International Journal of Statistics and Applied Mathematics 2018; 3(5): 56-61

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(5): 56-61

www.mathsjournal.com
Received: 19-07-2018
Accepted: 23-08-2018

Olanrewaju O Rasaki
Department of Statistics, University of Ibadan, Ibadan, Nigeria

Folorunsho A Serifat
Department of Statistics, University of Ibadan, Ibadan, Nigeria

Correspondence
Olanrewaju O Rasaki
Department of Statistics, University of Ibadan, Ibadan, Nigeria

Generalized autoregressive score (GAS) functions under Gaussian and student-t distributions

Olanrewaju O Rasaki and Folorunsho A Serifat

Abstract
This article described and worked-out the score functions otherwise known as the in information matrixes of Generalized Autoregressive model with time-varying parameters when the error term is assumed to follow Gaussian and student-"t" conditional distributions for location and heavy-tails affected series respectively.

Keywords: autoregressive**;** time-varying parameters; student-"t" distribution, Gaussian distribution

1. Introduction
In linear time series analysis, the assumption of Gaussian and student-t distributions as imposing error terms (white noises) are indispensible for effective, efficient and consistent estimation of score functions and their standard errors of estimates with respect to their marginal distributions (Tobias *et al*, 2017) [9]. Classification of linear time series models e.g., Autoregressive (AR), Moving Average (MA), Autoregressive Moving Average (ARMA) etc. in which their time-varying observations do follow either a location or heavy-tailed scale changes are dynamically driven by score functions of the estimates via their conditional marginal distributions (Harvey, 2013) [6]. Location scaled changes is conventionally modeled with Gaussian based error term while the heavy-tailed time change goes for the student-t distribution as the error term (Wintenberger, 2013) [10].
Bayesian approach has been frequently used as an alternative approach in lieu of the classical approach for parameter estimation of these score functions and their standard errors via the marginal and conditional distributions due to the former's theoretical rigor of its parameter estimation (D'Agostino *et al.*, 2013 [6]; Pesaran *et al.* 2011 [8]). Petrella (2014) [7] affirmed the shortcomings in the Bayesian approach to restriction and imposition in order to achieve stationary of the process. Blasquesa *et al.* (2014) [1] claimed this imposition to do lead to large inefficiency and limit in ability to capture tails behavior if present in the time-varying observations. This behavior might be constituted by the effect of skewness, kurtosis and outliers. In order to overcome this problem, this paper duel on the theoretical parameter estimation of driven score functions of generalized autoregressive model with Gaussian and student-t distributional error terms for both location and heavy-tailed affected series respectively.

2. Preliminaries
2.1 Generalized Autoregressive Score (GAS) model with normal distributed time-varying parameter
Monache & Petrella (2014) [7] proposed that an autoregressive model of order "p" with time-varying parameters could be defined as:

$$y_t = \phi_{0,t} + \phi_{1,t} y_{t-1} + \cdots + \phi_{t,p} y_{t-p} + \varepsilon_t \qquad \rightarrow 1$$

With white noise (error term) $\varepsilon_t \sim (0, \sigma_t^2) \qquad t = 1 \cdots\cdots n$

The updating rule can be defined to be variation of the vector of time-varying parameters to be $f_t = (\phi_t^{'}, \sigma_t^2)$, $\phi_t = (\phi_{0,t}, \phi_{1,t}, \cdots, \phi_{p,t})^{'}$ such that equation (1) is nothing but a Markov process of the first order of

$$f_{t+1} = \omega + Kf_t + r_t,\ r_t \sim (0, \Sigma_t) \qquad \rightarrow 2$$

Where ω is a vector of constants; K and Σ_t are the matrices containing the hyper-parameters (updated location and scale parameter respectively). f_t Represents the time-varying parameters while θ represents the fixed coefficient.
The Generalized Autoregressive Score (GAS) model for the time-varying parameters that will be driven by the scale-function of the conditional likelihood of f_t give the immediate past of $t-1$, $f_{t/t-1} = (\phi_{t/t-1}^{'}, \sigma_{t-1}^2)$ is

$$f_{t+1/t} = \omega + Kf_{t/t+1} + Gb_t \qquad \rightarrow 3$$

Such that $Y_{t-1} = \{y_{t-1}, y_{t-2}, \cdots, y_1\}$ and $\omega, K,$ and G are defined same in equation (2) with the driving mechanism called the score-function.

