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TRANSITIVE MAPS ON TOPOLOGICAL SPACES

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In the present paper, we consider the problem of existence of nonequivalent definitions of topological transitivity, which is a classical problem of the topological dynamics. In particular, we use the fact that all available definitions of this kind imply a condition imposed on the dynamical system. The main result of our investigations is the complete classification of these dynamical systems.

1. Introduction

Let X be a topological space and let $f\colon X \to X$ be a continuous map (for the sake of brevity, we write $f \in S(X)$, where $f \in S(X)$ is the space of all continuous maps from X into itself, and use (X, f) to denote the corresponding dynamical system specified by iterations of this map). Consider the following two properties:

(TT) for any pair of nonempty open sets[3] U and V in X, there exists a natural number n such that

$$f^n(U) \bigcap V \neq \varnothing,$$

(DO) there exists a point $x_0 \in X$ such that its orbit $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is everywhere dense in X.

It should be noted that, generally speaking, there are a lot of topological spaces that do not admit maps with these properties. For example, the space

$$X = \{0\} \bigcup \left\{ \frac{1}{n} \colon n \in \mathbb{N} \right\}$$

(with the ordinary metric on the closed interval $[0, 1]$) does not admit continuous maps with property (TT), and any nonseparable space does not admit continuous maps with property (DO). At the same time, if these properties take place, property (TT) is usually regarded as the definition of *topological transitivity*. On the other hand, some authors use property (DO) instead of property (TT) for this purpose. In the present section, we say that a dynamical system is topologically transitive if it possesses property (TT).

An arbitrary point with everywhere dense orbit is called a *transitive point.* A point which is not transitive is called *intransitive.* By tr_f and intr_f we denote the sets of transitive and intransitive points of the dynamical system (X, f), respectively.

In cases where we speak about properties of the dynamical system (X, f) or use the notation for these properties, we often automatically use the same properties and notation for the corresponding map f with no special suggestions.

In general, properties (TT) and (DO) are independent. Indeed, consider the space

$$X = \{0\} \bigcup \left\{ \frac{1}{n} \colon n \in \mathbb{N} \right\}$$

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[3] Here and in what follows, we use the term "opene set" instead of "nonempty open set."

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with the ordinary metric and $f\colon X\to X$ defined in such a way that

$$f(0)=0 \quad \text{and} \quad f\left(\frac{1}{n}\right)=\frac{1}{n+1}, \quad n=1,2,\ldots.$$

It is easy to see that the map f is continuous. The point $x_0=1$ is a transitive point for (X,f). However, the system is not topologically transitive $\left(\text{e.g., consider } U=\left\{\frac{1}{2}\right\} \text{ and } V=\{1\}\right)$. This means that property (DO) does not imply property (TT).

Property (TT) does not imply property (DO) too. In order to prove this, it suffices to consider, e.g., the space $I=[0,1]$ and the standard tent map $g(x)=1-|2x-1|$ on I. Let X be the set of all periodic points of g and let $f=g_{|X}$ (a point x is *periodic* for g if $g^n(x)=x$ for some natural n). Then the system (X,f) does not satisfy condition (DO) because the set X is everywhere dense in I, whereas the orbit of every periodic point is finite. Nevertheless, condition (TT) is satisfied because, for any nondegenerate interval J in I, there exists a positive integer k such that $g^k(J)=I$. Thus, whenever J_1 and J_2 are open intervals in I, there exists a periodic orbit g intersecting J_1 and J_2. This yields condition (TT) for (X,f).

It should be noted that under additional conditions imposed on the phase space (or on the map), the definitions (TT) and (DO) are equivalent. Indeed, the following result by S. Silverman is true [8]: If a metric space X does not have isolated points, then (DO) implies (TT). If X is a separable Baire second category space, then (TT) implies (DO). In arbitrary metric spaces, properties (TT) and (DO) are equivalent for surjective maps. If a compact metric space X admits a transitive map (i.e., there exists a continuous map f in this space satisfying (TT)), then the space X does not have isolated points if and only if X is infinite.

The following theorem is true (see [1, 6]):

Theorem 1.1. *Let (X,f) be a dynamical system, X be a compact Hausdorff space, and $f\colon X\to X$ be continuous. Then the following properties are equivalent:*

f is a topologically transitive map;

for any pair of opene sets U and V in X, there exists a positive n such that $f^n(U)\bigcap V\neq\varnothing$;

for any opene set U in X, $\overline{\bigcup_{n=1}^{\infty} f^n(U)}=X$;

for any opene set U in X, $\overline{\bigcup_{n=0}^{\infty} f^n(U)}=X$;

for any pair of opene sets U and V in X, there exists a positive n such that $f^{-n}(U)\bigcap V\neq\varnothing$;

for any pair of opene sets U and V in X, there exists n such that $f^{-n}(U)\bigcap V\neq\varnothing$;

for any opene set U in X, $\overline{\bigcup_{n=1}^{\infty} f^{-n}(U)}=X$;

for any opene set U in X, $\overline{\bigcup_{n=0}^{\infty} f^{-n}(U)}=X$;

if $E\subset X$ is a closed set and $f(E)\subset E$, then either $E=X$ or E is nowhere dense in X;

if $U\subset X$ is an open set in $f^{-1}(U)\subset U$, then either $U=\varnothing$ or U is nowhere dense in X;

the set tr_f is G_δ-dense;

the map f is surjective and the set tr_f is nonempty.

Furthermore, it follows from all conditions presented above that the set tr_f is nonempty.

In addition, if (X,f) is given in the space with no isolated points, then the last condition is also equivalent to all previous conditions.

A natural question arises: What relations between the indicated properties are true in the case of a general topological space?

In what follows, let X be a topological space and let $f \in S(X)$. The following statement was proved in [7] (to a large extent, namely that work motivated us to write the present paper):

Theorem 1.2. *If there exists a point $x \in X$ such that the orbit of the point $f(x)$ is everywhere dense in X, then the map f is topologically transitive.*

By virtue of this theorem, we can see that when we characterize transitive properties of a map, it is important to take into account "the zero time." In view of this fact, we use the following two properties as a definition of topological transitivity (by Theorem 1.1, these properties are equivalent for dynamical systems on compact Hausdorff spaces):

$(\mathrm{TT})_{\mathbb{N}}$ for any pair of opene sets U and V in X, there exists a positive (integer) number n such that

$$f^n(U) \bigcap V \neq \varnothing,$$

$(\mathrm{TT})_{\mathbb{N}_0}$ for any pair of opene sets U and V in X, there exists a nonnegative (integer) number n such that

$$f^n(U) \bigcap V \neq \varnothing.$$

Relatively, if property $(\mathrm{TT})_{\mathbb{N}}$ holds, then we call the dynamical system (X, f) (topologically) $\mathbb{N}$-transitive (or simply topologically transitive). If property $(\mathrm{TT})_{\mathbb{N}_0}$ holds, then we call this system $\mathbb{N}_0$-transitive. It is obvious that $\mathbb{N}$-transitive dynamical systems are $\mathbb{N}_0$-transitive.

For convenience, let us introduce a series of new notation. For a subset $A \subset X$, its orbit is defined as $\bigcup_{n\in\mathbb{N}_0} f^n(A)$ and denoted by $f^+(A)$, the set of all its preimages $\bigcup_{n\in\mathbb{N}_0} f^{-n}(A)$ is denoted by $f^-(A)$, and their union $f^+(A) \bigcup f^-(A)$ is denoted by $f^{\pm}(A)$. For any set $A = \{x\}$ consisting of one point, we write (x) instead of $(\{x\})$.

The following statements indicate relations between (TT) and (DO) in definitions of transitivity (their proof is represented in Sec. 2):

Proposition 1.1. *The following properties are equivalent:*

(i) f *is an* $\mathbb{N}_0$*-transitive map*;

(ii) *for any opene set* U *in* X, $\overline{f^+(U)} = X$;

(iii) *for any opene set* U *in* X, $\overline{f^-(U)} = X$.

Proposition 1.2. *The following properties are equivalent:*

(i) f *is an* $\mathbb{N}$*-transitive map*;

(ii) *for any opene set* U *in* X, $\overline{f^+\left(f(U)\right)} = X$;

(iii) *for any opene set* U *in* X, $\overline{f^-\left(f^{-1}(U)\right)} = X$.

In the case of homeomorphisms on noncompact (metric) spaces, in order to define a transitive (or minimal) homeomorphism (as a rule, by means of property (DO)), one can use the property that the full orbit of a point is everywhere dense in the space. As a certain analogy, consider the following property of transitivity:

$(\mathrm{TT})_{\mathbb{Z}}$ for any pair of opene sets U and V in X, there exists an integer n such that $f^n(U) \bigcap V \neq \varnothing$.

It is obvious that properties $(\mathrm{TT})_{\mathbb{N}}$ and $(\mathrm{TT})_{\mathbb{N}_0}$ of a dynamical system imply its $\mathbb{Z}$-transitivity (i.e., property $(\mathrm{TT})_{\mathbb{Z}}$ for this system).

Introducing this definition, we are inspired also by the following reasonings: "Set-theoretic" definitions and criteria for both types of transitivity, which are introduced above, reveal a symmetry between the behavior of images and preimages, i.e., between "the future" and "the past." Namely for this reason, we study a class of dynamical systems specified by the definition, which is analogous to the definition of topological transitivity and, in addition, symmetric in itself.

The main purpose of the present paper is a classification of $\mathbb{Z}$-transitive dynamical systems. In Secs. 2–7, we investigate properties of the defined classes of transitive dynamical systems. Before proceeding with these sections, we encourage you to read auxiliary sections 8–10, which are useful for understanding the material of the main sections. In Sec. 8, we represent some properties of orbits and invariant sets. In Sec. 9, we describe properties of so-called *meeting time sets* and *wandering sets* (for definitions, see Sec. 2). In both these sections, we consider self-maps of spaces with no additional structures on them. In Sec. 10, we represent some auxiliary topological results. In particular, we give generalizations of the results from Secs. 8 and 9 to the case of a topological space and a continuous map.

2. Main Relations Between Different Transitivity Properties of Dynamical Systems

For subsets A and B of the space X, we define so-called *meeting time set* (see also [2]) for the set A with respect to the set B:

$$n_f(A,B) := \{n \in \mathbb{N}_0 \mid A \bigcap f^{-n}(B) \neq \varnothing\}, \qquad N_f(A,B) := \{n \in \mathbb{Z} \mid A \bigcap f^{-n}(B) \neq \varnothing\}.$$

It is easy to see that $n_f(A,B) = \{n \in \mathbb{N}_0 \mid f^n(A) \bigcap B \neq \varnothing\}$ and $N_f(A,B) = \{n \in \mathbb{Z} \mid f^n(A) \bigcap B \neq \varnothing\}$. We omit the subscript of n_f and N_f if it is clear from the context for which map f these sets are defined.

It is clear that $n(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are maps with arguments in $2^X \times 2^X$ and values in $2^{\mathbb{Z}}$. In what follows, along with ordinary set-theoretic operations with sets of integers we also perform arithmetic operations with these sets. For example, if $M, N \subset \mathbb{Z}$ and $k \in \mathbb{Z}$, then $M + N := \{m + n \mid m \in M,\ n \in N\}$, and $M + k := \{m + k \mid m \in M\}$. We also use the multiplication operation for $M, N \subset \mathbb{Z}$ and $k \in \mathbb{Z}$.

By using the introduced notation, we rewrite definitions of these transitivity properties as follows: A system is $\mathbb{N}$-, $\mathbb{N}_0$-, and $\mathbb{Z}$-transitive if, for any pair of opene sets U and V in X, $n(U,V)\backslash\{0\} \neq \varnothing$, $n(U,V) \neq \varnothing$, and $N(U,V) \neq \varnothing$, respectively.

Let us proceed to the proof of generalized versions of Propositions 1.1 and 1.2.

Proposition 2.1. *The following properties are equivalent:*

(i) f *is an* $\mathbb{N}_0$*-transitive map;*

(ii) for any opene set U *in* X, $\overline{f^+(U)} = X$;

(iii) for any opene set U *in* X, $\overline{f^-(U)} = X$;

(iv) if $f(A) \subset A$, *then* A *is either everywhere dense or nowhere dense;*

(v) if $F \neq X$ *is a closed set and* $f(F) \subset F$, *then* $\operatorname{int} F = \varnothing$;

(vi) if $f^{-1}(B) \subset B$, *then* $\operatorname{int} B$ *is either empty or everywhere dense;*

(vii) if U *is an opene set and* $f^{-1}(U) \subset U$, *then* U *is an everywhere dense set.*

Proof. It is easy to see that properties (ii) and (iii) are equivalent to the following properties, respectively:

$$\text{for any pair of opene sets } U \text{ and } V \text{ in } X, \ f^{+}(U) \bigcap V \neq \varnothing$$

and

$$\text{for any pair of opene sets } U \text{ and } V \text{ in } X, \ f^{-}(U) \bigcap V \neq \varnothing.$$

It is obvious that these properties are equivalent to $\mathbb{N}_0$-transitivity.

