzy metric spaces. If \(\ast\) is weaker than \(\ominus\), then \((Z = X \coprod Y, U, \ast)\) with (2.2) is a fuzzy metric space.

**Proof:** It is easy to check that \(U(x, y, t)\) satisfies the conditions (i)-(iii) and (v) in Definition 2.2 or Definition 2.3. It remains to be shown that \(U(x, y, t)\) satisfies the triangle inequality (iv) in Definition 2.2 or Definition 2.3. In cases \(x, y, z \in X\) or \(x, y, z \in Y\), it is obvious that \(U(x, y, z)\) satisfies the triangle inequality since both \(M(x, y, t)\) and \(N(x, y, t)\) satisfy the triangle inequality. Now assume \(x, z \in X\) and \(y \in Y\). Since \(X\) and \(Y\) are compact, it follows from Lemma 2.5 that for \(t, s > 0\), there exist \(z_0 \in X\) such that
\[ M_0(x, t + s) = M(X, z_0, t + s) \]
Then we have
\[ M_0(x, z, t) \ast \min\{M_0(z, s), N_0(y, s)\} \leq M(x, z, t) \ast M(z, z_0, s) \]
\[ \leq M(x, z, t) \ast \min\{M_0(z, s), N_0(y, s)\} \leq M_0(x, t + s). \]
Also since \(\ast\) is weaker than \(\ominus\), we have
\[ M(x, z, t) \ast \min\{M_0(z, s), N_0(y, s)\} \]
\[ \leq M(x, z, t) \otimes N_0(y, s) \leq N_0(y, s) \leq N_0(y, t + s). \]
Thus,
\[
U(x, z, t) * U(z, y, s) = \min \{M_0(x, t + s), N_0(y, t + s)\} = U(x, y, t + s).
\]
Next, let \( M_0(x, t) = M(x, x_0, t) \) and \( M_0(z, s) = M(z, z', s) \) for some \( x_0, z' \in X \). Since \( M(z, z', s) = \inf \{M(z, x', s)|x' \in X\} \leq M(z, x_0, s) \),
\[
M_0(x, t) * M_0(z, s) = M(x, x_0, t) * M(z, z', s) \leq M(x, x_0, t) * M(z, s, z, s) \leq M(x, z, t + s).
\]
Moreover if \( N_0(y, s) \leq M_0(z, s) \), then
\[
M_0(x, t) * N_0(y, s) \leq M_0(x, t) * M_0(z, s) \leq M(x, z, t + s).
\]
Thus, we have
\[
U(x, y, t) * U(y, z, s) = \min \{M_0(x, t), N_0(y, t)\} * \min \{M_0(z, s), N_0(y, s)\} \leq M (x, z, t + s) = U(x, y, t + s).
\]
Hence \( U(x, y, t) \) satisfies the triangle inequality (iv).

When \( x \in X \) and \( y, z \in Y \), the same argument shows that \( U(x, y, t) \) satisfies the triangle inequality.

The metric \( U(x, y, t) \) defined by (2.2) is called the joined metric determined by \( M \) and \( N \).

**Corollary 2.10:** Let \((X, M, *)\) and \((Y, N, \otimes)\) be two compact fuzzy metric spaces. If \( \otimes \) is weaker than \(*\), then \((X, Y, U, *)\) with (2.2) is a fuzzy metric space.

**Corollary 2.11:** Let \((X, M, *)\) and \((Y, N, \otimes)\) be two compact fuzzy metric spaces with the same continuous \( t \)-norm. Then \((X, Y, U, *)\) with (2.2) is a fuzzy metric space.

**Example 2.12:** Let \( X = \{x_1, x_2, x_3\} \) with discrete metric \( \rho(x_i, x_j) = \begin{cases} 1 & \text{if } x_i = x_j \\ 0 & \text{if } x_i \neq x_j \end{cases} \)
Define
\[
M(x_i, x_j, t) = \frac{t}{t + \rho(x_i, x_j)}
\]
and let \( Y = \{p\} \) be a single point with \( N = 1 \). If \(*\) is the multiplicative continuous \( t \)-norm, i.e., \( ab = ab \), then \((X, M, *)\) and \((Y, N, \otimes)\) are fuzzy metric spaces. Let \( U(x, y, t) \) be the joined metric given by (2.2) so that \((X, Y, U, *)\) is a fuzzy metric space. Note that since for \( x \in X \),
\[
M_0(x, t) = \inf \{M(x, x', t)|x' \in X\} = \frac{t}{t + 1} < 1
\]
and \( N_0 = N = 1 \), we have
\[
U(x, y, t) = \begin{cases} M(x, y, t) & x, y \in X \\ \frac{t}{t + 1} & x \in X, y \in Y \end{cases}
\]

**Definition 2.13:** Let \((X, M, *)\) and \((Y, N, \otimes)\) be two fuzzy metric spaces. A map \( f : X \rightarrow Y \) is said to be an isometric embedding if \( f \) is one-to-one and for each \( x = (0) \),
\[
N(f(x_n), f(x_{n+1}), t) = M(x_n, x_{n+1}, t).
\]

**Example 2.14:** Let \((X, M, *)\) and \((Y, N, \otimes)\) be two compact fuzzy metric spaces. Assume \(*\) is weaker than \( \otimes \) so that \((X, Y, U, *)\) with (2.2) is a fuzzy metric space by Theorem 2.9. It is easy to see that the inclusions
\[
t_x : X \rightarrow Z, \quad t_y (x) = x
\]
and
\[
t_y : Y \rightarrow Z, \quad t_y (y) = y
\]
are fuzzy isometric embeddings.