$$b_t = B_t \times \nabla_t$$

Such that $\nabla_t = \dfrac{\partial(\log p(y_t / G_t; \theta))}{\partial f_{t/t-1}}$; $B_t = I_{t-1}^{-1} = -\mathrm{E}\left[\dfrac{\partial^2 \log p(y_t / G_t; \theta)}{\partial f_{t/t-1} \partial f_{t/t-1}^{'}}\right]^{-1}$

With I_{t-1}^{-1} the information matrix, $G_t = [F_t, Y_{t-1}]$ and $F_t = \{f_{t/t-1}, f_{t-1/t-2}, \cdots, f_{1/0}\}$ for the vector parameters θ_t .

2.2 Score function of the time-varying normally distributed parameters' estimation.
Equation (1) can be re-written in a matrix form as $y_t = J^{'} \phi_{t/t-1} + \varepsilon_t$ such that $\varepsilon_t / Y_{t-1} \sim N(0, f_{t/t-1})$ for $t = 1 \cdots\cdots n$ and $\sigma_{t/t-1}^2 = f_{t/t-1}$, where $J^{'} = (1, y_{t-1}, \cdots\cdots, y_{t-p})$ & $\phi_{t/t-1} = (\phi_{0,t/t-1}, \phi_{1,t/t-1}, \cdots\cdots, \phi_{p,t/t-1})^{'}$

Since $\varepsilon_t = y_t - J^{'} \phi_{t/t-1}$, $\mu_t = J^{'} \phi_{t/t-1}$

$$p(y_t / G_t; \theta_t) = \frac{1}{\sqrt{2\pi f_{t/t-1}^2}} e^{-\frac{(y_t - \mu_t)^2}{2 f_{t/t-1}}} \quad -\infty < y_t < \infty \qquad \rightarrow 4$$

In a matrix form and taking the conditional likelihood

$$\ell(y_t / G_t; \theta_t) = \ln p(y_t / G_t; \theta_t)$$

$$= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log f_{t/t-1} - \frac{(y_t - J^{'}\phi_{t/t-1})^{'}(y_t - J^{'}\phi_{t/t-1})}{2 f_{t/t-1}}$$

So,

$$\nabla_t = \frac{\partial \ln p(y_t / G_t; \theta_t)}{\partial f_{t/t-1}} = -\frac{1}{2 f_{t/t-1}} + \frac{(y_t - J^{'}\phi_{t/t-1})^{'}(y_t - J^{'}\phi_{t/t-1})}{2 f_{t/t-1}^2}$$

But,

$$(y_t - J^{'}\phi_{t/t-1})^{'}(y_t - J^{'}\phi_{t/t-1}) = y_t^{'} y_t - y_t^{'} J^{'}\phi_{t/t-1} - \phi_{t/t-1}^{'} J y_t + \phi_{t/t-1}^{'} J J^{'} \phi_{t/t-1},$$

Recall $y_t = J^{'}\phi_{t/t-1} + \varepsilon_t$

$$=(J'\phi_{t/t-1}+\varepsilon_t)'(J'\phi_{t/t-1}+\varepsilon_t)-(J'\phi_{t/t-1}+\varepsilon_t)'J'\phi_{t/t-1}-\phi'_{t/t-1}(J'\phi_{t/t-1}+\varepsilon_t)+\phi'_{t/t-1}JJ'\phi_{t/t-1}$$