(ii) $\Rightarrow$ (iv). Let $f(A) \subset A$. According to Proposition 10.2, $f\left(\overline{A}\right) \subset \overline{A}$. By using Proposition 8.2, we obtain $f^{+}\left(\overline{A}\right) = \overline{A}$ and, hence, $\overline{f^{+}\left(\operatorname{int}\overline{A}\right)} \subset \overline{A}$. If $\operatorname{int}\overline{A} \neq \varnothing$, then $X = \overline{f^{+}\left(\operatorname{int}\overline{A}\right)} \subset \overline{A}$.

(v) $\Rightarrow$ (ii). Let U be an opene set. By using Proposition 10.2, we establish that $f\left(\overline{f^{+}(U)}\right) \subset \overline{f^{+}(U)}$ and $\varnothing \neq U \subset \operatorname{int}\overline{f^{+}(U)}$ and, hence, $\overline{f^{+}(U)} = X$.

(iv) $\Rightarrow$ (v) is obvious. To go from (vi) [and (vii)] to (iv) [and (v)], respectively, it suffices to consider the sets $F = X \backslash U$ and $A = X \backslash B$ and apply Proposition 8.2.

Proposition 2.2. *The following properties are equivalent:*

(i) (X, f) *is an* $\mathbb{N}$*-transitive system;*

(ii) for any opene set U *in* X, $\overline{f^{+}(f(U))} = X$;

(iii) for any opene set U *in* X, $\overline{f^{-}(f^{-1}(U))} = X$;

(iv) (X, f) *is an* $\mathbb{N}_0$*-transitive system and, for any opene set* U *in* X, $f^{-1}(U) \neq \varnothing$;

(v) (X, f) *is an* $\mathbb{N}_0$*-transitive system and* $\overline{f(X)} = X$;

(vi) for any opene sets U *and* V *in* X, *the set* $n(U, V)$ *is infinite.*

Proof. Equivalence of items (i), (ii), and (iii) is proved in the same way as in Proposition 2.1. It is also obvious that items (iv) and (v) are equivalent to each other and follow from item (i); item (vi) imply item (i).

(iv) $\Rightarrow$ (iii). For any opene U, the set $f^{-1}(U)$ is opene. Therefore, $\overline{f^{-}(f^{-1}(U))} = X$.

(iv) $\Rightarrow$ (vi). Let there exist $N \in \mathbb{N}$ such that $n(U, V) \subset [0, N]$. Since $f^{-N-1}(V) \neq \varnothing$, we have

$$\varnothing \neq n\left(U, f^{-N-1}(V)\right) = (n(U, V) \backslash [0, N]) - N - 1,$$

which is a contradiction.

We say that $A \subset X$ is a *wandering* set if $N(A, A) \subset \{0\}$. Topologically transitive systems also have properties presented below that follow from properties of the previous statement.

Corollary 2.1. *(i) If* f *is an* $\mathbb{N}$*-transitive map, then* $\overline{f^{n}(X)} = X$ *for any* $n \in \mathbb{N}_0$.

(ii) If f *is an* $\mathbb{N}$*-transitive map, then* $\overline{f^{-}(f^{-n}(U))} = \overline{f^{+}(f^{-n}(U))} = \overline{f^{-}(f^{n}(U))} = \overline{f^{+}(f^{n}(U))} = X$ *for any opene* U *and any* $n \in \mathbb{N}_0$.

(iii) In $\mathbb{N}$*-transitive systems, there are no opene wandering sets.*

Proof. Item (i) is a direct consequence of the previous statement.

(ii) It is obvious that the first three sets are everywhere dense. If $\overline{f^{+}\left(f^{n}(U)\right)} \neq X$, then $V := X \setminus \overline{f^{+}\left(f^{n}(U)\right)}$ is an opene set and $n(U,V) \subset [0,n]$, which contradicts the statement.

(iii) If U is an opene wandering set, then $n\left(U,U\right) \setminus \{0\} = \varnothing$, which gives $U = \varnothing$.

Let us proceed to studying properties of $\mathbb{Z}$-transitive dynamical systems and their relations with $\mathbb{N}$- and $\mathbb{N}_0$-transitivity.

Proposition 2.3. *The following properties are equivalent:*

(i) f *is a* $\mathbb{Z}$*-transitive map;*

(ii) for any opene set U *in* X, $\overline{f^{\pm}(U)} = X$;

(iii) for arbitrary opene U *and* V *in* X, $f^{-}(U) \bigcap f^{-}\left(V\right) \neq \varnothing$;

(iv) X *cannot contain two disjunctive inversely invariant sets, which are opene;*

(v) X *cannot be represented as a union of two proper invariant subsets, which are closed;*

(vi) for any opene U, V, *and* W *in* X, *"the triangle inequality" is true:* $N(U,W) \subset N(U,V) + N(V,W)$.

Proof. Both conditions (i) and (ii) are equivalent to the condition that $f^{\pm}(U) \bigcap V \neq \varnothing$ for any opene U and V. Conditions (iii) and (iv) are equivalent because each inverse orbit of any opene set is inversely invariant and opene and each inversely invariant set is an inverse obit for itself. Conditions (iv) and (v) are dual conditions.

(i) $\Rightarrow$ (iii). Actually, $\mathbb{Z}$-transitivity implies the following stronger condition: For any opene U and V in X, $f^{-}(U) \bigcap V \bigcup f^{-}\left(V\right) \bigcap U \neq \varnothing$.

(iii) $\Rightarrow$ (i). Consider arbitrary opene U and V in X. By condition, there exist $m,n \in \mathbb{N}_0$ such that $f^{-n}(U) \bigcap f^{-m}\left(V\right) \neq \varnothing$. If, for example, $n \geq m$, $n \in n\left(f^{-m}\left(V\right), U\right) \subset n(V,U) + m$, then $n(V,U) \neq \varnothing$. Similarly, if $m \geq n$, then $n(U,V) \neq \varnothing$ and, hence, $N(U,V) \neq \varnothing$ in any case.

(vi) $\Rightarrow$ (i). Since $0 \in N\left(U,U\right) \subset N(U,V) + N(V,U)$, we have $N(U,V) \neq \varnothing$.

(i) $\Rightarrow$ (vi). Let $n \in n(U,W)$. Then $S := f^{-n}\left(W\right) \bigcap U$ is opene. Therefore, there exists $m \in N\left(S,V\right) \subset N\left(f^{-n}\left(W\right), V\right) \bigcap N(U,V) \subset \left(N(W,V) + n\right) \bigcap N(U,V)$. This yields $m \in N(U,V)$, $m - n \in N(W,V)$ and, hence, $n(U,W) \subset N(U,V) + N(V,W)$. Similarly, it follows from $n(W,U) \subset N(W,V) + N(V,U)$ that $-n(W,U) \subset N(U,V) + N(V,W)$ and, hence, $N(U,W) = n(U,W) \bigcup \left(-n(W,U)\right) \subset N(U,V) + N(V,W)$.

The following statement and its corollary are taken from [4]:

Proposition 2.4. *If* f *is a* $\mathbb{Z}$*-transitive map,* U *and* V *are arbitrary opene sets in* X, *then there exists an opene set* W *such that* $N(W,W) \subset N\left(U,U\right) \bigcap N(V,V)$.

Proof. Let $n \in N(U,V)$. If $n \geq 0$, then $W := f^{-n}\left(V\right) \bigcap U$ is opene and

$$N(W,W) \subset N\left(f^{-n}\left(V\right), f^{-n}\left(V\right)\right) \bigcap N\left(U,U\right) \subset N(V,V) \bigcap N\left(U,U\right).$$

If $n \leq 0$, then, for $W := f^{n}(U) \bigcap V$, the inclusion is verified by analogy with the previous case.

Corollary 2.2. *If* f *is an* $\mathbb{N}$*-transitive map,* $U_1, U_2, \ldots, U_n$ *are arbitrary opene sets in* X, *then*

$$\bigcap_{k=1}^{n} n(U_k, U_k) \neq \{0\}.$$

Proof. For $n = 1$, this statement is reduced to Corollary 2.1. Further proof is performed by induction with regard for Proposition 2.4.

Proposition 2.5. *If f is a $\mathbb{Z}$-transitive map and $Y \subset X$ is a canonically closed*[1] *invariant set, then $g := f_{|Y}$ is also $\mathbb{Z}$-transitive.*

Proof. Let $U := \operatorname{int} Y$ and let V and W be opene sets in Y. Since Y is a canonically closed set, by virtue of Proposition 10.4 sets $V \bigcap U$ and $W \bigcap U$ are opene in X and $\varnothing \neq N(V \bigcap U, W \bigcap U) \subset N(V, W)$ ($N_f = N_g$, see Proposition 9.6). Since V and W are arbitrary, $\mathbb{Z}$-transitivity of g is stated.

It will be shown later that it is very important that the phase space X includes subsets of a special form, which we call atoms. We say that a subset $A \subset X$ is a *topological atom* (briefly, an *atom*) if, for any $B \subset A$, we have either $\overline{A} = \overline{B}$ or $\overline{A} = \overline{A \setminus B}$. It is obvious that all singletons and $\varnothing$ are topological atoms.

For example, we prove that the notions of $\mathbb{N}$- and $\mathbb{N}_0$-transitivity do not coincide on very pathological spaces only. In particular, in the case where X is not an atom, the notions of $\mathbb{N}$- and $\mathbb{N}_0$-transitivity do coincide. We also show that there is a close relation between the notions of $\mathbb{N}$- and $\mathbb{Z}$-transitivity in nonpathological spaces. Namely, the following theorem, which is the main result of the present paper, is true:

Theorem 2.1 (relation between $\mathbb{Z}$-transitivity and $\mathbb{N}$-transitivity). *Let (X, f) be a $\mathbb{Z}$-transitive system. If V is the union of all opene atoms (which is representable in the form of at most countable union), then $Y := \overline{X \setminus \overline{V}}$ is an invariant (possibly nonempty) set and $\left(Y, f_{|Y}\right)$ is an $\mathbb{N}$-transitive system.*

The proof of this theorem is represented after Theorem 7.2.

Corollary 2.3. *In spaces that contain no opene atoms, the notions of $\mathbb{N}$-transitivity and $\mathbb{Z}$-transitivity coincide.*

A typical example of such spaces is a Hausdorff space without isolated points.

3. Topological Atoms

This section is devoted to investigation of specific features of atoms in topological spaces.

Proposition 3.1 (characterization of atoms). *Let $A \subset X$. The following statements are equivalent:*

(i) *A is an atom;*

(ii) *$\overline{A}$ is an atom;*

(iii) *if E and F are closed subsets in X and $A \subset E \bigcup F$, then either $A \subset E$ or $A \subset F$;*

(iv) *if $F_1, F_2, \ldots, F_n$ are closed subsets in X and $A \subset \bigcup_{k=1}^{n} F_k$, then there exists $k \in \overline{1,n}$ such that $A \subset F_k$;*

(v) *if $(F_n)_{n=1}^{\infty}$ is a sequence of closed sets in X and $A \subset \overline{\bigcup_{n=1}^{\infty} F_n}$, then either there exists $n \in \mathbb{N}$ such that $A \subset F_n$ or $A \subset \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} F_k}$;*

(vi) *if U and V are opene subsets in X and $A \bigcap U \neq \varnothing$ and $A \bigcap V \neq \varnothing$, then $V \bigcap A \bigcap U \neq \varnothing$;*

(vii) *if $U_1, U_2, \ldots, U_n$ are opene subsets in X and $A \bigcap U_k \neq \varnothing$ for all $k \in \overline{1,n}$, then $A \bigcap \bigcap_{k=1}^{n} U_k \neq \varnothing$;*

Proof. (i) $\Rightarrow$ (iii). Let $A \subset E \bigcup F$. However, $A \not\subset E$. Then $\overline{A} \neq \overline{A \bigcap E}$. Therefore, $\overline{A} = \overline{A \setminus E} \subset F$.

[1] In other words, it is the closure of its interior. Analogously, an open set is called *canonically open* if it is the interior of its closure.

(iii) $\Rightarrow$ (i). If A is not an atom, then there exists $B \subset A$ such that $\overline{A} \neq \overline{B} =: E$, $\overline{A} \neq \overline{A \backslash B} =: F$, and $A \subset \overline{A} = E \bigcup F$.

(iii) $\Rightarrow$ (v). Assume that $A \not\subset F_n$ for any $n \in \mathbb{N}$. We prove that $A \subset E_n := \overline{\bigcup_{k=n}^{\infty} F_k}$ for any $n \in \mathbb{N}$. We prove this by induction. We have $E_1 = \overline{\bigcup_{k=1}^{\infty} F_k} \supset A$. Let $A \subset E_m$. We also have $E_m = E_{m+1} \bigcup F_m$ and, hence, $A \subset E_{m+1}$ since $A \not\subset F_m$. Thus, $A \subset \bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} F_k}$.

(v) $\Rightarrow$ (iv). It suffices to set $F_k = \varnothing$ for $k > n$.

(iv) $\Rightarrow$ (vii). Assume that $A \bigcap \bigcap_{k=1}^{n} U_k = \varnothing$. Then $A \subset X \backslash \bigcap_{k=1}^{n} U_k = \bigcup_{k=1}^{n} F_k$, where $F_k := X \backslash U_k$. This implies that there exists $k \in \overline{1, n}$ such that $A \subset F_k = X \backslash U_k$. Thus, $A \bigcap U_k = \varnothing$, which is a contradiction.

(vii) $\Rightarrow$ (vi). The proof is obvious.