Next we shall consider fuzzy contractive maps and fixed point theory for maps between fuzzy metric spaces.

**Definition 2.15** [9]: Let \((X, M, *)\) be a fuzzy metric space.

1. A map \( f : X \rightarrow X \) is said to be fuzzy contractive if there exists \( k \in (0, 1) \) such that
\[
\frac{1}{M(f(x), f(y), t)} - 1 \leq k \left( \frac{1}{M(x, y, t)} - 1 \right)
\]
for each \( x, y \in X \) and \( t > 0 \). In this case \( k \) is called the contractive constant of \( f \).

2. A sequence \((x_n)\) in \( X \) is said to be fuzzy contractive if there exists \( k \in (0, 1) \) such that
\[
\frac{1}{M(x_n, x_{n+1}, t)} - 1 \leq k \left( \frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \right)
\]
for all \( t > 0, n \in N \).

**Definition 2.16** [4]: A sequence \((x_n)\) in a fuzzy metric space \((X, M, *)\) is Cauchy if for each \( \varepsilon \in (0, 1) \) and each \( t > 0 \), there exists \( n_0 \in N \) such that
\[
M(x_n, x_m, t) > 1 - \varepsilon, \text{ for all } n, m \geq n_0.
\]

Gregori and Sapena extended the Banach fixed point theorem to fuzzy contractive maps of complete fuzzy metric spaces.

**Theorem 2.17** [9]: Let \((X, M, *)\) be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. If \( f : X \rightarrow X \) is a fuzzy contractive map, then
\[ f \text{ has a unique fixed point.} \]

If \((X,d)\) is a classical complete metric space and \((X,M_d,\ast)\) is the standard fuzzy metric space induced by the metric \(d\) on \(X\) with \(a\ast b = ab\), then the sequence \((x_n)\) in \(X\) is contractive in \((X,d)\) if and only if \((x_n)\) is fuzzy contractive in \((X,M_d,\ast)\). Thus, we have the following corollary.

**Corollary 2.18** [9]: Let \((X,M_d,\ast)\) be the standard fuzzy metric space and let \(f : X \to X\) be a fuzzy contractive map. Then \(f\) has a unique fixed point.

Applying this result to the disjoint union of two fuzzy metric spaces constructed in Theorem 2.7, we can show that there do not exist fuzzy contractive maps on the disjoint union of two fuzzy metric space with the fuzzy metric defined by (2.2).

Let \((X,M,\ast)\) and \((Y,N,\otimes)\) be two compact fuzzy metric spaces such that \(\ast\) is weaker than \(\otimes\). Then it follows from Theorem 2.9 that the disjoint union \((Z = X \coprod Y, M, \ast)\) is a fuzzy metric space. If \((x_n)\) and \((y_n)\) are sequences in \(X\) and \(Y\), respectively which are fuzzy contractive, then it is easy to see that both \((x_n)\) and \((y_n)\) are also fuzzy contractive in \(Z = X \coprod Y\) with the same contractive constant by Example 2.14.

**Lemma 2.19**: Let \((X,M,\ast)\) and \((Y,N,\otimes)\) be two compact fuzzy metric spaces such that \(\ast\) is weaker than \(\otimes\). Let \((X \coprod Y, U, \ast)\) be the disjoint union of compact fuzzy metric spaces such that \(\ast\) is weaker than \(\otimes\). Let \((X \coprod Y, U, \ast)\) be the disjoint union.

If an any fuzzy contractive sequence in \(X\) and \(Y\) is Cauchy, then any fuzzy contractive sequence in \((X \coprod Y, U, \ast)\) is Cauchy.

**Proof**: Let \((z_n)\) be a fuzzy contractive sequence in the disjoint union \((X \coprod Y, U, \ast)\) with contractive constant \(k\) so that

\[
\frac{1}{U(z_{n+1}, z_{n+2}, t)} - 1 \leq k \left( \frac{1}{U(z_n, z_{n+1}, t)} - 1 \right) \quad (2.3)
\]

for all \(t > 0, n \in N\). If there exists a positive integer \(n_0 \in N\) such that \(z_n \in X\) for all \(n \geq n_0\), by letting \(x_n = z_{n+n_0}\), \((x_n)\) is a fuzzy contractive sequence in \(X\) with contractive constant \(k\). By assumption, \((x_n)\) is Cauchy in \(X\) and so for each \(\varepsilon \in (0,1)\) and each \(t > 0\), there exists \(n_1 \in N\) such that

\[
M(x_n, x_m, t) = U(x_n, x_m, t) > 1 - \varepsilon
\]

for all \(n, m \geq n_1\). Hence \((z_n)\) is Cauchy in \(X \coprod Y\).

