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A NOTE ON ANALOGUES OF TANGENT POLYNOMIALS

CEMILE BILDIRICI, MEHMET ACIKGOZ and SERKAN ARACI

Department of Mathematics
Faculty of Arts and Science
University of Gaziantep
27310 Gaziantep, Turkey
E-mail: cemile_bildirici27@hotmail.com
acikgoz@gantep.edu.tr

Faculty of Economics
Administrative and Social Sciences
Hasan Kalyoncu University
27410 Gaziantep, Turkey
E-mail: mtsrkn@hotmail.com

Abstract

A Witt's formula for Tangent numbers was recently introduced and connected with Genocchi, Euler and Eulerian numbers [D. S. Kim et al., A note on Eulerian polynomials, Abstr. Appl. Anal. Vol. 2012 (2012), Article ID 269640, 10 pages]. Afterwards, a Witt's formula for Tangent polynomials was constructed [C. S. Ryoo, A note on the Tangent numbers and polynomials, Adv. Studies Theor. Phys. 7(9) (2013), 447-454]. So, we define the generating function of a new generalization of Tangent polynomials and derive interesting results which we state in the present paper.

1. Introduction

In the complex plane, the Bernoulli polynomials are known as

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t - 1}e^{xt}, \quad |t| < 2\pi. \tag{1.1}$$

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In the case $x = 0$ in (1.1), we have $B_n(0) := B_n$ is called n-th Bernoulli numbers and these numbers satisfy the following relation:

$$(B+1)^n - B_n = \delta_{1,n},$$

where we have used $B^n := B_n$ that is known as a technique of umbral calculus in the mathematics and $\delta_{1,n}$ stands for Kronecker delta (see [1, 2, 9, 13-16, 22-27]).

The Euler polynomials are introduced by

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt}, \; |t| < \pi. \tag{1.2}$$

In the special case $x = 0$, we have $E_n(0) := E_n$ that stands for n-th Euler numbers.

The Euler numbers can be computed by the rule:

$$(E+1)^n + E_n = 2\delta_{0,n},$$

where we have used $E^n := E_n$, symbolically cf. [1], [3], [7], [9], [13], [14-16], [22-27], [28]. The Genocchi polynomials are defined by

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt}, \; |t| < \pi. \tag{1.3}$$

In the case $x = 0$, we have $G_n(0) := G_n$ that means n-th Genocchi numbers cf. [2], [4], [8], [9] [10], [22-27].

Numerous properties of Bernoulli, Euler and Genocchi numbers and polynomials have many applications in analytic number theory, physics and the other related areas. Despite their being already one hundred years old, these numbers and polynomials are still today enveloped in an aura of mystery within scientific community. In recent years, some mathematicians have introduced new techniques for obtaining not only new but also interesting identities for Bernoulli polynomials, Euler polynomials, Genocchi polynomials and their various generalizations (see [1-29]). For instance, by using orthogonal property of known special polynomials, Kim et al. obtained novel identities of Bernoulli, Euler, Hermite, Gegenbauer polynomials cf. [14], [15], [16]. By using the generating function of q-Genocchi polynomials and the theory of Laurent series, Araci [2] derived new interesting identities of q-Genocchi polynomials that deal mainly with p-adic analysis and number theory. The new generalizations of Bernoulli, Euler and Genocchi polynomials were studied by Kim [12], Lee et al. [20] and [21], Seo et al. [29], Rim and Jeong [28], Araci et al. [5-7]. In [10], Jang also studied Witt's-type formula, explicit formula, multiplication formula and recurrence formula for w-Genocchi polynomials.

The following integral that called fermionic p-adic invariant integral was firstly defined by Kim [12]:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n\to\infty} \sum_{x=0}^{p^n-1} (-1)^x f(x).$$

From the above, we have, for $f_1(x) := f(x+1)$,

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

which is well-known an identity due to Kim [12]. The fermionic integral is a very good tool for obtaining the new generalizations of generating functions of Euler polynomials, Genocchi polynomials and Frobenius-Euler polynomials. Also, Ryoo [22-27] defined Tangent polynomials in terms of fermionic p-adic invariant integral on Z_p as follows:

$$\sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+2y)t} d\mu_{-1}(y) = \frac{2}{e^{2t}+1} e^{xt}, \ |t| < \frac{\pi}{2}.$$

By the same motivation of the above last identity, we extend it to

$$\sum_{n=0}^{\infty} C_{n,a}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+ay)t} d\mu_{-1}(y) = \frac{2}{e^{at}+1} e^{xt}, \ |t| < \frac{\pi}{a}, \tag{1.4}$$

where $a \in \mathfrak{R}^+$.

Corollary 1. $C_{n,a}(x)$ *that called generalized Tangent polynomials holds the following*:

(1) $\lim_{a \to 1} C_{n,a}(x) = E_n(x)$,

(2) $\lim_{a \to 2} C_{n,a}(x) = T_n(x)$,

(3) $C_{n,a}(x) = \int_{\mathbb{Z}_p} (x + ay)^n d\mu_{-1}(y)$,

(4) $C_{n,a}(0) := C_{n,a}$ *are called generalized Tangent numbers*.