(vi) $\Rightarrow$ (iii). We have to assume the opposite and consider $U = X \backslash E$ and $V = X \backslash F$.

(i) $\Leftrightarrow$ (ii). A and $\overline{A}$ simultaneously either are subsets or are not subsets of closed sets. Therefore, all inclusions in (iii) hold simultaneously for A and $\overline{A}$.

Corollary 3.1. *If A is an atom, then for any open set U such that $A \bigcap U \neq \varnothing$, the inclusion $\overline{A} \subset \overline{U}$ is valid, but the converse statement is not true.*

Proof. $V := X \backslash \overline{U}$ is an open set. Since $U \bigcap A \bigcap V = \varnothing$, we have $\varnothing = A \bigcap V = A \backslash \overline{U}$. This yields $A \subset \overline{U}$.

Let us prove that the converse statement is not true. Consider $X = Y \times Y$, where $Y = [-1, 1]$ is a space equipped with cofinite topology. It is easy to see that Y is an atom. Therefore, by virtue of Proposition 3.1, X is also an atom. Thus, any opene set in X is everywhere dense. Thus, all subsets of X satisfy condition (i) of Proposition 3.1. At the same time, it follows from Proposition 3.1 that $A = \{(x, y) \in X \mid xy = 0\}$ is not an atom because $E = \{(x, y) \subset X \mid y = 0\}$ and $F = \{(x, y) \subset X \mid x = 0\}$ are two closed sets such that each of them does not contain A whereas their union contains it.

Nonetheless, under certain conditions, the converse implication of Corollary 3.1 holds.

Proposition 3.2. *Let $A \subset X$.*

(i) If $\operatorname{int} \overline{A} \neq \varnothing$, then A is an atom if and only if $\overline{A} \subset \overline{U}$ holds for any opene U that intersects A.

(ii) If $\operatorname{int} A \neq \varnothing$, then A is an atom if and only if $\overline{A} = \overline{U}$ holds for any opene U contained in A.

(iii) If A is an atom, then any open subset of A (in particular, $\operatorname{int}(A)$) is also an atom.

(iv) If A is an atom and it is not nowhere dense, then $\overline{A}$ is canonically closed.

Proof. *Necessity* of (i) is proved in Corollary 3.1 and this result obviously implies *necessity* of (ii). Item (iii) is implied by(ii) and Proposition 3.1.

Sufficiency of (i). Let U and V be open and such that $A \bigcap U \neq \varnothing$ and $A \bigcap V \neq \varnothing$. According to Proposition 10.4, we have

$$\operatorname{int} \overline{U \bigcap A \bigcap V} = \operatorname{int} \left(\overline{U \bigcap A} \bigcap \overline{V} \right) = \operatorname{int} \overline{U \bigcap A} \bigcap \operatorname{int} \overline{V} = \operatorname{int} \overline{U} \bigcap \operatorname{int} \overline{A} \bigcap \operatorname{int} \overline{V} \supset \operatorname{int} \overline{A} \neq \varnothing.$$

Therefore, $U \bigcap A \bigcap V \neq \varnothing$. By using Proposition 3.1, we obtain atomicity of A.

Sufficiency of (ii). It is obvious that $A \subset \overline{\operatorname{int} A}$. Let U be an open set that intersects A. It follows from Proposition 10.4 that $V := U \bigcap \operatorname{int} A$ is an opene set contained in A. Under the conditions stated above, we have $\overline{A} = \overline{V} \subset \overline{U}$. Thus, it follows from (i) that A is an atom.

(iv) It follows from Proposition 10.4 that $A \bigcap \operatorname{int}\left(\overline{A}\right) \neq \varnothing$. Therefore, $\operatorname{int}\left(\overline{A}\right) \subset \overline{A} \subset \overline{\operatorname{int}\left(\overline{A}\right)}$.

On the class of open atoms, we introduce the following equivalence relation: $S \sim T$ if and only if $\overline{S} = \overline{T}$.

Corollary 3.2. *(i) For opene atoms S and T, $S \sim T$ if and only if $S \bigcap T \neq \varnothing$.*

(ii) The class that contains an opene atom T is the class of opene subsets of $\operatorname{int} \overline{T}$ (therefore, any union and any finite intersection preserve belonging to a class).

Proof. (i) *Sufficiency*. It follows from Proposition 3.1 that if $S \bigcap T \neq \varnothing$, then $\overline{S} = \overline{S \bigcap T} = \overline{T}$. *Necessity* follows from Proposition 10.4. Condition (ii) follows from Proposition 10.4 and (i).

Proposition 3.3. *If Y is a closed subset of X, then $A \subset Y$ is an atom in Y if and only if A is an atom in X.*

Proof follows from Proposition 3.1 and the fact that if Y is a closed subset of X, then closures of A in X and Y coincide.

Proposition 3.4. *Let Y be a topological space.*

(i) If $f \in C(X, Y)$ and $A \subset X$ is an atom, then $f(A)$ is also an atom.

(ii) If $A \subset X \times Y$ is such that $\operatorname{int} \overline{A} \neq \varnothing$, then A is an atom if and only if its projections $\operatorname{pr}_X A$ and $\operatorname{pr}_Y A$ to X and Y, respectively, are atoms.

Proof. (i) Let $U, V \subset Y$ be open sets such that $f(A) \bigcap U \neq \varnothing \neq f(A) \bigcap V$. Then $f^{-1}(U) \bigcap A \neq \varnothing \neq f^{-1}(V) \bigcap A$ and $f^{-1}(U)$ and $f^{-1}(V)$ are open. Then, by using Proposition 3.1, we obtain

$$f^{-1}(U) \bigcap A \bigcap f^{-1}(V) \neq \varnothing$$

and, hence, $V \bigcap f(A) \bigcap U \neq \varnothing$. According to Proposition 3.1, $f(A)$ is an atom.

(ii) *Necessity* is a consequence of (i) because projections are continuous maps.

Sufficiency. Let $B := \operatorname{pr}_X A$ and $C := \operatorname{pr}_Y A$ be atoms and let U be an open set that intersects A. It is easy to see that there exist open sets V and W in X and Y, respectively, such that $V \times W \subset U$ and $V \times W \bigcap A \neq \varnothing$ because these rectangles form a base of the topology. It is obvious that $B \bigcap V \neq \varnothing$. By virtue of Corollary 3.1, this yields $\overline{B} \subset \overline{V}$. By analogy, we prove that $\overline{C} \subset \overline{W}$. Therefore, $\overline{A} \subset \overline{B} \times \overline{C} \subset \overline{V} \times \overline{W} \subset \overline{U}$. Thus, according to Proposition 3.2, A is an atom.

The condition $\operatorname{int} \overline{A} \neq \varnothing$ in Proposition 3.4 (ii) is essential. As a counterexample, we can use the set from Corollary 3.1. Also note that Propositions 3.1 and 3.4 (i) prove that atoms can be treated as a certain strengthening of the notion of connected sets.

4. Topological Atoms in Dynamical Systems

Let (X, f) be a dynamical system and let T be an open atom. In order to classify the behavior of topological atoms in a dynamical system, we introduce some definitions. We say that T is a *wandering atom* if T is a wandering set. Otherwise, we say that T is a *periodic atom*. For a periodic atom T, its *period* is defined as $\min\left(n(T, T) \backslash \{0\}\right)$

(which, obviously, exists). We say that T a *quasiperiodic atom* if there exists $n \in \mathbb{N}$ such that $f^n(T) \subset \overline{T}$ (the least n is called a *quasiperiod*). It is obvious that this condition is equivalent to $f^n\left(\overline{T}\right) \subset \overline{T}$. In particular, this implies that if an atom is equivalent to a quasiperiodic one, then this atom is also quasiperiodic with the same quasiperiod.

Proposition 4.1. *Any periodic atom is quasiperiodic and its quasiperiod does not exceed its period.*

Proof. Let T be a periodic atom with period n. Then $f^n(T) \bigcap T \neq \varnothing$. Since $f^n(T)$ is an atom and T is open, we have $f^n(T) \subset \overline{T}$ by virtue of Corollary 3.1.

In fact, we can state in this case that the quasiperiod coincides with the period according to item (v) of the following statement:

Proposition 4.2. *Let T be a quasiperiodic atom with quasiperiod n. Then:*

(i) $\overline{f^+(T)} = \bigcup_{k=0}^{n-1} \overline{f^k(T)}$;

(ii) $\overline{f^k(T)} \not\subset \overline{f^l(T)}$ *for* $0 \leq k < l < n$;

(iii) $N(T,T) \subset n\mathbb{Z}$;

(iv) $S := T \backslash \overline{f^n(T)}$ *is a wandering atom;*

(v) *if T is periodic, then its period is equal to n.*

Proof. (i) Since $f^n\left(\overline{T}\right) \subset \overline{T}$, we have $f^+\left(\overline{T}\right) = \bigcup_{k=0}^{n-1} f^k\left(\overline{T}\right)$ according to Proposition 8.2. Thus, it follows from Proposition 10.3 that $\overline{f^+(T)} = \overline{f^+\left(\overline{T}\right)} = \overline{\bigcup_{k=0}^{n-1} f^k\left(\overline{T}\right)} = \bigcup_{k=0}^{n-1} \overline{f^k(T)}$.

(ii) Suppose this is not true, i.e., $\overline{f^k(T)} \subset \overline{f^l(T)}$ for certain $0 \leq k < l < n$. Then $\overline{f^{n+k-l}(T)} = \overline{f^{n-l}\left(f^k(T)\right)} \subset \overline{f^{n-l}\left(f^l(T)\right)} = \overline{f^n(T)} \subset \overline{T}$. Since $k < l$, we have $n + k - l < n$, which contradicts the statement that n is a quasiperiod of T.

(iii) For any $k \in \overline{1, n-1}$, we have $f^{-k}(T) \bigcap T = \varnothing$ because its period is greater than $n-1$ even in the case when T is periodic. Since $f^{-k}(T)$ is an open set and $f^{ln}(T) \subset \overline{T}$ for any $l \in \mathbb{N}_0$, we have $f^{-k}(T) \bigcap f^{ln}(T) = \varnothing$. This yields $T \bigcap f^{-ln-k}(T) = \varnothing$ and, hence, $n(T,T) \subset n\mathbb{N}_0$ and $N(T,T) \subset n\mathbb{Z}$.

(iv) It is obvious that S is an open atom. Moreover, for any $k \in \mathbb{N}$,

$$S \bigcap f^{kn}(S) \subset S \bigcap \overline{f^{kn}(T)} \subset S \bigcap \overline{f^n(T)} = \varnothing.$$

Therefore, $n\left(S,S\right) \bigcap n\mathbb{N} = \varnothing$. At the same time, $n\left(S,S\right) \subset n(T,T) \subset n\mathbb{N}_0$ and, hence, $n\left(S,S\right) = \{0\}$.

(v) If n is not a period of T, then $T \bigcap \overline{f^n(T)} = \varnothing$. With regard for item (iv), we establish that $T = T \backslash \overline{f^n(T)}$ is a wandering atom, which is a contraction.

Now consider the classification of atoms in more details by introducing the following notions: We say that an atom T is *strongly periodic* (*wandering*) if any open atom equivalent to T is also periodic (wandering). It is clear that any strongly periodic (wandering) atom is also periodic (wandering). The other periodic (wandering) atoms are called *weakly periodic* (*wandering*).

Proposition 4.3. *(i)* *T is a strongly wandering atom if and only if* $\operatorname{int} \overline{T}$ *is a wandering atom.*

(ii) *If T is a quasiperiodic atom, then T is a strongly periodic atom if and only if* $\overline{f^n(T)} = \overline{T}$ *(n is a quasiperiod).*

Proof. (i) By virtue of Corollary 3.2, we obtain $\operatorname{int}\overline{T}\sim T$ (*necessity*). If $S\sim T$, then $S\subset\operatorname{int}\overline{T}$. It follows from $n\left(\operatorname{int}\overline{T},\operatorname{int}\overline{T}\right)=\{0\}$ that $n\left(S,S\right)=\{0\}$ *(sufficiency)*.

(ii) *Necessity*. If $\overline{f^n(T)}\neq\overline{T}$, then, by virtue of Proposition 4.2, $S:=T\backslash\overline{f^n(T)}$ is an wandering atom equivalent to T.

Sufficiency. According to Proposition 10.1, the equality $\overline{S}=\overline{T}$ implies that $\overline{f^n(S)}=\overline{f^n(T)}=\overline{T}=\overline{S}$. Therefore, we obtain $f^n(S)\bigcap S\neq\varnothing$ by Proposition 10.4.

Item (i) indicates that a canonically open atom is wandering because it is strongly wandering.

Corollary 4.1. *If T is a strongly periodic atom, then $\overline{f^{kn+l}(T)}=\overline{f^l(T)}$ for any $k,l\in\mathbb{N}_0$.*

Proof. We have to apply Proposition 10.1 to $f^{kn}(T)$ and T (for the map f^l).

5. Simple Dynamical Systems

We say that the dynamical system (X,f) is a *simple dynamical system* if there exists a quasiperiodic atom T with quasiperiod n such that $X=\overline{f^+(T)}$. In this section, let (X,f) be a simple dynamical system and, for any $k\in\overline{0,n-1}$, let $T_k:=\operatorname{int}\overline{f^k(T)}$ be open atoms.