Now assume that both \((z_n) \cap X\) and \((z_n) \cap Y\) are infinite sets. We may assume, without loss of generality, that \(z_n, z_{n+2}, z_{n+3} \in X\) and \(z_{n+1} \in Y\). Then

\[
U(z_n, z_{n+1}, t) = \min\{d_M(z_n, t), N_0(z_{n+1}, t)\}
\]

by definition, and there exist \(x \in X\) and \(y \in Y\) such that

\[
M_0(z_n, t) = M(y, x, t) \quad \text{and} \quad N_0(z_{n+1}, t) = N(z_{n+1}, y, t)
\]

respectively. If \(M_0(z_n, t) \leq N_0(z_{n+1}, t)\), then we can see from (2.3) that the triple \(z_n, x, z_{n+2} \in X\) satisfy

\[
\frac{1}{M(x, z_{n+2}, t)} - 1 \leq k \left( \frac{1}{M(z_n, x, t)} - 1 \right) \quad (2.4)
\]

If \(M_0(z_n, t) > N_0(z_{n+1}, t)\), then

\[
M_0(z_n, t) < N_0(z_{n+1}, t)
\]

Since \(U(z_n, z_{n+1}, t) < U(z_{n+2}, z_{n+1}, t)\) from (2.3). Thus, if \(M_0(z_n, t) = M(z_{n+2}, x', t)\) for some \(x' \in X\), then

\[
M(z_{n+2}, x', t) < N_0(z_{n+1}, t)
\]

\[
M_0(z_n, t) = M(z_n, x, t) \leq \inf\{M(z_n, x', t) : x' \in X\}
\]

\[
< M(z_n, z_{n+2}, t)
\]

Therefore, the triple \(z_n, z_{n+2}, x' \in X\) satisfy the inequality (2.4). Consequently in case that both \((z_n) \cap X\) and \((z_n) \cap Y\) are infinite sets, we can construct a sequence in \(X\) and a sequence in \(Y\) by adding appropriate points which become fuzzy contractive sequences in \(X\) and \(Y\), respectively. Hence, by assumption, the fuzzy contractive sequence \((z_n)\) in the disjoint union \((X \coprod Y, U, \ast)\) is Cauchy.

**Theorem 2.20**: Let \(f : (X, M, \ast) \to (X, M, \ast)\) and \(g : (Y, N, \otimes) \to (Y, N, \otimes)\) be maps between compact fuzzy metric spaces. Assume any fuzzy contractive sequence in \(X\) and \(Y\) is Cauchy and \(\ast\) is weaker than \(\otimes\). Then the map \(h : (X \coprod Y, U, \ast) \to (X \coprod Y, U, \ast)\) with (2.2) defined by

\[
h(z) = \begin{cases} f(z) & z \in X \\ g(z) & z \in Y \end{cases}
\]

cannot be fuzzy contractive with any constant \(k \in (0,1)\).

**Proof**: Suppose that \(h : (X \coprod Y, U, \ast) \to (X \coprod Y, U, \ast)\) is a fuzzy contractive map. It follows from Lemma 2.19 that any fuzzy contractive sequence in \(X \coprod Y\) is Cauchy and so it follows from Theorem 2.17 that \(h\) has a unique fixed point. On the other hand, since \(h\) is
a fuzzy contractive map, so are both \( f \) and \( g \) with the same contractive constant. Thus it follows also from Theorem 2.17 that \( f \) and \( g \) have unique fixed points \( x_0 \) and \( y_0 \), respectively. Then \( h \) has at least two fixed points in \( X \bigcup Y \), which is a contradiction.

**Theorem 2.21:** There exists a fuzzy metric space such that for any \( k \in (0,1) \) there are no fuzzy contractive maps on it with contractive constant \( k \).

**Proof:** Take two compact metric spaces \( (X,d_X) \) and \( (Y,d_Y) \) and let \( (X,M_{d_X,*}) \) and \( (Y,M_{d_Y,*}) \) be the standard fuzzy metric spaces induced by \( d_X \) and \( d_Y \), respectively, where \( a*b = ab \). Let \( Z = X \bigcup Y \) with \( U \) in (2.2).

Suppose \( h: (Z,U,*) \rightarrow (Z,U,*) \) is a fuzzy contractive map. Defining \( f: X \rightarrow X \) and \( g: Y \rightarrow Y \) by \( f(x) = h(x) \) and \( g(y) = h(y) \), respectively, the maps \( f: X \rightarrow X \) and \( g: Y \rightarrow Y \) are fuzzy contractive with the same contractive constant as \( h \). It follows from Theorem 2.20 that \( h \) cannot be fuzzy contractive which is a contradiction.

### 3. Fuzzy Diameter

Recall that if \( (X,M,*) \) is a fuzzy metric space, then the ball \( B(x,r,t) \) with center \( x \in X \) and radius \( r, 0 < r < 1, t > 0 \) defined by

\[
B(x,r,t) = \{ y \in X : M(x,y,t) > 1 - r \}
\]

is an open set with respect to the topology \( \tau_M \) generated by the fuzzy metric \( (M,*) \) on \( X \) (cf. [4]). From this fact, we introduce the notion of fuzzy diameter for fuzzy metric spaces.

**Definition 3.1:** Let \( (X,M,*) \) be a fuzzy metric space. For \( t > 0 \), we define \( diam_t(X) \) by

\[
diam_t(X) = \text{sup}\{1 - M(x,y,t): x, y \in X\}
\]

and call it the fuzzy diameter of \( X \) with \( t > 0 \).