$$=\phi'_{t/t-1}JJ'\phi_{t/t-1}+\phi'_{t/t-1}J\varepsilon_t+\varepsilon'_tJ'\phi_{t/t-1}+\varepsilon'_t\varepsilon_t-\phi'_{t/t-1}JJ'\phi_{t/t-1}-\varepsilon'_tJ'\phi_{t/t-1}-\phi'_{t/t-1}JJ'\phi_{t/t-1}+\phi'_{t/t-1}J\varepsilon_t+\phi'_{t/t-1}JJ'\phi_{t/t-1}$$

$$=\phi'_{t/t-1}J\varepsilon_t+\varepsilon'_tJ'\phi_{t/t-1}+\varepsilon'_t\varepsilon_t-\varepsilon'_tJ'\phi_{t/t-1}+\phi'_{t/t-1}J\varepsilon_t = 2\phi'_{t/t-1}J\varepsilon_t+\varepsilon'_t\varepsilon_t$$

So,

$$\nabla_t = -\frac{1}{2f_{t/t-1}}+\frac{2(\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t}{2f^2_{t/t-1}} = \frac{\left[2(\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t-f_{t/t-1}\right]}{2f^2_{t/t-1}}$$

For b_t to be a scale vector, $\mathrm{E}(\nabla_t)=0$

$$\mathrm{E}(\nabla_t)=\frac{1}{2f^2_{t/t-1}}\left[(2\phi'_{t/t-1}J+\varepsilon'_t)\mathrm{E}(\varepsilon_t)-\mathrm{E}(f_{t/t-1})\right]$$

$$=\frac{1}{2f^2_{t/t-1}}\left[2\phi'_{t/t-1}J\,\mathrm{E}(\varepsilon'_t\varepsilon_t)-\mathrm{E}(f_{t/t-1})\right]$$

$$Since\ \ \mathrm{E}(\varepsilon_t)=0,\quad \mathrm{E}(\varepsilon'_t\varepsilon_t)=\sigma^2_t,\quad \mathrm{E}(f_{t/t-1})=\sigma^2_t$$

$$\mathrm{E}(\nabla_t)=\left[\,0+\sigma^2_t-\sigma^2_t\,\right]=0$$

Also, $I_{t-1}=-\mathrm{E}\left[\frac{\partial^2\log p(y_t/G_t;\theta_t)}{\partial f_{t/t-1}\partial f'_{t/t-1}}\right]=\mathrm{E}\left[-\frac{1}{2f^2_{t/t-1}}+\frac{(2\phi'_{t/t-1}J\varepsilon'_t)\varepsilon_t}{f^3_{t/t-1}}\right]$, $\sin ce\ \ \mathrm{E}(\varepsilon_t)=0$

$I_{t-1} = \frac{1}{2f^2_{t/t-1}}$, So $B_t=I^{-1}_{t-1}=2f^2_{t/t-1}$

Then, $b_t=B_t\times\nabla_t=2f^2_{t/t-1}\times\frac{\left[(2\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t-f_{t/t-1}\right]}{2f^2_{t/t-1}} = \left[(2\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t-f_{t/t-1}\right]$

The observation-driven model updates are: $\nabla_t = \frac{\left[2(\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t-f_{t/t-1}\right]}{2f^2_{t/t-1}}$,

$$I_{t-1} = \frac{1}{2f^2_{t/t-1}}$$

So, equation (3) becomes

$$f_{t/t+1}=\omega+Kf_{t/t+1}+G\left[(2\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t-f_{t/t-1}\right]$$

The updated parameters equal; $\phi_{t+1/t}=\phi_{t/t-1}+k_\phi((2\phi'_{t/t-1}J+\varepsilon'_t)\varepsilon_t-f_{t/t-1})$, $f_{t+1/t}=f_{t+1/t}+k_\phi(2f^2_{t/t-1})$

2.3 Score function of GAS model with heavy-tails (student-t) time-varying parameters.

From equation (1), $y_t=J'\phi_{t/t-1}+\varepsilon_t$ such that $\varepsilon_t/Y_{t-1}\approx NID(0,\ f_{t/t-1},\ v_t)$

$$p(y_t/G_t;\theta_t)=\frac{\Gamma\left(\frac{v_t+1}{2}\right)}{\Gamma\left(\frac{v_t}{2}\right)f_t\sqrt{\pi v_t}}\left(1+\frac{(y_t-\mu_t)^2}{v_t\,f^2_{t/t-1}}\right)^{-\frac{v_t+1}{2}}\quad -\infty<y_t<\infty\ \rightarrow 5$$