Proposition 5.1. *(i) $T_l=X\backslash\bigcup_{k\in\overline{0,n-1},k\neq l}\overline{T_k}$ for any $l\in\overline{0,n-1}$, i.e., atoms T_k are disjunctive for different k and their union is an everywhere dense set in X.*

(ii) $\partial\overline{f^l(T)}=\overline{f^l(T)}\bigcap\bigcup_{k\in\overline{0,n-1}\backslash\{l\}}\overline{T_k}$ for any $l\in\overline{0,n-1}$.

(iii) The dynamical system (X,f) is $\mathbb{Z}$-transitive.

Proof. (i) Due to Proposition 4.2 we have $\overline{T_l}\not\subset\overline{T_k}$ for $0\leq l<k<n$. Thus, $\overline{T_l}\neq\overline{T_k}$ and, in accordance with Corollary 3.2, this implies disjunction. The density of the union is obvious.

(ii) The statement of this item is implied by (i) and $\partial\overline{f^l(T)}=\overline{f^l(T)}\backslash T_l$.

(iii) Let U be an opene set in X. Since $\bigcup_{k=0}^{n-1}f^k(T)$ is everywhere dense in X, there exists $m\in\overline{0,n-1}$ such that $f^m(T)\bigcap U\neq\varnothing$. This yields $\overline{f^m(T)}\subset\overline{U}$. For any $k\in\overline{0,m}$, we also have $f^k(T)\bigcap f^{k-m}(U)\neq\varnothing$. Since $f^{k-m}(U)$ is opene and $f^k(T)$ is atomic, we conclude that $\overline{f^k(T)}\subset\overline{f^{k-m}(U)}$. It is also obvious that $\overline{f^k(T)}\subset\overline{f^{k-m}(U)}$ for any $k\in\overline{m,n-1}$. Thus, $X=\bigcup_{k=0}^{n-1}\overline{f^k(T)}\subset\bigcup_{k=0}^{n-1}\overline{f^{m-k}(U)}\subset\overline{f^{\pm}(U)}$, which yields $\mathbb{Z}$-transitivity of (X,f).

Proposition 5.2. *(i) $f^{-1}(T)\neq\varnothing$ if and only if T is a periodic atom.*

(ii) If an atom T is periodic, then $\overline{f^{-l}(T)}=\overline{f^{n-l}(T)}$ for any $l\in\overline{1,n-1}$.

Proof. Let $f^{-1}(T)\neq\varnothing$. Then $f^{-1}(T)$ is an opene set. Therefore, $\bigcup_{k=0}^{n-1}f^k(T)\bigcap f^{-1}(T)\neq\varnothing$. Thus, $n\left(T,f^{-1}(T)\right)\neq\varnothing$, which yields the periodicity of T. If T is a periodic atom, then $f^{-l}(T)$ is an opene set for any $l\in\overline{1,n-1}$. Therefore, $f^{-l}(T)\bigcap\bigcup_{k=0}^{n-1}f^k(T)\neq\varnothing$. Since $n\left(T,f^{-l}(T)\right)\bigcap[0,n-1]=\{n-l\}$, we have $f^{-l}(T)\subset X\backslash\bigcup_{0\leq k<n,k\neq n-l}f^k(T)$. Since $f^{-l}(T)$ is open, the last relation implies that

$$f^{-l}(T)\subset X\backslash\overline{\bigcup_{0\leq k<n,l\neq n-l}f^k(T)}=T_{n-l}.$$

By using Proposition 3.2, we obtain $\overline{f^{-l}(T)}=\overline{f^{n-l}(T)}$.

Since characteristics of the system are independent of the atom from the equivalence class, which is chosen to be a "generating" element, similar properties also take place in the case where T is a weakly wandering atom. The following statement unites these two cases:

Proposition 5.3. *(i)* $f^{-1}\left(\operatorname{int}\overline{T}\right) \neq \varnothing$ *if and only if* T *is not a strongly wandering atom.*

(ii) If T *is not a strongly wandering atom, then* $T_l \neq \varnothing$ *for all* $l \in \overline{0, n-1}$ *and* $f\left(\partial T_l\right) \subset \partial T_{l+1} \pmod n$.

(iii) If T *is not a strongly wandering atom and* $S \subset X$ *is an open atom, then* $N\left(S, S\right) \subset n\mathbb{Z}$.

Proof. (i), (ii). According to Propositions 4.3 and 5.2, T is not a strongly wandering atom if and only if $\operatorname{int}\overline{T}$ is a periodic atom. This is true if and only if $f^{-1}\left(\operatorname{int}\overline{T}\right) \neq \varnothing$. Even a more general statement is true: In this case, $T_l = \operatorname{int}\overline{f^{l-n}\left(\operatorname{int}\overline{T}\right)} \neq \varnothing$ for any $l \in \overline{0, n-1}$.

According to Proposition 5.1, $\partial T_l = \overline{T_l} \bigcap \bigcup_{k \in \overline{0,n-1}, k \neq l} \overline{T_k}$. Therefore,

$$f\left(\partial T_l\right) \subset f\left(\overline{T_l}\right) \bigcap f\left(\bigcup_{k \in \overline{0,n-1}, k \neq l} \overline{T_k}\right) \subset \overline{T_{l+1}} \bigcap \bigcup_{k \in \overline{0,n-1}, k \neq l} \overline{T_{k+1}} = \partial T_{l+1}.$$

Here, the addition of unity is understood modulo n. We use the inclusion $\overline{f\left(T_{n-1}\right)} \subset \overline{T_0}$, which is true by Proposition 4.3.

(iii) Let S be an opene atom. Then there exists $m \in \overline{0, n-1}$ such that $S \subset \overline{f^m(T)}$. This also yields $S \subset T_m$. By using Proposition 10.3, we obtain $n\left(S, S\right) \subset n\left(f^m(T), T_m\right)$. Since, by virtue of Proposition 5.1, $T_m = X \backslash \bigcup_{k \in \overline{0,n-1}, k \neq m} \overline{T_k} = X \backslash \bigcup_{k \in \mathbb{N}_0, k \neq m \bmod n} \overline{f^k(T)}$, we have also $n\left(f^m(T), T_m\right) \subset n\mathbb{N}_0$. This yields $N\left(S, S\right) \subset n\mathbb{Z}$.

For systems of considered class, item (ii) of the previous statement indicates in particular that n depends only on X.

Proposition 5.4. *(i) If* T *is a periodic atom, then it is possible that there exists a strongly wandering atom in* X.

(ii) If T *is a wandering atom, then it is possible that there exists a strongly periodic atom in* X*. It is also possible that* $f\left(\partial\overline{f^l(T)}\right) \not\subset \partial\overline{f^{l+1}(T)}$ *for a certain* $l \in \mathbb{N}_0$.

Proof. (i) Let $X = \{1, 2, 3, 4\}$. Closed sets are $\varnothing$, X, $\{2, 3, 4\}$, $\{1, 3, 4\}$, $\{2, 4\}$, $\{3, 4\}$, and $\{4\}$; the map

$$f(x) := \begin{cases} x + 1, & x < 4, \\ 4, & x = 4. \end{cases}$$

The continuity of f is easily verified, $T := \{1, 3\}$ is a periodic atom with period 2, and $S = \{2\}$ is a canonically open wandering atom.

(ii) Let $X = \{1, 2, 3\}$. Closed sets are $\varnothing$, X, $\{1, 2\}$, $\{2, 3\}$, $\{3\}$, and $\{2\}$; the map

$$f(x) := \begin{cases} 3, & x < 3, \\ 2, & x = 3. \end{cases}$$

The continuity of f is easily verified, $T := \{1\}$ is a wandering atom but $S := \{3\}$ is a strongly periodic atom with period 2. We also have $f\left(\partial\overline{f^2(T)}\right) = f\left(\partial\{2\}\right) = f\left(\{2\}\right) = \{3\} \not\subset \varnothing = \partial\overline{f^3(T)}$.

The described pathologies are absent in the following class:

Proposition 5.5. *(i) T is a strongly periodic atom if and only if (X, f) is an $\mathbb{N}$-transitive system.*

(ii) If T is a strongly periodic atom and $S \subset X$ is an open atom, then $N(S, S) = n\mathbb{Z}$.

(iii) If T is a strongly periodic atom, then $f\left(\partial\overline{f^l(T)}\right) \subset \partial\overline{f^{l+1}(T)}$ for any $l \in \mathbb{N}_0$.

Proof. (i) *Necessity*. Let U be an opene set in X. Then there exists $l \in \overline{0, n-1}$ such that $f^l(T) \bigcap U \neq \varnothing$. Therefore, for any $k \in \overline{0, n-1}$, we have $\overline{f^{k+l}(T)} \subset \overline{f^k(U)}$ and, hence,

$$X = \bigcup_{k=0}^{n-1} \overline{f^k(T)} = \bigcup_{k=l}^{l+n-1} \overline{f^k(T)} \subset \bigcup_{k=0}^{n-1} \overline{f^k(U)} \subset \overline{f^+(U)},$$

which means transitivity of (X, f). *Sufficiency* is implied by the absence of wandering opene sets in transitive systems (it was proved in Corollary 2.1).

(ii) Under the conditions of the statement, there exists $m \in \overline{0, n-1}$ such that $\overline{S} = \overline{f^m(T)}$. For any $k \in \mathbb{N}$, the relations $\overline{f^{kn}(S)} = \overline{f^{kn+m}(T)} = \overline{f^m(T)} = \overline{S}$ are true. According to Proposition 10.4, we have $f^{kn}(S) \bigcap S \neq \varnothing$. Thus, $n\mathbb{N}_0 \subset N(S, S)$ and, hence, $n\mathbb{Z} \subset N(S, S)$. The inclusion to the opposite direction was already proved in Proposition 5.2.

(iii) The statement of this item is implied by Corollary 4.1 and Proposition 5.3.

Corollary 5.1. *If X is an atom, then (X, f) is an $\mathbb{N}$-transitive system if and only if $\overline{f(X)} = X$ and if and only if $N(U, V) = \mathbb{Z}$ for any opene sets U and V in X.*

It remains to consider just one class of simple dynamical systems. General properties of these systems are described in items (i) and (ii) of the proposition presented below. Item (iii) gives possible pathologies. The most regular systems of this class are characterized in item (iv).

Proposition 5.6. *If T is a strongly wandering atom, then:*

(i) there exists $m \in \overline{0, n-1}$ such that $T_k \neq \varnothing$ for any $k \in \overline{0, m}$ and $\operatorname{int} \overline{f^k(T)} = \varnothing$ for any $k > m$;

(ii) there are no strongly periodic atoms in the dynamical system (X, f);

(iii) it is possible that $m < n-1$; for a certain $l \in \mathbb{N}_0$, $f\left(\partial\overline{f^l(T)}\right) \not\subset \partial\overline{f^{l+1}(T)}$; periodic atoms can exist;

(iv) if $m = n-1$, then $f\left(\partial\overline{f^l(T)}\right) \subset \partial\overline{f^{l+1}(T)}$ for all $l \in \mathbb{N}_0$ and there are no periodic atoms in the system.

Proof. (i) Let $m \in \overline{0, n-1}$ be the least number from the set of possibly existing numbers such that $\operatorname{int} \overline{f^{m+1}(T)} = \varnothing$ (these numbers always exist because $\operatorname{int} \overline{f^n(T)} = \varnothing$). We prove that $\operatorname{int} \overline{f^k(T)} = \varnothing$ for any $k > m$. If $m < n-1$, then $f^{m+1}(T) \subset \bigcup_{l \in \overline{0,n-1}} \overline{T_l}$. Since $f^{m+1}(T)$ is an atom, Proposition 3.1 implies that there exists $l \in \overline{0, n-1}$ such that $f^{m+1}(T) \subset \overline{T_l}$. Since Proposition 4.2 leads to $f^{m+1}(T) \not\subset \overline{f^l(T)}$ for $l \in \overline{m+2, n-1}$ and $T_{m+1} = \varnothing$, we have $l \in \overline{0, m}$. Thus, we can conclude that T_l is a quasiperiodic atom. Indeed, by using Proposition 4.2, we obtain $\overline{f^+(T_l)} = \bigcup_{i=l}^m \overline{T_i}$ and, hence,

$$X = \overline{f^+(T)} = \bigcup_{i=0}^{l-1} \overline{T_i} \bigcup \overline{f^+(T_l)} = \bigcup_{i=0}^{m} \overline{T_i}.$$

Finally, by using Proposition 5.1, we establish that $\operatorname{int} \overline{f^k(T)} = \varnothing$ for any $k \in \overline{m+1, n-1}$.