Note that by definition, we have \( diam_0(X) = 1 \) and

\[
\lim_{t \to 0} diam_t(X) = 1, \quad \text{and for} \quad 0 < t_1 < t_2, \quad diam_{t_1}(X) \geq diam_{t_2}(X).
\]

**Example 3.2:** Consider the fuzzy metric spaces induced by a discrete metric space which is compact and a standard metric on \( R \), respectively.

1. Let \( X = \{ x_1, \ldots, x_n \} \) be a finite set with the discrete metric \( \rho \) defined by

\[
\rho(x_i, x_j) = \begin{cases} 1 & (x_i = x_j) \\ 0 & (x_i \neq x_j). \end{cases}
\]

Let \( M(x,y,t) \) be the standard fuzzy metric induced by \( \rho \) with multiplicative \( t \)-norm \( * \) so that \( (X,M,*) \) is a fuzzy metric space. Then it is easy to see that

\[
diam_t(X) = \frac{1}{t+1}.
\]

2. Let \( X = R \) be the real numbers with the metric \( d(x,y) = |x - y| \) and let \( M(x,y,t) = \frac{t}{t+d(x,y)} \) be the standard fuzzy metric induced by \( d \) with multiplicative \( t \)-norm \( * \). Then for each \( t > 0 \), we have

\[
diam_t(R) = 1.
\]

It follows from Example 3.2 that the diameter of a discrete fuzzy metric space with a standard fuzzy metric is less than 1 and for a noncompact metric space, the diameter of the induced fuzzy metric space is equal to 1. This means that on a noncompact fuzzy metric space induced from a noncompact metric space, for a given large positive real number \( \lambda \), there are points whose distance is almost equal to \( \lambda \).

For a compact fuzzy metric space, we have the following theorem.

**Theorem 3.3:** Let \( (X,M,*) \) be a fuzzy metric space which is compact with respect to the topology induced by the fuzzy metric. Then there exists a \( t > 0 \) such that \( diam_t(X) < 1 \).

**Proof:** Since \( X \) is compact, it is in particular F-bounded in the sense of [4], i.e., there exists a \( t > 0 \) and \( 0 < r_0 < 1 \) such that \( M(x,y,t) > 1 - r_0 \) for all \( x, y \in X \). Thus,

\[
diam_t(X) \leq r_0 < 1.
\]

**Theorem 3.4:** Let \( (X,d) \) be a compact metric space and let \( M(x,y,t) \) be the standard fuzzy metric space induced by \( d \) with multiplicative \( t \)-norm \( * \). Then

\[
diam_t(X) = \frac{diam(X)}{t+diam(X)},
\]

where \( diam(X) = diam(X,d) \) is the classical diameter of the metric space \( (X,d) \).

If the metric space \( (X,d) \) is noncompact, then we can show similarly to Example 3.2 that \( diam(X) = 1 \). Using the notion of fuzzy diameter for fuzzy metric spaces, we can construct a fuzzy metric on the disjoint union of the two given fuzzy metric spaces.

**Theorem 3.5:** Let \( (X,M,*) \) and \( (Y,N,\otimes) \) be two
fuzzy metric spaces. Let \( Z = X \sqcup Y \) be the disjoint union. For \( x, y \in Z \) and \( t > 0 \), define \( V(x, y, t) \) by

\[
V(x, y, t) = \begin{cases} 
\frac{M(x, y, t)}{N(x, y)} & x, y \in X \\
\min\{1 - \text{diam}(X), 1 - \text{diam}(Y)\} & x \in X, y \in Y
\end{cases}
\]  

(3.1)

and \( V(y, x, t) = V(x, y, t) \) for \( x \in X \) and \( y \in Y \).

(1) If * is weaker than \( \otimes \), then \((Z, V, *)\) is a fuzzy metric space.

(2) If \( \otimes \) is weaker than *, then \((Z, V, \otimes)\) is a fuzzy metric space.

**Proof:** As in the proof of Theorem 2.9, it suffices to show that \( V \) satisfies the triangle inequality.

(1) Note that for \( 0 < t_1 < t_2 \),

\[
1 - \text{diam}_1(X) \leq 1 - \text{diam}_2(X)
\]

and

\[
1 - \text{diam}_1(Y) \leq 1 - \text{diam}_2(Y).
\]

This implies that for \( x, z \in X, y \in Y \) and \( t, s > 0 \),

\[
V(x, z, t)*V(z, y, s) \leq V(x, y, t + s).
\]

Also note that \( 1 - \text{diam}(X) \leq M(x, z, t) \) for any \( x, z \in X \). It follows from this fact that

\[
V(x, y, t)*V(y, z, s) \leq V(x, z, t + s) = M(x, z, t + s).
\]

Hence \( V \) satisfies the triangle inequality.

(2) The proof is similar to that of (1).

Let \((X, d_X)\) and \((Y, d_Y)\) be two compact metric spaces and let \( Z = X \sqcup Y \) be the disjoint union. For \( x, y \in Z \), define \( d(x, y) \) by

\[
d(x, y) = \begin{cases} 
d_X(x, y) & x, y \in X \\
d_Y(x, y) & x, y \in Y \\
\max\{\text{diam}(X), \text{diam}(Y)\} & x \in X, y \in Y
\end{cases}
\]

Then \( d \) is a metric on \( Z = X \sqcup Y \). Let \( M_d(x, y, t) \) be the standard fuzzy metric on \( Z \) induced by \( d \) with multiplicative \( t \)-norm *.