For location parameter μ_t , scale parameter f_t, v_t degree of freedom, where $\theta_t = (\mu_t, \phi_t, f_t, v_t)'$
In a matrix form,

$$p(y_t / G_t;\theta_t) = \frac{\Gamma\left(\frac{v_t+1}{2}\right)}{\Gamma\left(\frac{v_t}{2}\right) f_t \sqrt{\pi v_t}} \left(1 + \frac{\left(2\phi'_{t/t-1}J + \varepsilon'_t\right)\varepsilon_t}{v_t\, f^2_{t/t-1}}\right)^{-\frac{v_t+1}{2}}$$

The maximum likelihood,

$$\ell(y_t / G_t;\theta_t) = \log\left[\Gamma\left(\frac{v_t+1}{2}\right)\right] - \log\left[\Gamma\left(\frac{v_t}{2}\right)\right] - \frac{1}{2}\log\pi - \frac{1}{2}\log f^2_{t/t-1} v_t - \frac{v_t+1}{2}\left(1 + \frac{\left(2\phi'_{t/t-1}J + \varepsilon'_t\right)\varepsilon_t}{v_t\, f^2_{t/t-1}}\right) \rightarrow 6$$

$$\ell(y_t / G_t;\theta_t) = a + h_t + m_t$$

Such that,

$$a = \log\left[\Gamma\left(\frac{v_t+1}{2}\right)\right] - \log\left[\Gamma\left(\frac{v_t}{2}\right)\right] - \frac{1}{2}\log\pi - \frac{1}{2}\log v_t \rightarrow 7$$

$$h_t = -\frac{1}{2}\log f^2_{t/t-1} \rightarrow 8$$

$$m_t = -\frac{v_t+1}{2}\log\left[1 + \frac{\left(2\phi'_{t/t-1}J + \varepsilon'_t\right)\varepsilon_t}{v_t\, f^2_{t/t-1}}\right] \rightarrow 9$$

Creal *et al*. (2011, 2013) [2, 3] and Harvey (2013, 2014) [5, 6] affirmed that the Degree of Freedom v_t must be strictly greater than two.

Let $\frac{\varepsilon_t}{f^2_{t/t-1}} = \gamma_t$, so $m_t = -\frac{v_t+1}{2}\log\left[1 + \frac{\left(2\phi'_{t/t-1}J + \varepsilon'_t\right)\gamma_t}{v_t}\right]$, Since $\nabla_t = \frac{\partial\ell(y_t / G_t;\theta_t)}{\partial f_{t/t-1}}$

∇_t can be partition into two functions of ∇_ϕ & ∇_f such that ∇_ϕ depends on γ_t and m_t while ∇_f depends on h_t, m_t, and γ_t.

$\nabla_\phi = \frac{\partial m_t}{\partial\gamma_t}\times\frac{\partial\gamma_t}{\partial\phi'_t}$, $\nabla_f = \frac{\partial h_t}{\partial f^2_{t/t-1}} + \frac{\partial m_t}{\partial f^2_{t/t-1}} = \frac{\partial h_t}{\partial f^2_{t/t-1}} + \frac{\partial m_t}{\partial\gamma_t}\bullet\frac{\partial\gamma_t}{\partial f^2_{t/t-1}}$, using the function of the function rule

Recall, $\varepsilon_t = y_t - J'\phi_{t/t-1} \Rightarrow \gamma_t = \frac{y_t - J'\phi_{t/t-1}}{f^2_{t/t-1}}$, so $\frac{\partial\gamma_t}{\partial\phi'_t} = -\frac{J'}{f^2_{t/t-1}}$,