Now if $k = np + r,\ 0 \le r < n,\ p \in \mathbb{N}$, then

$$f^k(T) = f^{np+r}(T) = f^r\left((f^n)^p(T)\right) \subset f^r\left(\partial\overline{T}\right) = f^r\left(\overline{T}\bigcap\bigcup_{l\in\overline{1,m}}\overline{f^l(T)}\right) \subset \overline{f^r(T)}\bigcap\bigcup_{l\in\overline{1,m}}\overline{f^{r+l}(T)}.$$

It follows from Propositions 3.4 and 3.1 that there exists $l \in \overline{1,m}$ such that $f^k(T) \subset \overline{f^r(T)}\bigcap\overline{f^{r+l}(T)}$. This yields $\operatorname{int}\overline{f^k(T)} \subset T_r\bigcap\operatorname{int}\overline{f^{r+l}(T)}$. If $l \in \overline{1, m-r}$, then $\operatorname{int}\overline{f^{r+l}(T)} = T_{r+l}$. If $l \in \overline{m-r+1, n-r-1}$, then $\operatorname{int}\overline{f^{r+l}(T)} = \varnothing$. For $l \in \overline{n-r, m}$, we have $\operatorname{int}\overline{f^{r+l}(T)} \subset T_{r+l-n}$. It is easy to see that, for these l, these sets do not intersect T_r and, hence, $\operatorname{int}\overline{f^k(T)} = \varnothing$.

(ii) By virtue of Proposition 5.1, it suffices to verify only T_k. We have $\overline{f^l(T_k)} = \overline{f^l(f^k(T))} = \overline{f^{l+k}(T)}$ for $l \in \mathbb{N}_0$. The set $\overline{f^{l+k}(T)}$ is nowhere dense in X if $l > m-k$ and is contained in T_{k+l} if $l \in \overline{1, m-k}$. Therefore, $\overline{f^{l+k}(T)} \neq T_k$ and, hence, Proposition 4.3 implies that T_k is not a strongly periodic atom.

(iii) Let $X = \{1,2,3,4,5,6\}$. Closed sets are $\varnothing,\ X,\ \{5\},\ \{4,5\},\ \{5,6\},\ \{1,5,6\},\ \{3,5,6\},\ \{2,4,5\}$, and all possible their unions; the map

$$f(x) := \begin{cases} x+1, & x < 5, \\ 5, & x = 5, \\ 4, & x = 6. \end{cases}$$

The continuity of f is easily verified. $T := \{1\}$ is a canonically open atom. Since $\overline{T} = \{1,5,6\},\ f^5(T) = \{5\} \subset \partial T = \{5,6\},\ T$ is a strongly wandering atom, which is quasiperiodic with quasiperiod $n = 5$ at the same time. Since $f^4(T) = \{4\} \Rightarrow \overline{f^4(T)} = \{4,5\}$, we have $\operatorname{int}\overline{f^4(T)} = \varnothing$. Moreover, $S = \{2,4\}$ is an open atom, which is periodic with period 2, and $f\left(\partial\overline{T}\right) = f(\{5,6\}) = \{5,4\} \not\subset \{5\} = \partial\{2,5,4\} = \partial\overline{f(T)}$.

(iv) By analogy with the proof of Proposition 5.3, we can show that $f(\partial T_l) \subset \partial T_{l+1} \pmod n$ for any $l \in \overline{0, n-1}$. Therefore, this result and the equality $\partial\overline{f^l(T)} = \overline{f^l(T)}$ for $l \ge n$ imply that $f\left(\partial\overline{f^l(T)}\right) \subset \partial\overline{f^{l+1}(T)}$. The absence of periodic atoms is implied by $\overline{f^+(f(T_l))} = \bigcup_{k=l+1}^{n-1}\overline{T_k}\bigcup\bigcup_{k=0}^{n-1}\partial T_k$.

6. Topological Atoms in Transitive Dynamical Systems

Proposition 6.1. *Let (X, f) be a $\mathbb{Z}$-transitive system.*

(i) If T is an open wandering set, then T is an atom.

(ii) If A is an invariant set, then $V := \operatorname{int}\left(f^{-1}(A)\backslash A\right)$ is an atom.

Proof. (i) If T is not an atom, then there exist nonintersecting opene subsets U and V such that $0 \notin N(U,V) \subset N(T,T) = \{0\}$. Thus, $N(U,V) = \varnothing$, which contradicts $\mathbb{Z}$-transitivity.

(ii) Since A is an invariant set, we have $f^+(f(V)) \subset A \subset X\backslash V$ and, hence, V is a wandering open set.

Proposition 6.2. *If X is not an atom and (X, f) is an $\mathbb{N}_0$-transitive system, then the preimage of any opene set is nonempty.*

Proof. Assume the contrary. Then there exist an opene set U in X such that $f^{-1}(U) = \varnothing \subset U$. Thus, U is an open wandering set (and therefore, according to Proposition 6.1, it is an atom) and, at the same time, U is an inversely invariant set. It follows from Proposition 2.1 that $\overline{U} = X$. Thus, by using Proposition 3.1, we conclude that X is an atom.

This statement and Proposition 2.2 imply the following result (which was already mentioned above):

Corollary 6.1. *If X is not an atom, then $\mathbb{N}$-transitivity is equivalent to $\mathbb{N}_0$-transitivity.*

It is worth noting that if X is an atom, then any (not necessarily continuous) map satisfies condition $(\mathrm{TT})_{\mathbb{N}_0}$.

Theorem 6.1. *(X, f) is an $\mathbb{N}$-transitive system if and only if (X, f) is a $\mathbb{Z}$-transitive system and this system has no wandering atoms.*

Proof. According to Corollary 2.1, only *sufficiency* has to be proved. If X is an atom, then X is a strongly periodic atom. Thus, according to Proposition 5.5, (X, f) is an $\mathbb{N}$-transitive system. Otherwise, by virtue of Corollary 6.1, it suffices to prove that (X, f) is an $\mathbb{N}_0$-transitive system. To this end, consider a closed set F such that $U := \mathrm{int}(F) \neq \varnothing$ and $f(F) \subset F$. By Proposition 2.1, it suffices to show that $F = X$. According to Proposition 6.1, $f^{-1}(U) \backslash F$ is an open atom that, by virtue of condition, must be empty. Hence, $f^{-1}(U) \subset F$. Since $f^{-1}(U)$ is an open set, we have $f^{-1}(U) \subset \mathrm{int}\, F = U$. This yields $\overline{f^{\pm}(U)} \subset F$, which gives $F = X$ with regard for Proposition 2.3.

Corollary 6.2. *Let the dynamical system (X, f) have an opene atom T. Then (X, f) is an $\mathbb{N}$-transitive system if and only if T is a strongly periodic atom and $X = \overline{f^{+}(T)}$.*

Proof. Since transitive systems contain no wandering atoms, T is a strongly periodic atom. Thus, $\overline{f^{+}(T)}$ is a closed invariant set with nonempty interior. Therefore, by Proposition 2.1, it coincides with X.

Recall that the product of the dynamical systems (X, f) and (Y, g) is defined as the dynamical system $(X \times Y, f \times g)$. By $(X, f)^2$, we denote $(X \times X, f \times f)$. Let us formulate several transitivity properties of products.

Proposition 6.3. *(i) The dynamical system $(X \times Y, f \times g)$ is $\mathbb{N}_0$ $(\mathbb{N}$, or $\mathbb{Z})$-transitive if and only if, for any opene sets U and V in X and any opene sets S and T in Y, $n_f(U,V) \bigcap n_g(S,T) \neq \varnothing$ $(n_f(U,V) \bigcap n_g(S,T)\backslash\{0\} \neq \varnothing$, or $N_f(U,V) \bigcap N_g(S,T) \neq \varnothing$, respectively).*

(ii) If Y is not an atom and $(X \times Y, f \times g)$ is a $\mathbb{Z}$–transitive system, then (X, f) is an $\mathbb{N}$-transitive system.

(iii) If X is an atom and (X, f) is an $\mathbb{N}$-transitive system, then $\mathbb{N}_0$ $(\mathbb{N}$, or $\mathbb{Z})$-transitivity of $(X \times Y, f \times g)$ is equivalent to the corresponding transitivity of (Y, g).

Proof. (i) *Necessity* follows from Proposition 9.5 because $U \times S$ and $V \times T$ are opene sets in $X \times Y$. *Sufficiency* follows from the fact that such a rectangle can be inscribed into any opene set in $X \times Y$.

(ii) By virtue of item (i), (X, f) is a $\mathbb{Z}$-transitive system. Let W be any opene subset of X. Since Y is not an atom, it contains open disjunctive sets U and V. Therefore, $0 \notin N_g(U,V)$ but $\varnothing \neq N_f(W,W) \bigcap N_g(U,V)$. This yields $N_f(W,W) \neq \{0\}$ and, hence, W is not wandering. Thus, by virtue of Theorem 6.1, (X, f) is an $\mathbb{N}$-transitive system.

(iii) The statement of this item is implied by (i) and Corollary 5.1.

For products of nonatomic systems, $\mathbb{N}$-transitivity and $\mathbb{Z}$-transitivity are closely related.

Theorem 6.2. *(i) If X and Y are not atoms, then $\mathbb{N}$-transitivity and $\mathbb{Z}$-transitivity are equivalent properties for the system $(X \times Y, f \times g)$.*

(ii) If $(X, f)^2$ is an $\mathbb{N}$-transitive system, then X either contains no opene atoms or consists of just one atom.

Proof. (i) Let $(X \times Y, f \times g)$ be a $\mathbb{Z}$-transitive system. It is necessary to prove that it is $\mathbb{N}$-transitive. By virtue of Theorem 6.1 and Proposition 3.4, it suffices to consider the case where opene atoms are in X and Y. On the other hand, (X, f) and (Y, g) are $\mathbb{N}$-transitive systems due to Proposition 6.3. According to Corollary 6.2, this means that they are orbits of strongly periodic atoms. Let S and T be these atoms and let m and n be their periods, respectively. According to Proposition 4.3, $S \times T$ is a strongly periodic atom with period $LCM(m, n)$. We also have $\varnothing \neq N_f\left(S, f^{-1}(S)\right) \bigcap N_g(T, T) \subset (n\mathbb{Z} - 1) \bigcap m\mathbb{Z}$, which is possible if and only if m and n are coprime integers. It is easy to see that, in this case, the dynamical system is the closure of the orbit of this atom. Thus, by Proposition 5.5, the system is $\mathbb{N}$-transitive.

(ii) It follows from (i) that, in the presence of atoms in X, $(X, f)^2$ can be $\mathbb{N}$-transitive only if X consists of n atoms. Moreover, n is coprime with itself, i.e., $n = 1$.

Proposition 6.4. *Let (X, f) be a $\mathbb{Z}$-transitive system and let T and S be open atoms. Then either there exists $n \in \mathbb{N}_0$ such that $S \subset \overline{f^{-n}(T)}$ or there exists $n \in \mathbb{N}_0$ such that $T \subset \overline{f^{-n}(S)}$.*

Proof. $\mathbb{Z}$-transitivity implies that $N(S, T) \neq \varnothing$, i.e., there exists $n \in N(S, T)$. Let $n \geq 0$. Then

$$S \bigcap f^{-n}(T) \neq \varnothing,$$

which yields $S \subset \overline{f^{-n}(T)}$ since S is an atom. By analogy, we establish that $T \subset \overline{f^{-|n|}(S)}$ for $n \leq 0$.

7. Wandering Atoms in $\mathbb{Z}$-Transitive Dynamical Systems

In this section, let (X, f) be a $\mathbb{Z}$-transitive dynamical system. By τ, we denote a class of wandering open atoms (including an empty set). Also, let $U := \bigcup_{T \in \tau} T$.

Proposition 7.1. *Let $T \in \tau$. Then:*

(i) $f^{-1}(T) \in \tau$;

(ii) for $n \in \mathbb{N}$, $f^{-n}(T) \neq \varnothing$ if and only if $n \in N(U, T)$.

Proof. (i) Since the preimage of an open wandering set is an open and, by virtue of Proposition 9.7, wandering set, according to Proposition 6.1, it is an atom.

(ii) *Sufficiency* is obvious. We prove *necessity*. Since $f^{-n}(T) \subset U$, it follows from $f^{-n}(T) \neq \varnothing$ that $n \in N(U, T)$.

Corollary 7.1. $f^{-1}(U) \subset U$.

Proof. $f^{-1}(U) = \bigcup_{T \in \tau} f^{-1}(T)$. Since, for any $T \in \tau$, we have $f^{-1}(T) \in \tau$, we have also $f^{-1}(T) \subset U$, which yields $f^{-1}(U) \subset U$.

Proposition 7.2. *If π is a class of equivalent open wandering atoms and $\varnothing \neq \theta \subset \pi$, then $S := \bigcup_{T \in \theta} T \in \pi$.*

Proof. Since $\pi \neq \{\varnothing\}$, we have $N(S, S) = \bigcup_{T, W \in \theta} N(T, W) = \bigcup_{T, W \in \theta} \{0\} = \{0\}$. Thus, S is a wandering atom that contains at least one atom from π, which means that this atom is equivalent to it.

The last statement shows, in particular, that any equivalence class of wandering open atoms contains a maximal element with respect to inclusion. This class can also be considered as a class of open subsets of such maximal elements.

Proposition 7.3. *Let* $T, S, W \in \tau \backslash \{\varnothing\}$.

(i) *If* $T \sim S$, *then* $f^{-1}(T) \sim f^{-1}(S)$.

(ii) $|N(T,S)| = 1$.

(iii) *If* $N(T,S) = \{n\}$ *for some* $n \in \mathbb{Z}$, *then* $\overline{f^n(T)} = \overline{S}$.

(iv) $N(T,S) = N(T,W) + N(W,S)$.

(v) $N(T,W) = N(S,W)$ *if and only if* $T \sim S$.