Now let \( M(x, x', t) \) and \( N(y, y', t) \) be the standard fuzzy metrics on \( X \) and \( Y \) induced by \( d_X \) and \( d_Y \), respectively, with multiplicative \( t \)-norm *.

Let \( Z = X \sqcup Y \) be the disjoint union with the fuzzy metric \( V \) defined by (3.1). The following theorem shows that these two fuzzy metric spaces are fuzzy isometric.

**Theorem 3.6:** \((Z, M_d, *)\) and \((Z, V, *)\) are fuzzy isometric.

**Proof:** Let \( f : Z \to Z \) be the identity map. If \( x, y \in X \) or \( x, y \in Y \), it is obvious that

\[
M_d(x, y, t) = V(x, y, t) = \frac{t}{t + d_X(x, y)}
\]
or

\[
M_d(x, y, t) = V(x, y, t) = \frac{t}{t + d_Y(x, y)}.
\]

Now assume \( x \in X \) and \( y \in Y \) so that

\[
M_d(x, y, t) = \frac{t}{t + \max\{\text{diam}(X), \text{diam}(Y)\}}
\]

and \( V(x, y, t) = \min\{1 - \text{diam}(X), 1 - \text{diam}(Y)\} \).

Since, by Theorem 3.4

\[
\min\{1 - \text{diam}(X), 1 - \text{diam}(Y)\}
\]

we have

\[
M_d(x, y, t) = V(x, y, t) \quad \text{for} \quad x \in X \quad \text{and} \quad y \in Y.
\]

Next we shall prove that on the disjoint union of two fuzzy metric spaces, there exists another fuzzy metric other than the fuzzy metrics discussed above. Let \((X, M, *)\) and \((Y, N, \otimes)\) be fuzzy metric spaces and assume that both \( X \) and \( Y \) contain at least two elements. Choose \( m \) elements \( x_1, x_2, \ldots, x_m \) in \( X \) and \( m \) elements \( y_1, y_2, \ldots, y_m \) in \( Y \) \((m \geq 2)\), respectively and define

\[
W(x, y, t) = \begin{cases} 
M(x, y, t) & x, y \in X \\
N(x, y) & x, y \in Y \\
\min\{M(x, y, t), N(x, y, t)\} & x \in X, y \in Y
\end{cases}
\]

and \( W(x, y, t) = W(x, y, t) \) for \( x \in X, y \in Y \).

The following theorem shows that the disjoint union \( Z = X \sqcup Y \) together with \( W \) is a fuzzy metric space.

**Theorem 3.7:** \((Z = X \sqcup Y, W, *)\) is a fuzzy metric space if * is weaker than \( \otimes \).

**Proof:** It is easy to see that \( W \) satisfies \((i) - (iii)\) and \((v)\) in Definition 2.3. It is enough that \( W \) satisfies the triangle inequality. For \( x, z \in X \) and \( y \in Y \), and for any \( i = 1, 2, \ldots, m \),

\[
M(x, z, t)*M(z, x_i, s) \leq M(x, x_i, t + s)
\]

and

\[
M(x, z, t)*N(y, y_i, s) \leq N(y, y_i, s) \leq N(y, y_i, t + s).
\]

So,
\( W(x,z,t)\ast W(z,y,s) \leq \min \{ M(x,x,t), N(y,y,t) \} = W(x,y,t+s) \).

Next, if \( M(x,x,t) = W(x,y,t) \) and \( M(z,z,s) = W(z,y,s) \), then
\[ M(x,x,t) \ast M(z,z,s) \leq M(x,x,t) \ast M(z,z,s) \leq M(x,z,t+s). \]

In the other three cases, we can easily show that \( W \) satisfies the triangle inequality and so we have
\[ W(x,y,t) \ast W(z,y,s) \leq W(x,z,t+s) = M(x,z,t+s). \]

**Remark 3.8:** Using similar arguments as in Theorem 2.19 and Theorem 2.20, we can also prove that any fuzzy contractive sequence in \( X \) in a metric space \( (X,d) \), if we define \( a \ast b = \min \{a,b\} \) and multiplicative \( a \ast b = ab \). These continuous \( t \)-norms can be used to make a fuzzy metric space from a given classical metric space. For instance, in a metric space \( (X,d) \), we define \( a \ast b = ab \) and
\[ M(x,y,t) = \frac{t}{t + d(x,y)}, \]
then \((X,M,\ast)\) is a fuzzy metric space. Even if we take \( a \ast b = \min \{a,b\} \), \((X,M,\ast)\) still becomes a fuzzy metric space ([4]). Using these continuous \( t \)-norms, we can construct fuzzy product metric spaces from two given fuzzy metric spaces. Let \((X,M,\ast)\) is a fuzzy metric space and let
\[ X \times X = \{(x,y): x, y \in X\}. \]

For \((x,y),(u,v) \in X \times X\) and \( t > 0 \), define
\[ P((x,y),(u,v),t) \]
by
\[ P((x,y),(u,v),t) = \min \{ M(x,u,t), M(y,v,t) \}. \]

Then we can easily show that \( P((x,y),(u,v),t) \) satisfies the conditions in Definition 2.3 and so it defines a metric on \( X \times X \) with the same continuous \( t \)-norm *.