So,

$$\frac{\partial m_t}{\partial\gamma_t} = -\frac{v_t+1}{2}\left[\frac{\frac{(2\phi'_{t/t-1}J' + \varepsilon_t)}{v_t}}{\frac{v_t + (2\phi'_{t/t-1}J + \varepsilon_t)}{v_t}\gamma_t}\right] = -\frac{v_t+1}{2}\left[1 + \frac{(2\phi'_{t/t-1}J' + \varepsilon'_t)\gamma_t}{v_t}\right]^{-1}\frac{(2\phi'_{t/t-1}J + \varepsilon'_t)}{v_t}$$

$$= -\frac{v_t+1}{2}\frac{(2\phi'_{t/t-1}J' + \varepsilon_t)}{v_t}\left[\frac{v_t}{v_t + (2\phi'_{t/t-1}J' + \varepsilon'_t)\gamma_t}\right] = -\frac{v_t+1}{2}\left[\frac{(2\phi'_{t/t-1}J' + \varepsilon'_t)}{\left[v_t + (2\phi'_{t/t-1}J' + \varepsilon_t)\right]\gamma_t}\right]$$

$$=-\frac{v_t+1}{2}\left[\frac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\frac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon_t)}{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon_t\gamma_t)}\right]=-\frac{v_t+1}{2}\left[\frac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t}{v_t\gamma_t}+\frac{1}{\gamma_t}\right]$$

But, $B_t=I^{-1}\begin{bmatrix}I_{\phi\phi,t}\\ I_{ff,t}\end{bmatrix}^{-1}$ where $I_{\phi\phi,t}=-\mathrm{E}(H_{\phi\phi,t}),\quad I_{ff,t}=-\mathrm{E}(H_{ff,t})$

$$H_{\phi\phi,t}=\frac{\partial\nabla_{\phi\phi,t}}{\partial\phi^{'}_{t/t-1}}=-\frac{v_t+1}{2}\left[\frac{J^{'}}{v_t\gamma_t}\right]$$

Also, $\nabla_{t/t-1}=\dfrac{\partial h_t}{\partial f^2_{t/t-1}}+\dfrac{\partial m_t}{\partial\gamma_t}\bullet\dfrac{\partial\gamma_t}{\partial f^2_{t/t-1}}=-\dfrac{1}{2f^2_{t/t-1}}+\dfrac{v_t+1}{2}\left[\dfrac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\dfrac{1}{\gamma_t}\right]\times\dfrac{2\gamma_t}{f_{t/t-1}}$

$$H_{ff,t}=\frac{\partial\nabla_{ff,t}}{\partial f_{t/t-1}}=\frac{1}{f^3_{t/t-1}}-v_t+1\left[\frac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\frac{1}{\gamma_t}\right]\times\frac{\gamma_t}{f^2_{t/t-1}}$$

So, $I_{\phi\phi,t}=-\mathrm{E}(H_{\phi\phi,t})=\dfrac{v_t+1}{2}\left[\dfrac{J^{'}}{v_t\gamma_t}\right]=\dfrac{v_t+1}{2}\left[\dfrac{J^{'}f^2_{t/t-1}}{v_t\varepsilon_t}\right]$

$$I_{ff,t}=-\mathrm{E}(H_{ff,t})=v_t+1\left[(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)+\frac{1}{\gamma_t}\right]\frac{\gamma_t}{f^2_{t/t-1}}-\frac{1}{f^3_{t/t-1}}=v_t+1\left[(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)+\frac{f^2_{t/t-1}}{\varepsilon_t}\right]\frac{\varepsilon_t}{f^2_{t/t-1}}-\frac{1}{f^3_{t/t-1}}$$