(vi) *If* $f^{-1}(T) \neq \varnothing$, *then* $N\left(S, f^{-1}(T)\right) = N(S,T) - 1$.

Proof. (i) If $f^{-1}\left(T \bigcap S\right) \neq \varnothing$, then $f^{-1}(T) \bigcap f^{-1}(S) = f^{-1}\left(T \bigcap S\right) \neq \varnothing$. Therefore, $f^{-1}(T) \sim f^{-1}(S)$. Further, $N\left(T \bigcap S, f^{-1}(T)\right) \subset N(T,T) - 1 = \{-1\}$ (by analogy, $N\left(T \bigcap S, f^{-1}(S)\right) \subset \{-1\}$). Thus, if $f^{-1}\left(T \bigcap S\right) = \varnothing$, then $N\left(T \bigcap S, f^{-1}(T)\right) = \varnothing = N\left(T \bigcap S, f^{-1}(S)\right)$. With regard for $\mathbb{Z}$-transitivity of the map f, we obtain $f^{-1}(T) = \varnothing = f^{-1}(S)$, i.e., $f^{-1}(T) \sim f^{-1}(S)$.

(ii) Let $m, n \geq 0$ and let $n, m \in N(S,T)$. Then $f^{-m}(T) \sim S \sim f^{-n}(T)$. This yields $f^{-m}(T) \bigcap f^{-n}(T) \neq \varnothing$ and $m = n$ (the last statement follows from Proposition 9.7). The case $m, n \leq 0$ is proved analogously. Now let $n \geq 0$ and $m \leq 0$. Then $f^{-n}(T) \sim S$ and $f^m(S) \sim T$, i.e., $f^{m-n}(T) \sim T$ and $f^{m-n}(T) \bigcap T \neq \varnothing$. This yields $m = n$. Thus, $|N(T,S)| < 2$. Since $|N(T,S)| > 0$, we have $|N(T,S)| = 1$.

(iii) It follows from Proposition 10.4 that $\overline{S} = \overline{X \bigcap S} = \overline{\overline{f^{\pm}(T)} \bigcap S} = \overline{f^{\pm}(T) \bigcap S} = \overline{f^n(T) \bigcap S} \subset \overline{f^n(T)}$. Simultaneously, $f^n(T)$ is an atom (for $n \geq 0$ according to Proposition 3.4 and for $n \leq 0$ according to Proposition 7.1) that intersects with S. Thus, by virtue of Corollary 3.1, $\overline{f^n(T)} \subset \overline{S}$.

(iv) The statement of this item follows from Proposition 2.3 since each of these sets consists of just one element.

(v) $T \sim S$ if and only if $\{0\} = N(T,S)$. This is equivalent to $N(T,W) = -N(W,S)$ or $N(T,W) = N(S,W)$.

(vi) The proof of this item is analogous to the proof of (iv).

Corollary 7.2. *For opene sets* V *and* W, *the condition* $|N(V,W)| = 1$ *is equivalent to* $V, W \in \tau$.

Proof. *Sufficiency* was proved in the main statement. *Necessity*. Since $1 \leq |N(V,V)| \leq \left|N(V,W) + N(W,V)\right| = 1$, we have $|N(V,V)| = 1$. According to Proposition 6.1, this implies that $V \in \tau$. By analogy, we prove that $W \in \tau$.

Further, by R, we denote $N(T^*, U)$ for a fixed $T^* \in \tau$.

Proposition 7.4. *(i)* R *is a (possibly unbounded) segment in* $\mathbb{Z}$.

(ii) $\tau_n := \{S \in \tau \mid N(T^*, S) = \{n\}\}$ *is a nonempty equivalence class of atoms for any* $n \in R$.

(iii) $\tau = \bigcup_{n \in R} \tau_n \bigcup \{\varnothing\}$ *and* $U = \bigcup_{n \in R} T_n$, *where* T_n *are maximal elements in* τ_n.

Proof. (i) Let $m < n$ and let $m, n \in N(T^*, U)$. This means that there exist $S, W \in \tau$ for which $\{m\} = N(T^*, S)$ and $\{n\} = N(T^*, W)$. This yields $\{n - m\} = N(S,W) \subset N(U,W)$ and $n - m \in \mathbb{N}$. Therefore, for any $k \in \overline{0, n-m}$, we have $\varnothing \neq f^{-k}(W) \subset U$. Thus, $[m,n] \bigcap \mathbb{Z} \subset N\left(T^*, \bigcup_{k=0}^{n-m} f^{-k}(W)\right) \subset N(T^*, U)$, i.e., R is a segment.

(ii) The statement of this item follows from Proposition 7.3. The statement of item (iii) is obvious.

Note that the obtained enumeration of classes depends on the fixed T^*. Nevertheless, it is clear that a set of indices is a segment independently of T^*. In particular, we can choose T^* such that the segment is the most convenient to use (e.g., $\overline{1,N}$; $-\mathbb{N}$; $\mathbb{N}_0 \colon \mathbb{Z}$). The quantity $N(\cdot,\cdot)$ can be considered as an oriented distance between equivalence classes in τ. The number of these classes is at most countable.

Further, let $T_n := \varnothing$ for $n < \inf R$. It is easy to see that the set $f^{-1}(T_n)$ is everywhere dense in T_{n-1} for any $n \in R$.

If the system does not contain opene wandering atoms, then assume that $R := \varnothing$. In this case, according to Proposition 6.1, (X, f) is an $\mathbb{N}$-transitive system.

Proposition 7.5. *Among T_n, there exist at most finitely many elements that are not canonically open. There are no such elements if at least one of the following conditions is satisfied:*

(i) $\sup R = +\infty$. *In this case,* $X = \overline{U}$.

(ii) There exist a closed invariant set F that does not intersect $\bigcup_{n\in R} \operatorname{int}\overline{T_n}$, and $m \in R$ such that $f(T_m) \subset F$. In this case, $m = \sup R$ and $X = \overline{U} \bigcup F$.

Proof. (i) Let $T \in \tau_m$. Then $\overline{f^n(T)} = \overline{T_{m+n}}$ for any $n \in R - m$. Hence, $X = \overline{f^{\pm}(T)} \subset \overline{\bigcup_{n\in R} T_n} = \overline{U}$ and the system has no periodic atoms. Thus, all wandering atoms are canonically open. Elements T_n, as maximal elements of their classes, are canonically open by virtue of Proposition 4.3.

(ii) Since $f(F) \subset F$, we have $\overline{f^+(f(T_m))} \subset F$. Therefore, $X = \overline{f^{\pm}(T_m)} = \overline{f^-(T_m)} \bigcup \overline{f^+(f(T_m))} \subset \overline{f^-(T_m)} \bigcup F$. Since F does not intersect $\bigcup_{n\in R} \operatorname{int}\overline{T_n}$, we have $U \subset \overline{f^-(T_m)}$. This yields $m = \sup R$. For any $n \in R$, $\overline{f^+\left(f\left(\overline{T_n}\right)\right)} = \bigcup_{k=n+1}^{m} \overline{T_k} \bigcup F$. It follows from Proposition 10.4 that this set does not intersect $\operatorname{int}\overline{T_n}$. Thus, $\operatorname{int}\overline{T_n}$ is a wandering set and, hence, $\operatorname{int}\overline{T_n} = T_n$.

We now prove the main statement: Among T_n, there exist at most finitely many elements that are not canonically open. Let $n \in R$ be such that $\operatorname{int}\overline{T_n}$ is a periodic atom with period m. It follows from (i) directly that $\sup R < +\infty$. It suffices to prove that, for $k > m$, T_{n-k} is a canonically open atom or, what is the same, $\operatorname{int}\overline{T_{n-k}}$ is a wandering atom.

First, we prove that $n\left(T_n, \operatorname{int}\overline{T_{n-k}}\right) = \varnothing$. Let $f^l(T_n) \bigcap \operatorname{int}\overline{T_{n-k}} \neq \varnothing$ for $l \in \overline{0, m-1}$. By virtue of Corollary 3.1, we obtain $\overline{f^l(T_n)} \subset \overline{T_{n-k}}$, which yields $\overline{f^m(T_n)} \subset \overline{T_{m+n-k-l}}$. Since $m+n-k-l < n$, it follows from Propositions 10.1 and 10.4 that $\overline{f^m\left(\operatorname{int}\overline{T_n}\right)} \bigcap \operatorname{int}\overline{T_n} = \overline{f^m(T_n)} \bigcap \operatorname{int}\overline{T_n} \subset \overline{T_{m+n-k-l}} \bigcap \operatorname{int}\overline{T_n} = \varnothing$, which contradicts the statement that $\operatorname{int}\overline{T_n}$ is periodic with period m. By using Proposition 4.2, we obtain $\overline{f^+(T_n)} \bigcap \operatorname{int}\overline{T_{n-k}} = \bigcup_{l=0}^{m-1} \overline{f^l(T)} \bigcap \operatorname{int}\overline{T_{n-k}} = \varnothing$ and, hence, $n\left(T_n, \operatorname{int}\overline{T_{n-k}}\right) = \varnothing$.

By using Proposition 10.3, we obtain $n\left(\operatorname{int}\overline{T_{n-k}}, \operatorname{int}\overline{T_{n-k}}\right) = n\left(T_{n-k}, \operatorname{int}\overline{T_{n-k}}\right) \subset N\left(T_{n-k}, \operatorname{int}\overline{T_{n-k}}\right) \subset N\left(T_{n-k}, T_n\right) + N\left(T_n, \operatorname{int}\overline{T_{n-k}}\right) = k + N\left(T_n, \operatorname{int}\overline{T_{n-k}}\right) = k - n\left(\operatorname{int}\overline{T_{n-k}}, T_n\right) = k - n\left(T_{n-k}, T_n\right) = \{0\}$. Thus, $\operatorname{int}\overline{T_{n-k}}$ is a wandering atom.

Proposition 7.6. *Let $Z_+ := \bigcap_{m=0}^{+\infty} \overline{\bigcup_{n=m}^{+\infty} T_n}$ and $Z_- := \bigcap_{m=0}^{-\infty} \overline{\bigcup_{n=m}^{-\infty} T_n}$ ($Z_+ = \varnothing$ for $\sup R < +\infty$ and $Z_- = \varnothing$ for $\inf R > -\infty$). Then $Z_{\pm}$ are closed invariant sets.*

Proof. $f(Z_{\pm}) \subset \bigcap_{m=0}^{\pm\infty} f\left(\overline{\bigcup_{n=m}^{\pm\infty} T_n}\right)$

$$\subset \bigcap_{m=0}^{\pm\infty} \overline{f\left(\bigcup_{n=m}^{\pm\infty} T_n\right)} = \bigcap_{m=0}^{\pm\infty} \overline{\bigcup_{n=m}^{\pm\infty} f(T_n)} \subset \bigcap_{m=0}^{\pm\infty} \overline{\bigcup_{n=m}^{\pm\infty} T_{n+1}} = Z_{\pm}.$$

This statement is given here, in particular, because the set Z_- provides an example of a set that satisfies condition (ii) of Proposition 7.5.

Proposition 7.7. *Let $p \in R$ be such that T_n is a canonically open atom for any $n \in R$ and $n \leq p$. In addition, suppose that either $p+1 \notin R$ or T_{p+1} is not a canonically open atom. Let $V := \bigcup_{n \in R, n \leq p} T_n$ and let $F := \overline{X \backslash \overline{V}}$. Then F is an invariant set. If this set is not empty, then $\overline{f^{+}\left(f\left(T_p\right)\right)} = F$.*

Proof. To prove invariance, it suffices to show that either $f\left(X \backslash \overline{V}\right) \subset F$ (because, in this case, $f(F) \subset \overline{f\left(X \backslash \overline{V}\right)} \subset F$) or $S := f^{-1}\left(\operatorname{int} \overline{V}\right) \bigcap X \backslash \overline{V} = \varnothing$ (because $\operatorname{int} \overline{V} = X \backslash \overline{X \backslash \overline{V}} = X \backslash F$).

Assume that $S \neq \varnothing$. This immediately yields $V \neq \varnothing$. We have

$$f^{-1}(V) = \bigcup_{n \in P} f^{-1}\left(T_n\right) \subset \bigcup_{n \in P} T_{n-1} \subset V.$$

By using Proposition 8.2, we obtain $f^{-}(V) = V \subset X \backslash S$. This yields $n(S, V) = \varnothing$ and, as a corollary, $n\left(S, T_p\right) = \varnothing$. It is clear that S is an open set. Thus, by virtue of $\mathbb{Z}$-transitivity, we get $n\left(T_p, S\right) \neq \varnothing$. It is also easy to see that $f^{-}\left(T_p\right) \subset V \subset \overline{f^{-}\left(T_p\right)}$, which yields $\overline{f(V)} \subset \overline{V} \bigcup \overline{f\left(T_p\right)}$ by virtue of Proposition 10.1. Thus, $l := \min n\left(T_p, S\right) = \min n\left(f^{-}\left(T_p\right), S\right) = \min n\left(\overline{V}, S\right)$.