In fact, it is easy to see that \( P((x,y),(u,v),t) > 0 \) for any \((x,y),(u,v) \in X \times X \) and for \( t > 0 \). If \( P((x,y),(u,v),t) = 1 \), then \( M(x,u,t) = 1 = M(y,v,t) \) and so \((x,y) = (u,v)\) and vice versa. Also, the metric
\[ M \]
is symmetric on \( X \), so is \( P \) on \( X \times X \), i.e.,
\[ P((x,y),(u,v),t) = P((u,v),(x,y),t) \text{ for } t > 0. \]

On the other hand, for \((x,y),(u,v),(a,b) \in X \times X \), and for \( t, s > 0 \), it follows from the definition of \( M \) and continuous \( t \)-norm * that
\[ M(x,u,t) \ast M(u,a,s) \leq M(x,a,t+s) \]
and
\[ M(y,v,t) \ast M(v,b,s) \leq M(y,b,t+s). \]

If \( M(v,b,s) \leq M(u,a,s) \), then
\[ M(x,u,t) \ast M(v,b,s) \leq M(x,u,t) \ast M(u,a,s) \leq M(x,a,t+s) \]
and if \( M(u,a,s) \leq M(y,v,t) \), then
\[ M(x,u,t) \ast M(v,b,s) \leq M(y,v,t) \ast M(v,b,s) \leq M(y,b,t+s). \]

Thus, in any case, we obtain
\[ \min \{ M(x,u,t), M(y,b,s) \} = \min \{ M(a,u,s), M(v,b,s) \} \]
\[ \leq \min \{ M(x,a,t+s), M(y,b,t+s) \} \]
\[ = P((x,y),(a,b),t+s). \]

Finally, for each \((x,y),(u,v) \in X \times X\), the map
\[ P((x,y),(u,v),t): (0, \infty) \rightarrow [0,1] \]
is continuous.

Therefore, \((X \times X, P,\ast)\) is a fuzzy metric space. We call \((X \times X, P,\ast)\) the fuzzy product metric space of \((X,M,\ast)\). If we define
\[ Q((x,y),(u,v),t) = M(x,u,t) \cdot M(y,v,t), \]
where \( \cdot \) is the usual product, then we can easily show that \((X \times X, Q,\ast)\) is also a fuzzy metric space.

Next, we shall consider how to construct fuzzy product space from given two distinct fuzzy metric spaces \((X,M,\ast)\) and \((Y,N,\otimes)\). In this case, we can define for \((x,y),(u,v) \in X \times Y\) and for \( t > 0 \)
\[ Q((x,y),(u,v),t) = M(x,u,t) \cdot N(y,v,t), \]
the product \((X \times Y, P,\ast)\) or \((X \times Y, Q,\ast)\) is not usually a fuzzy metric space.

However, if one of the continuous \( t \)-norms * or \( \otimes \) is weaker than the other, then the following theorem shows that \( P((x,y),(u,v),t) \) satisfies the triangle inequality and so the product \((X \times Y, P)\) becomes a fuzzy metric space with weaker continuous \( t \)-norm.

**Theorem 4.1:** Let \((X,M,\ast)\) and \((Y,N,\otimes)\) be two fuzzy metric spaces and for \((x,y),(u,v) \in X \times Y\) and for \( t > 0 \)
\[ P((x,y),(u,v),t) = \min \{ M(x,u,t), N(y,v,t) \}. \]

(1) If * is weaker than \( \otimes \), then \((X \times Y, P,\ast)\) is a
fuzzy metric space.

(2) If $\otimes$ is weaker than $\ast$, then $(X \times Y, P, \otimes)$ is a fuzzy metric space.

Proof: Let $(x, y), (u, v), (a, b) \in X$. If
\[
\min \{M(x, u, t), N(y, v, t)\} = M(x, u, t)
\]
and
\[
\min \{M(u, a, s), N(v, b, s)\} = M(u, a, s)
\]
for $t, s > 0$, then
\[
M(x, u, t) \ast M(u, a, s) \leq M(x, a + t + s).
\]
Since $\ast$ is weaker than $\otimes$, we also have from the definition of a continuous $t$-norm
\[
M(x, u, t) \ast M(u, a, s) \leq N(y, v, t) \ast N(v, b, s)
\]
for $t, s > 0$. Thus we get
\[
\min \{M(x, u, t), N(y, v, t)\} \ast \min \{M(u, a, s), N(v, b, s)\} \leq \min \{M(x, a + t + s), N(y, b + t + s)\}
\]
and so
\[
P((x, y), (a, b), s) \leq P((x, y), (a, b), t) + s.
\]
For the other three cases
(i). \[
\min \{M(x, u, t), N(y, v, t)\} = M(x, u, t)
\]
\[
\min \{M(u, a, s), N(v, b, s)\} = N(v, b, s),
\]
(ii). \[
\min \{M(x, u, t), N(y, v, t)\} = N(y, v, t)
\]
\[
\min \{M(u, a, s), N(v, b, s)\} = M(u, a, s),
\]
(iii). \[
\min \{M(x, u, t), N(y, v, t)\} = N(y, v, t)
\]
\[
\min \{M(u, a, s), N(v, b, s)\} = N(v, b, s),
\]
We can similarly show that
\[
P((x, y), (a, b), s) \leq P((x, y), (a, b), t) + s.
\]
Hence $(X \times Y, P, \ast)$ is a fuzzy metric space.