So, $B_t=I^{-1}=\begin{bmatrix}I_{\phi\phi,t}\\ I_{ff,t}\end{bmatrix}=\begin{bmatrix}\dfrac{2v_t\,\varepsilon_t}{(v_t+1)J^{'}f^2_{t/t-1}} & 0\\ 0 & \left[I_{ff,t}\right]^{-1}\end{bmatrix}$

$$b_t=I_t^{-1}\,\nabla_t$$

$$b_t=\begin{bmatrix}\dfrac{2v_t\,\varepsilon_t}{(v_t+1)J^{'}f^2_{t/t-1}} & 0\\ 0 & \dfrac{f^4_{t/t-1}}{(v_t+1)}\left[(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)+\dfrac{f^2_{t/t-1}}{\varepsilon_t}\right]^{-1}-f^3_{t/t-1}\end{bmatrix}\begin{bmatrix}-\dfrac{v_t+1}{2}\left[\dfrac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\dfrac{1}{\gamma_t}\right]\\ -\dfrac{1}{2f^2_{t/t-1}}+\dfrac{(v_t+1)\gamma_t}{f_{t/t-1}}\left[\dfrac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\dfrac{1}{\gamma_t}\right]\end{bmatrix}$$

The observation-driven model updates are $I_t,\ \nabla_t$

$$b_t=\begin{bmatrix}-\dfrac{v_t\varepsilon_t}{J^{'}f^2_{t/t-1}}\left(\dfrac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\dfrac{1}{\gamma_t}\right)\\ \dfrac{f^4_{t/t-1}}{(v_t+1)}\left[(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)+\dfrac{f^2_{t/t-1}}{\varepsilon_t}\right]^{-1}-f^3_{t/t-1}\left(-\dfrac{1}{2f^2_{t/t-1}}+\dfrac{(v_t+1)\gamma_t}{f_{t/t-1}}\left[\dfrac{(2\phi^{'}_{t/t-1}J^{'}+\varepsilon^{'}_t)}{v_t\gamma_t}+\dfrac{1}{\gamma_t}\right]\right)\end{bmatrix}\rightarrow 10$$

Equation (10) is the (2x1) matrix of the Score function for GAS with heavy-tails using student-“t” distribution.

3. Conclusions

Having find-out that estimation of multivariate normal and student-t distributions’ score functions are usually left out in GAS related articles, working papers, manuscripts and write-ups due to its rigor. This led to the estimation of the score functions for easy computational and estimating of the function.

4. Acknowledgments

I appreciate Oladebo Omotayo for his useful and constructive comments; suggestions and insight in the course of the manuscript.

5. Competing interests

Author declared, no potential conflict of interest exist.

6. References

1. Blasques F, Koopman SJ, Lucas A. Maximum likelihood estimation for correctly specified generalized autoregressive score models: feedback effects, contraction conditions and asymptotic properties. Tinbergen institute discussion paper, 2014, 1-29:Tl, 2014-074/III.
2. Creal D, Koopman SJ, Lucas A. Generalized autoregressive score models with applications. Journal of Applied Econometrics. 2013; 28(5):777-795.
3. Creal D, Koopman SJ, Lucas A. A dynamic multivariate heavy-tailed model for time-varying volatilities and correlations. Journal of Business and Economic Statistics. 2011; 29(4):552-563.
4. D' Agostino A, Gambatti L, Giannone D. Macro-economic forecasting and structural change. Journal of Applied Econometrics. 2013; 28:82-101.
5. Harvey AC, Luati A. Filtering with heavy-tails. Journal of the American Statistical Association. 2014; 109(507):1112-1122.
6. Harvey AC. Dynamic models for volatility and heavy-tails: with applications to financial and economic time series. Cambridge University Press, 2013.
7. Monache DD, Petrella I. Adaptive models and heavy tails. Birkbeck working papers in economics and finance. Birkbeck University of London, 2014. ISSN 1745-8587.
8. Pesaran MH, Pick A. Forecast combination across estimation window in the presence of structural breaks. Journal of Business Economics and Statistics. 2011; 29:30-318.
9. Tobias L, Fokianos K, Fried R. An R-package for analysis of count time series following generalized linear models. Journal of Statistical Software. 2017; 85(5):1-51.
10. Wingtenberger O. Continuous invertibility and stable QML estimation of EGARCH (1, 1) model. Journal of Statistics. 2013; 40(4):846-867.