Let $E := \bigcup_{k=1}^{l} \overline{f^k\left(T_p\right)} \bigcup \overline{V}$. By virtue of Proposition 3.4 and Corollary 3.1, we have $f^l\left(T_p\right) \subset \overline{S}$ and $f(E) \subset \overline{V} \bigcup \bigcup_{k=1}^{l} \overline{f^k\left(T_p\right)} \bigcup \overline{f\left(f^l\left(T_p\right)\right)} \subset E \bigcup \overline{f\left(\overline{S}\right)} \subset E \bigcup \overline{\operatorname{int} \overline{V}} = E$. This yields $\overline{f^{+}\left(T_p\right)} = E$, which, together with $\overline{f^{-}\left(T_p\right)} = \overline{V} \subset E$, gives $X = \overline{f^{\pm}\left(T_p\right)} = E = \bigcup_{k=-\infty}^{l} \overline{f^k\left(T_p\right)}$.

Since $f(S) \subset \overline{V}$, we have $n(S, S) \backslash \{0\} = n(f(S), S) + 1 \subset n\left(\overline{V}, S\right) + 1 \subset [l+1, +\infty)$. This yields $n\left(T_p, f^{-l}(S)\right) \bigcap [0, l] \subset n\left(f^l\left(T_p\right), S\right) \bigcap [0, l] \subset n\left(\overline{S}, S\right) \bigcap [0, l]$. According to Proposition 10.3, this is equivalent to $n(S, S) \bigcap [0, l] = \{0\}$. This result, together with $N\left(T_p, S\right) \bigcap (-\infty, l-1] = \varnothing$, yields

$$N\left(T_p, f^{-l}(S)\right) \bigcap (-\infty, l] = \{0\}.$$

Since $f^{-l}(S)$ is an open set, this implies that $f^{-l}(S) \subset X \backslash \overline{\bigcup_{k \leq, k \neq 0} f^{-k}\left(T_p\right)} = \operatorname{int} \overline{T_p} = T_p$. We have $n(S, S) \backslash \{0\} = n(S, S) \backslash [0, l-1] = n\left(S, f^{-l}(S)\right) + l \subset n\left(S, T_p\right) + l = \varnothing$. Thus, S is a wandering opene set and $S \subset T_{p+l},\ p+l = \sup R$.

At the same time, we have $f\left(T_k\right) \subset \overline{T_{k+1}}$ for $k \in R \backslash \{p, p+l\}$ and $f\left(T_{p+l}\right) \subset \overline{V}$. Thus,

$$f^{-1}\left(\operatorname{int} \overline{T_{p+1}}\right) = f^{-1}\left(X \backslash \overline{\bigcup_{k \in \overline{p+2, p+l}} T_k \bigcup V}\right) \subset X \backslash \overline{\bigcup_{k \in R \backslash \{p\}} T_k}.$$

Since $f^{-1}\left(\operatorname{int} \overline{T_{p+1}}\right)$ is an open set, it is contained in $X \backslash \overline{\bigcup_{k \in R \backslash \{p\}} T_k} = \operatorname{int} \overline{T_p} = T_p$. According to Proposition 10.3, this yields $n\left(\operatorname{int} T_{p+1}, f^{-1}\left(\operatorname{int} T_{p+1}\right)\right) \subset n\left(T_{p+1}, T_p\right) = \varnothing$ and, hence, $\operatorname{int} T_{p+1}$ is a wandering set. We obtain $T_{p+1} = \operatorname{int} T_{p+1}$, which contradicts the condition on p.

If $f\left(T_p\right) \subset \overline{V}$, then $\overline{f(V)} \subset \overline{V}$. Hence, $X = \overline{f^{\pm}(V)} \subset \overline{V}$. Thus, if $F \neq \varnothing$, then $f\left(T_p\right) \subset F$. Since $f(F) \subset F$, we have $\overline{f^{+}\left(f\left(T_p\right)\right)} \subset F$. At the same time, $\overline{f^{+}\left(f\left(T_p\right)\right)} \supset \overline{X \backslash f^{-}\left(T_p\right)} \supset \overline{X \backslash V} \supset F$. Thus, $\overline{f^{+}\left(f\left(T_p\right)\right)} = F$.

Wee can now summarize the obtained results in two theorems presented below, which can be regraded as a complete classification of $\mathbb{Z}$-transitive dynamical systems.

Theorem 7.1. *All equivalence classes of wandering opene atoms can be represented in the form of a sequence $\{\tau_n \mid n \in R\}$, where R is a (possibly unbounded) segment in $\mathbb{Z}$, so that $\tau_n \neq \varnothing$ if any $n \in R$ and the preimages of all elements τ_n are contained in τ_{n-1} for $n > \inf R$. The following properties occur:*

(i) The preimages of all elements τ_l are empty for $l := \inf R \in \mathbb{Z}$.

(ii) The atoms T_n, which are maximal elements of τ_n, possess the following properties: $f\left(\overline{T_n}\right) \subset \overline{T_{n+1}}$ and $f\left(\partial T_n\right) \subset \partial T_{n+1}$ for $n < \sup R$.

(iii) $U = \bigcup_{n\in R} T_n$, $X = \overline{U}$, and all T_n are canonically open atoms for $n \in R$ as $\sup R = +\infty$.

(iv) Let $f\left(T_m\right) \subset Z_-$ or $f\left(T_m\right) \subset Y$ for some $m \in R$, where $Y := \overline{X \backslash \overline{U}}$ is an invariant set. Then all T_n are canonically open atoms for $n \in R$ and $m = \sup R$.

(v) Let some of T_n are non canonically open. Then $m := \sup R < +\infty$ and there exist $q \in R$ and $k \in \mathbb{N}_0$ such that $F := \overline{X \backslash \overline{\bigcup_{n\in R, n<q} T_n}}$ is invariant and coincides with $\bigcup_{n=q}^{m+k} \overline{T_n}$, where T_n are strongly periodic atoms for $n \in \overline{m+1, m+k}$.

Proof. As a method of division of atoms into classes, we can take the method suggested in the proof of Proposition 7.4. Then the main statement (item (i)) and almost the whole item (ii) are obtained by this method with regard for Proposition 7.3. Items (iii) and (iv) are corollaries of Proposition 7.5. Let us complete the proof of item (ii). If $n \in R$ and $n \neq \sup R$, then $f^{-1}\left(T_{n+1}\right) \subset T_n$. Therefore, $f\left(\partial T_n\right) \subset f\left(\overline{T_n}\right) \bigcap X \backslash T_{n+1} \subset \overline{T_{n+1}} \backslash T_{n+1} = \partial T_{n+1}$.

(v) If not all T_n are canonically open, then let q be equal to the least element of the set $\left\{n \in R \mid \operatorname{int} \overline{T_n} \neq T_n\right\}$, which is finite by Proposition 7.5. It follows from Proposition 7.7 that $F := \overline{X \backslash \overline{\bigcup_{n\in R, n<q} T_n}}$ is an invariant set and $F = \overline{f^+\left(T_q\right)}$. Since T_q is a weakly wandering atom, it is a quasiperiodic opene atom. Thus, there exists $k \in \mathbb{N}_0$ such that $F = \bigcup_{n=0}^{m+k-q} \overline{f^n\left(T_q\right)}$ and, hence, $\left(F, f_{|F}\right)$ is simple. By virtue of Proposition 5.3 and the canonical closeness of F, the atom $T_n := \operatorname{int} \overline{f^{n-q}\left(T_q\right)}$ is opene (in X) for $n \in \overline{m+1, m+k}$. This atom is not equivalent to any wandering atom and, hence, does not intersect U. Thus, it is strongly periodic.

The results of the last theorem (more exactly, its main statement and items (i) and (ii)) show how we can structure the space consistently with respect to the action of given map. Items (iii) and (iv) describe possible characteristics of the structure leading to "regularity" of the system. The last item gives a complete description of systems with pathologies. The following theorem describes in details the case that is really contained in item (iv):

Theorem 7.2. *Let $m := \sup R < +\infty$ and let all T_n be canonically open atoms for $n \in R$. Then:*

(i) the set $Y := \overline{X \backslash \overline{U}}$ is invariant and the dynamical system $\left(Y, f_{|Y}\right)$ is $\mathbb{N}$-transitive;

(ii) the segment R is empty if and only if $X = Y$;

(iii) the set Y is empty if and only if either there exists $n \in R$ such that $f\left(T_m\right) \subset \partial T_n$ or $f\left(T_m\right) \subset Z_-$;

(iv) let $R \neq \varnothing$ and $Y \neq \varnothing$, then $\overline{f^+\left(f\left(T_m\right)\right)} = Y$ and $f\left(T_m\right) \bigcap \operatorname{int} Y \neq \varnothing$.

Proof. (i) If all T_n are canonically open atoms for $n \in R$, then Proposition 7.7 implies that Y is an invariant canonically closed set. Thus, by Proposition 2.5, $\left(Y, f_{|Y}\right)$ is a $\mathbb{Z}$-transitive set. Simultaneously, if S is a wandering set open in Y, then $S \bigcap \operatorname{int} Y$ is a wandering set open in X. However, this contradicts the construction. Thus, $\left(Y, f_{|Y}\right)$ does not have wandering open sets. By Theorem 6.1, it is $\mathbb{N}$-transitive.

(ii) It is obvious that the segment R is empty if and only if the set U is empty. It is obvious that the latter is equivalent to the statement that $X = Y$.

(iii) Let the segment R be not empty. Since $N\left(T_m, U\right) = R - m$, we have $f\left(T_m\right) \subset X \backslash U = \overline{X \backslash \overline{U}} \bigcup \overline{U} \backslash U = Y \bigcup \left(\bigcup_{n\in R} \overline{T_n} \bigcup Z_-\right) \backslash U = \bigcup_{n\in R} \partial T_n \bigcup Z_- \bigcup Y$. By virtue of Proposition 3.1, either $f\left(T_m\right) \subset Y$, or there exists $n \in R$ such that $f\left(T_m\right) \subset \partial T_n$, or $f\left(T_m\right) \subset Z_-$. This means that *necessity* is proved. *Sufficiency* follows from Proposition 7.5 because, under the given conditions, the set $\bigcup_{n\in R} \partial T_n \bigcup Z_-$ is closed, invariant, and does not intersect $\bigcup_{n\in R} \operatorname{int} \overline{T_n}$ by virtue of Proposition 7.6 and Theorem 7.1.

(iv) The first part follows from Proposition 7.7. Assuming that $f(T_m) \bigcap \operatorname{int} Y = \varnothing$, we obtain $f(T_m) \subset \partial Y = Y \bigcap \overline{U}$. This yields $f(\partial Y) = f\left(Y \bigcap \overline{U}\right) \subset f(Y) \bigcap f\left(\overline{U}\right) \subset Y \bigcap \left(\overline{U} \bigcup \overline{f(T_m)}\right) \subset \partial Y \bigcup f(T_m) \subset \partial Y$. Therefore, $Y = \overline{f^+(f(T_m))} \subset \partial Y$, which contradicts the fact that Y is canonically open.

Note here that the result of item (iv) can be somewhat improved if we additionally suppose that Y is a union of a finite number of atoms (i.e., $\left(Y, f_{|Y}\right)$ is simple). Namely, if $Y = \bigcup_{k=m+1}^{m+n} \overline{T_k}$, where T_k are canonically open (in X) periodic atoms, then $f(T_m) \bigcap \bigcup_{k=m+1}^{m+n} T_k \neq \varnothing$. This statement is also proved by contradiction. Assuming the contrary, as in the proof of item (iv), we can show that the set $\bigcup_{k=m+1}^{m+n} \partial T_k$ is invariant. To this end, we have to use Proposition 5.3 and $\partial_X \overline{T_k} \subset \partial_Y T_k \bigcup \partial Y$.

In order to obtain more detailed classifications of atoms for elements described in item (v) of Theorem 7.1 and item (iii) of Theorem 7.2 (the first case), we can use the classification of simple dynamical systems given in the corresponding item (in the notation of the corresponding items, the restriction of the original system to F and $\bigcup_{k=n}^{m} \overline{T_k}$ is simple).

It is clear now that Theorem 2.1 follows from the theorems presented above. In the cases where all atoms of the system are wandering or, among the atoms of the system, there are periodic ones, this has been already proved in items (iii) and (v) of Theorems 7.1 and 7.2. If all atoms are either wandering or strongly periodic, then eliminating all wandering atoms, by virtue of Theorem 7.2, we obtain an $\mathbb{N}$-transitive system containing an atom. By Corollary 6.2, this implies that it is a closure of a union of finitely many open atoms. Thus, the entire space is a closure of a union of open atoms.

As was mentioned above, the final sections 8–10 are auxiliary for understanding the main results of the paper.

8. Some Properties of Maps of Sets into Themselves

In this and subsequent sections, let X be a set ("space") and f is a map of X into itself. In what follows, we often use the following lemma:

Lemma 8.1. *In the relation $f^m(f^n(A)) \vee f^{m+n}(A)$, the sign $\vee$ coincides with $=$, $\subset$, or $\supset$ depending on whether $mn \geq 0$, $m > 0$ and $n < 0$, or $m < 0$ and $n > 0$, respectively.*

Recall that the *orbit* of a set $A \subset X$ is defined as the set $f^+(A) := \bigcup_{n \in \mathbb{N}_0} f^n(A)$. We denote the set of all preimages of $A \subset X$ by $f^-(A) := \bigcup_{n \in \mathbb{Z}_0^-} f^n(A)$ and their union by $f^\pm(A) := \bigcup_{n \in \mathbb{Z}} f^n(A)$.