If $\otimes$ is weaker than $\ast$, we can similarly show that $(X \times Y, P, \otimes)$ is a fuzzy metric space.

Theorem 4.2: Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces and let $\ast$ be weaker than $\otimes$. If $(x_n)$ and $(y_n)$ are fuzzy contractive sequences in $X$ and $Y$ with the same contractive constant $k \in (0, 1)$, respectively, then the sequence $z_n = (x_n, y_n)$ is a fuzzy contractive sequence in $(X \times Y, P, \ast)$ with $(4.3)$.

Proof: It is easy to see that
\[
P((x_{n+1}, y_{n+1}), (x_{n+2}, y_{n+2}, t) = 1
\]
\[
= \min \{M(x_{n+1}, x_{n+2}, t), N(y_{n+1}, y_{n+2}, t)\} - 1
\]
\[
\leq k \left( \frac{1}{\min \{M(x_{n+1}, x_{n+2}, t), N(y_{n+1}, y_{n+2}, t)\}} - 1 \right).
\]

Theorem 4.3: Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces and let $\ast$ be weaker than $\otimes$. If $f : X \to X$ and $g : Y \to Y$ are fuzzy contractive maps with the same contractive constant, then the map $h : (X \times Y, P, \ast) \to (X \times Y, P, \ast)$ defined by $h(x, y) = (f(x), g(y))$ is fuzzy contractive.

Proof: Let $k$ be the contractive constant of $f$ and $g$.

Then it is easy to see that
\[
\frac{1}{P(h(x, y), h(x', y'), t)} - 1 = \frac{1}{\min \{M(f(x), f(x'), t), N(g(y), g(y'), t)\}} - 1
\]
\[
\leq k \left( \frac{1}{\min \{M(x, x', t), N(y, y', t)\}} - 1 \right).
\]

For metric spaces $(X, d_X), (Y, d_Y)$, let $X \times Y$ be the product metric space with the metric square norm. Then for fixed points $x_0 \in X$ and $y_0 \in Y$, respectively, it is easy to see that the inclusion maps $X \to X \times Y, x \to (x, y_0)$ and $Y \to X \times Y, y \to (x_0, y)$ are isometries, respectively. The following theorem shows that this property still holds for the fuzzy product metric space obtained from two fuzzy metric spaces. This also shows that there exists a family of fuzzy isometries between fuzzy metric spaces. To the authors' knowledge, not many examples for fuzzy isometries are known so far.

Theorem 4.4: Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces and let $\ast$ be weaker than $\otimes$. Let $(X \times Y, P, \ast)$ be the fuzzy metric space with $(4.1)$. Then for any point $y_0 \in Y$ fixed, the map
\[
f : X \to X \times Y, f(x) = (x, y_0)
\]
is a fuzzy isometric embedding of $(X, M, \ast)$ into $(X \times Y, P, \ast)$.

Proof: For $x, x' \in X$, since $N(y_0, y_0, t) = 1$, it follows that
\[
P(f(x), f(x'), t) = P((x, y_0), (x', y_0), t) = \min \{M(x, x', t), N(y_0, y_0, t)\}
\]
\[
= \min \{M(x, x', t), N(y_0, y_0, t)\} = M(x, x', t).
\]

Corollary 4.5: Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces and suppose that $\ast$ is weaker than $\otimes$. Let $(X \times Y, P, \ast)$ be the fuzzy metric space with $(4.1)$. Then for any point $x_0 \in X$ that is fixed, the map
\[
g : Y \to X \times Y, g(y) = (x_0, y)
\]
is a fuzzy isometric embedding of $(Y, N, \ast)$ into $(X \times Y, P, \ast)$.

When $Q((x, y), (u, v), t) = M(x, u, t) \cdot N(y, v, t)$, under the additional condition about $\ast$ which is similar to the
notion of weakness, we can show that $Q$ satisfies the triangle inequality and so $(X \times Y, Q, \ast)$ is a fuzzy metric space.

**Proposition 4.6:** Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces and for $(x, y), (u, v) \in X \times Y$ and for $t > 0$ let

$$Q((x, y), (u, v), t) = M(x, u, t) \cdot N(y, v, t).$$

If for any $a, b, c, d \in [0, 1]$

$$(a - b) \cdot (c - d) \leq (a - c) \cdot (b - d),$$

then $(X \times Y, Q, \ast)$ is a fuzzy metric space.

**Proof:** Straightforward computation together with (4.4) shows that $Q((x, y), (u, v), t)$ satisfies the triangle inequality and so $(X \times Y, Q, \ast)$ is a fuzzy metric space.

The condition (4.4) holds for the multiplicative continuous $t$-norm $a \ast b = a \cdot b$, but the minimum continuous $t$-norm $a \ast b = \min\{a, b\}$ does not satisfy the condition (4.4). In fact, if we take $a = b = \frac{1}{2}$, $c = \frac{1}{4}$ and $d = \frac{3}{4}$, then $(ab) \cdot (cd) = \frac{3}{16}$ and

$$(a - c) \cdot (b - d) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} < \frac{3}{16}.$$

**Theorem 4.7:** Suppose that $(X, M, \ast)$ is fuzzy isometric to $(X', M', \otimes)$ and $(Y, N, \ast)$ is fuzzy isometric to $(Y', N', \otimes)$.