In [12], it is well-known that

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l). \tag{1.5}$$

By (1.4), (1.5) and taking $f(x) = e^{axt}$, we have

$$\int_{\mathbb{Z}_p} e^{a(x+n)t} d\mu_{-1}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} e^{axt} d\mu_{-1}(x) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} e^{alt}$$

$$\sum_{m=0}^{\infty} C_{m,a}(na) \frac{t^m}{m!} + (-1)^{n-1} \sum_{m=0}^{\infty} C_{m,a} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(2 \sum_{l=0}^{n-1} (-1)^{n-1-l} a^m l^m \right) \frac{t^m}{m!}$$

$$\sum_{m=0}^{\infty} [C_{m,a}(na) + (-1)^{n-1} C_{m,a}] \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(2 \sum_{l=0}^{n-1} (-1)^{n-1-l} a^m l^m \right) \frac{t^m}{m!}.$$

When we equated the coefficients of $\frac{t^m}{m!}$ on the above, we see that

$$C_{m,a}(na) + (-1)^{n-1} C_{m,a} = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} a^m l^m. \tag{1.6}$$

By (1.6), we have

$$C_{m,a}(na) - C_{m,a} = 2 \sum_{l=0}^{n-1} (-1)^l a^m l^m, \text{ for } n \equiv 1 (\text{mod } 2)$$

and

$$C_{m,a}(na) - C_{m,a} = 2\sum_{l=0}^{n-1}(-1)^{l+1}a^m l^m, \text{ for } n \equiv 0 \pmod 2.$$

By (1.4), we have

$$\sum_{n=0}^{\infty} C_{n,a}(x)\frac{t^n}{n!} = 2\sum_{m=0}^{\infty}(-1)^m e^{(x+am)t}$$

$$= \sum_{n=0}^{\infty}\left(2\sum_{m=0}^{\infty}(-1)^m (x+am)^n\right)\frac{t^n}{n!}.$$

From the last identity, we have the following theorem:

Theorem 1. *For* $a \in \mathbb{R}^+$, *we have*

$$C_{n,a}(x) = 2\sum_{m=0}^{\infty}(-1)^m (x+am)^n.$$

Let $F_a(x, t) = \sum_{n=0}^{\infty} C_{n,a}(x)\frac{t^n}{n!}$ *and, by a simple calculation*

$$F_a(a-x, -t) = \frac{2}{e^{-at}+1}e^{(a-x)(-t)}$$

$$= \frac{2}{e^{at}+1}e^{xt} = F_a(x, t)$$

is derived. Hence, we have the following "*functional equation*"

$$F_a(a-x, -t) = F_a(x, t). \tag{1.7}$$

By (1.4) and (1.7), we have the following theorem.

Theorem 2. *The following equation holds true*:

$$(-1)^n C_{n,a}(a-x) = C_{n,a}(x).$$

We now give two theorems below without proof because it is easy to show by using the technique of generating function.

Theorem 3. *The following holds true*:

$$C_{n,a}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k C_{n-k,a}.$$

Theorem 4. *The following equality*

$$C_{n,a}(mx) = m^n \sum_{i=0}^{m-1} (-1)^i C_{n,a}\left(x + \frac{ai}{m}\right) \quad (m \equiv 1 \,(\mathrm{mod}\ 2))$$

is true.

2. Tangent-Zeta-Type Function and Related to Generalized Tangent Polynomials

For $s \in \mathbb{C}$, the Riemann-zeta function is known as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} \frac{1}{1 - p^{-s}}, \tag{1.8}$$

where the both sides are convergence for $\mathrm{Re}\,(s) > 1$. Note that the Bernoulli numbers interpolate by the Riemann-zeta function, which plays an important role in analytic number theory and has many applications in physics, probability and applied statistics. Firstly, L. Euler introduced the Riemann-zeta function in a real argument without using complex analysis. By (1.1) and (1.8), we have the following relation: For $n \in \mathbb{N}$,

$$\zeta(1 - n) = -\frac{B_n}{n}.$$

We define a new generalization of Tangent-zeta function by applying Mellin transformation to (1.4), as follows: For $s \in \mathbb{C}$

$$\zeta_a(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_a(x, -t) dt$$

$$= 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(x + am)^s}.$$

Definition 1. Let $s \in \mathbb{C}$, we have

$$\zeta_a(s, x) = 2\sum_{m=0}^{\infty} \frac{(-1)^m}{(x+am)^s}.$$

Corollary 2. *The following hold*:

(1) $\zeta_a(s, a) = \frac{1}{a^s}\zeta_E(s)$,

where $\zeta_E(s)$ *is Euler-zeta function defined by*

$$\zeta_E(s) = 2\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s},$$

(2) $\zeta_a(-n, x) = C_{n,a}(x)$,

(3) $\zeta_a(s, x) = \frac{1}{a^s}\zeta_E\left(s, \frac{x}{a}\right)$

where $\zeta_E(s, x)$ *is Hurwitz-Euler zeta function known as* $\zeta_E(s, x) = 2\sum_{m=0}^{\infty} \frac{(-1)^m}{(x+m)^s}$.

3. Conclusion

We gave a new generalization for Tangent polynomials due to Ryoo [22-27] and Kim et al. [19]. Moreover, we derived some interesting properties of new generalizations of Tangent polynomials.

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