Then the product fuzzy metric space $(X \times Y, P, \ast)$ is fuzzy isometric to $(X' \times Y', P', \otimes)$, where $P$ and $P'$ are fuzzy metrics defined by

$$P((x, y), (u, v), t) = \min\{M(x, u, t), N(y, v, t)\}$$

and

$$P'((x', y'), (u', v'), t) = \min\{M'(x', u', t), N'(y', v', t)\},$$

respectively.

**Proof:** Let $f : (X, M, \ast) \to (X', M', \otimes)$ and $g : (Y, N, \ast) \to (Y', N', \otimes)$ be fuzzy isometries with inverses $f^{-1}$ and $g^{-1}$, respectively. Define

$$h : (X \times Y, P, \ast) \to (X' \times Y', P', \otimes)$$

by $h(x, y) = (f(x), g(y))$.

Then $h$ is a bijection with inverse $h^{-1} = (f^{-1}, g^{-1})$. Moreover, since for $t > 0$

$$P'(h(x, y), h(u, v), t) = \min\{M'(f(x), u, t), N'(g(y), v, t)\} = \min\{M(x, u, t), N(y, v, t)\} = P((x, y), (u, v), t),$$

$h$ is a fuzzy isometry.

**5. Strong Fuzzy Isometry**

Recall that a surjective map $f : (X, M, \ast) \to (Y, N, \otimes)$ between fuzzy metric spaces is called a fuzzy isometry if

$$N(f(x_1), f(x_2), t) = M(x_1, x_2, t)$$

for all $t > 0$. An isometry between fuzzy metric spaces is a surjective map preserving metrics. Besides classical metric spaces, a fuzzy metric space has one more structure, i.e., continuous $t$-norm other than metric. So we can consider maps between fuzzy metric spaces preserving both metrics and continuous $t$-norms.

**Definition 5.1:** Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces. A surjective map $f : (X, M, \ast) \to (Y, N, \otimes)$ is called a strong fuzzy isometry if $f$ satisfies the following two conditions

(i) $N(f(x_1), f(x_2), t) = M(x_1, x_2, t)$ for all $t > 0$.

(ii) For any $x, y, x', y' \in X$ and $t, s > 0$,

$$N(f(x), f(y), t) \otimes N(f(x'), f(y'), s) = M(x, y, t) \ast M(x', y', s).$$

Note that a strong fuzzy isometry is automatically a fuzzy isometry. The identity map $id_x : (X, M, \ast) \to (X, M, \ast)$ is a strong fuzzy isometry. In fact, any fuzzy isometry $f : (X, M, \ast) \to (X, M, \ast)$ onto itself is a strong fuzzy isometry. If $(X, M, \ast)$ and $(Y, N, \otimes)$ are two fuzzy metric spaces with the same continuous $t$-norm $\ast$, then any fuzzy isometry $f : (X, M, \ast) \to (Y, N, \otimes)$ is a strong fuzzy isometry.

If $\ast$ is the multiplicative $t$-norm and $\otimes$ is the minimum $t$-norm, i.e., $a \ast b = ab, a \otimes b = \min\{a, b\}$, and $M$ is a metric on a set $X$ with respect to both $\ast$ and $\otimes$, then the identity map $id_x : (X, M, \ast) \to (X, M, \ast)$ is not a strong fuzzy isometry even though it is a fuzzy isometry.

**Theorem 5.2:** Let $(X, M, \ast)$ and $(Y, N, \otimes)$ be two fuzzy metric spaces such that $\ast$ is weaker than $\otimes$. Then we have the following.

(1) The inclusion $X \to (X \sqcup Y, U, \ast)$ with (2.2) is a strong fuzzy isometry.

(2) For any $y_0 \in Y$, the map $f : X \to (X \times Y, P, \ast)$ with (4.1) is a strong fuzzy isometry.

**6. Conclusions**

A basic method of investigating the structure of fuzzy metric spaces is to embed them into a bigger fuzzy metric space. The disjoint union and product space obtained from two given fuzzy metric spaces are basic examples into which given fuzzy metric spaces can be embedded. To make such an argument possible, the disjoint union and product space have natural fuzzy metrics on them. In
this paper, we have presented how to construct new fuzzy metrics on the disjoint union and the Cartesian product of two fuzzy metric spaces.

On the other hand, there is a metric, called the Hausdorff metric, on the set of nonempty closed and bounded subsets of a metric space $(X,d)$ (cf. [1]) and similarly, for a fuzzy metric space $(X,\mu,*),$ there is a fuzzy metric, called the Hausdorff fuzzy metric, on the set of nonempty compact subsets of $(X,\tau_M)$ [21]. The notion of a classical Hausdorff metric on the set of closed subsets of a metric space can be extended to the set of abstract metric spaces. The disjoint union and product space play an important role in obtaining basic properties for this generalized Hausdorff metric [10]. Thus, it is natural to consider the extension of the notion of a Hausdorff fuzzy metric on the set of nonempty compact subsets of $(X,\tau_M)$ to two abstract fuzzy metric spaces $(X,\mu,*),$ and $(Y,\nu,\oplus).$ One easy way to measure the distance between two abstract fuzzy metric spaces is to measure the distance between them as subsets after embedding them isometrically into their disjoint union or product space. In this case, the disjoint union and product space of two fuzzy metric spaces should play an important role as classical metric spaces.

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