Proposition 8.1. *The following properties of orbits are true:*

(i) $f(f^+(A)) = f^+(f(A)) \subset f^+(A) \subset f^{-1}(f^+(A))$; $f^-\left(f^{-1}(A)\right) = f^{-1}(f^-(A)) \subset f^-(A)$;
$f(f^\pm(A)) \subset f^\pm(A) \subset f^{-1}(f^\pm(A))$.

(ii) $f^-(A) = \{x \in X \mid f^+(X) \bigcap A \neq \varnothing\}$; $f^+(A) = \{x \in X \mid f^-(X) \bigcap A \neq \varnothing\}$;
$f^\pm(A) = \{x \in X \mid f^\pm(X) \bigcap A \neq \varnothing\}$.

(iii) If $B \subset X$, then $f^+(A \bigcup B) = f^+(A) \bigcup f^+(B)$, $f^-(A \bigcup B) = f^-(A) \bigcup f^-(B)$, *and*
$f^\pm(A \bigcup B) = f^\pm(A) \bigcup f^\pm(B)$.

(iv) $f^+(A) = (f^n)^+\left(\bigcup_{k=0}^{n-1} f^k(A)\right)$; $f^-(A) = (f^n)^-\left(\bigcup_{k=0}^{n-1} f^{-k}(A)\right)$ *for any $n \in \mathbb{N}$.*

Proof. Item (i) follows from Lemma 8.1, item (ii) is obvious, item (iii) follows from

$$f^n\left(A \bigcup B\right) = f^n(A) \bigcup f^n(B) \quad \text{for} \quad n \in \mathbb{Z},$$

and item (iv) follows from

$$f^+(A) = \bigcup_{m\in\mathbb{N}_0}\left(\bigcup_{k=0}^{n-1} f^{mn+k}(A)\right) = (f^n)^+\left(\bigcup_{k=0}^{n-1} f^k(A)\right)$$

and

$$f^-(A) = \bigcup_{m\in\mathbb{N}_0}\left(\bigcup_{k=0}^{n-1} f^{-mn-k}(A)\right) = (f^n)^-\left(\bigcup_{k=0}^{n-1} f^{-k}(A)\right).$$

The set $A \subset X$ is called *invariant (inversely invariant)* if $f(A) \subset A$ (respectively, $f^{-1}(A) \subset A$).

Proposition 8.2. *The following properties of invariant sets are true:*

(i) If $f(A) \subset A$, then $f^+(A) = A$ and if $f^{-1}(A) \subset A$, then $f^-(A) = A$.

(ii) $f(A) \subset A$ if and only if $f^{-1}(X\backslash A) \subset X\backslash A$.

(iii) The intersection of any number of (inversely) invariant sets is an (inversely) invariant set.

(iv) If $f^n(A) \subset A$, then $f\left(\bigcup_{k=0}^{n-1} f^k(A)\right) \subset \bigcup_{k=0}^{n-1} f^k(A)$ and $f^+(A) = \bigcup_{k=0}^{n-1} f^k(A)$.

(v) If $f^{-n}(A) \subset A$, then $f^{-1}\left(\bigcup_{k=0}^{n-1} f^{-k}(A)\right) \subset \bigcup_{k=0}^{n-1} f^{-k}(A)$ and $f^-(A) = \bigcup_{k=0}^{n-1} f^{-k}(A)$.

Proof. The first three items are obvious. We prove item (iv):

$$f\left(\bigcup_{k=0}^{n-1} f^k(A)\right) = \bigcup_{k=0}^{n-1} f^{k+1}(A) = f^n(A)\bigcup\bigcup_{k=1}^{n-1} f^k(A) \subset \bigcup_{k=0}^{n-1} f^k(A).$$

By virtue of Proposition 8.1, $f^+(A) = (f^n)^+\left(\bigcup_{k=0}^{n-1} f^k(A)\right) = \bigcup_{k=0}^{n-1} f^k(A)$. Item (v) is proved analogously.

9. Meeting Time Sets and Wandering Sets

Recall that, for $A, B \subset X$, we defined *meeting time sets* of A with respect to to B as

$$n_f(A,B) := \{n \in \mathbb{N}_0 \mid A\bigcap f^{-n}(B) \neq \varnothing\} = \{n \in \mathbb{N}_0 \mid f^n(A)\bigcap B \neq \varnothing\}$$

and

$$N_f(A,B) := \{n \in \mathbb{Z} \mid A\bigcap f^{-n}(B) \neq \varnothing\} = \left\{n \in \mathbb{Z} \mid f^n(A)\bigcap B \neq \varnothing\right\}.$$

As above, we omit the subscript in n_f and N_f if f is clear from the context.

Consider some properties of $n(\cdot,\cdot)$ and $N(\cdot,\cdot)$.

Proposition 9.1. *(i) $N(A,B) = n(A,B)\bigcup(-n(B,A)) = -N(B,A)$.*

(ii) $n(A,B) = N(A,B)\bigcap\mathbb{N}_0$; $n(B,A) = -N(A,B)\bigcap\mathbb{N}_0$.

(iii) $n(A,B) = \varnothing$ if and only if $f^+(A)\bigcap B = \varnothing$ and if and only if $A\bigcap f^-(B) = \varnothing$.

(iv) $N(A,B)=\varnothing$ *if and only if* $f^{\pm}(A)\bigcap B=\varnothing$ *and if and only if* $A\bigcap f^{\pm}(B)=\varnothing$.

(v) $X\backslash f^{+}(A)=\bigcup_{n(A,B)=\varnothing}B;\ X\backslash f^{-}(A)=\bigcup_{n(B,A)=\varnothing}B;\ X\backslash f^{\pm}(A)=\bigcup_{N(A,B)=\varnothing}B.$

Proposition 9.2. *Let* $C,D\subset X$. *Then:*

(i) $N\left(A\bigcup C,B\bigcup D\right)=N(A,B)\bigcup N\left(A,D\right)\bigcup N\left(C,B\right)\bigcup N\left(C,D\right)$;

(ii) $N\left(A\bigcap C,B\bigcap D\right)\subset N(A,B)\bigcap N\left(A,D\right)\bigcap N\left(C,B\right)\bigcap N\left(C,D\right)$;

(iii) $N\left(A,\varnothing\right)=N\left(\varnothing,A\right)=\varnothing$.

Proof. The proof of item (iii) is obvious. Items (i) and (ii) follow from

$$f^n(A)\bigcap\left(B\bigcup D\right)=\left(f^n(A)\bigcap B\right)\bigcup\left(f^n(A)\bigcap D\right)$$

and

$$f^n(A)\bigcap\left(B\bigcap D\right)=\left(f^n(A)\bigcap B\right)\bigcap\left(f^n(A)\bigcap D\right)$$

for any $n\ \in\ \mathbb{Z}$.

Corollary 9.1. *Similar properties are true for* $n\left(\cdot,\cdot\right)$.

It is easy to see that $N\left(\cdot,\cdot\right)$ and $n\left(\cdot,\cdot\right)$ are monotone with respect to natural orders on spaces of arguments and values. Consider properties of $N\left(\cdot,\cdot\right)$ and $n\left(\cdot,\cdot\right)$ with respect to the action of f and f^{-1}.

Proposition 9.3. *(i)* $N\left(f^{-1}(A),B\right)\subset N(A,B)+1\subset N\left(A,f\left(B\right)\right)$ *and*
$N\left(A,f^{-1}(B)\right)\subset N(A,B)-1\subset N\left(f(A),B\right)$.

(ii) $n(f(A),B)=(n(A,B)\backslash\{0\})-1$ *and* $n\big(A,f^{-1}\left(B\right)\big)=(n(A,B)\backslash\{0\})-1$.

Proof. The proof follows from Lemma 8.1.

Corollary 9.2. *(i)* $N\left(f(A),f\left(B\right)\right)\supset N(A,B)\supset N\left(f^{-1}(A),f^{-1}\left(B\right)\right)$.

(ii) $n\left(f(A),f\left(B\right)\right)\supset n(A,B)\supset n\left(f^{-1}(A),f^{-1}\left(B\right)\right)$.

Proposition 9.4. *Let* $n\in\mathbb{N}$. *Then*

$$N_{f^n}(A,B)=\left(N_f(A,B)\bigcap n\mathbb{Z}\right)/n\quad\text{and}\quad n_{f^n}(A,B)=\left(n_f(A,B)\bigcap n\mathbb{N}_0\right)/n.$$

Now let Y be a set and let g be a map of Y into itself. Consider the case of the direct product

$$\left(f\times g\right)\left((x,y)\right):=\left(f(X),g\left(y\right)\right).$$

Proposition 9.5. *Let* $A,B\subset X$ *and* $C,D\subset Y$.

(i) $N_{f\times g}\left(A\times C,B\times D\right)=N_f(A,B)\bigcap N_g\left(C,D\right)$.

(ii) $n_{f\times g}\left(A\times C,B\times D\right)=n_f(A,B)\bigcap n_g\left(C,D\right)$.

Proof. The proof follows from $(f \times g)(A \times C) = f(A) \times g(C)$.

Proposition 9.6. *If $Y \subset X$ is an f-invariant set and $g := f_{|Y}$, then, for $A, B \subset Y$, $n_g(A,B) = n_f(A,B)$ and $N_g(A,B) = N_f(A,B)$.*[2]

Proof. For any $n \in \mathbb{N}_0$, we have $f^n(A) = g^n(A)$. Therefore, $f^n(A) \bigcap B \neq \varnothing$ if and only if $g^n(A) \bigcap B \neq \varnothing$. Thus, $n_g(A,B) = n_f(A,B)$ and

$$N_g(A,B) = -n_g(B,A) \bigcup n_g(A,B) = -n_f(B,A) \bigcup n_f(A,B) = N_f(A,B).$$

Recall that an $A \subset X$ is a *wandering* set if $N(A,A) = \{0\}$. It is easy to see that this equality is equivalent to each one of the following equalities:

$$n(A,A) = \{0\};\ n(f(A),A) = \varnothing;\ n\left(A, f^{-1}(A)\right) = \varnothing;\ f^+(f(A)) \bigcap A = \varnothing;\ \textit{and}\ A \bigcap f^-\left(f^{-1}(A)\right) = \varnothing.$$

Proposition 9.7. *If A is a wandering set, then:*

(i) $f^{-1}(A)$ *is also a wandering set;*

(ii) *for arbitrary $n \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ such that $m \neq -n$, $f^{-n}(A) \bigcap f^m(A) = \varnothing$.*

Proof. The proof of item (i) follows from Corollary 9.2. The proof of item (ii) is based on the inclusion $N(A, f^{-n}(A)) \subset \{-n\}$.

10. Auxiliary Results from Topology

Proposition 10.1. *Let X and Y be topological spaces. The following conditions are equivalent:*

(i) $f \in C(X,Y)$;

(ii) *for any subset $A \subset X$, $f\left(\overline{A}\right) \subset \overline{f(A)}$;*

(iii) *for any subset $C \subset Y$, $f^{-1}(\operatorname{int} C) \subset \operatorname{int} f^{-1}(C)$;*

(iv) *if $\overline{A} = \overline{B}$, then $\overline{f(A)} = \overline{f(B)}$.*

Proof. Equivalence of items (i) and (ii) is known. Item (iii) is a dual condition for item (ii). It is easy to see that (iv) implies (ii). It remains to prove that (iii) $\Rightarrow$ (iv). Indeed, $\overline{f(A)} = \overline{f\left(\overline{A}\right)}$ because $f(A) \subset f\left(\overline{A}\right) \subset \overline{f(A)}$. Therefore, if $\overline{A} = \overline{B}$, then $\overline{f(A)} = \overline{f\left(\overline{A}\right)} = \overline{f\left(\overline{B}\right)} = \overline{f(B)}$.

Proposition 10.2. *Let $f \in S(X)$. Then the following properties hold:*

(i) *The closure of an invariant set is an invariant set. The interior of an inversely invariant set is an inversely invariant set.*

(ii) *For any subset $A \subset X$, $f\left(\overline{f^+(A)}\right) \subset \overline{f^+(A)}$, $f\left(\overline{f^\pm(A)}\right) \subset \overline{f^\pm(A)}$, and $f^{-1}(\operatorname{int} f^-(A)) \subset \operatorname{int} f^-(A)$.*

(iii) *For any open set U, $f^-(U)$ is an open set.*

[2] In these cases, we omit the subscript.

Proof. The proof of item (i) follows from Proposition 10.1. The proof of item (ii) follows from item (i) and Proposition 8.1. The statement of item (iii) is obvious.

Proposition 10.3. *Let $\overline{A} = \overline{B}$. Then:*

(i) If U is an open set, then $n(A,U) = n(B,U)$;

(ii) $\overline{f^{+}(A)} = \overline{f^{+}(B)}$.

Proof. (i) For $n \in \mathbb{N}_0$, $f^{-n}(U)$ is an open set. Thus, $f^{-n}(U) \bigcap A = \varnothing$ if and only if $f^{-n}(U) \bigcap \overline{A} = \varnothing$. This yields $n(A,U) = n\left(\overline{A},U\right) = n\left(\overline{B},U\right) = n(B,U)$. The proof of item (ii) follows from item (i). We have

$$X \backslash \overline{f^{+}(A)} = \bigcup\{U \colon U \text{ is open and } n(A,U) = \varnothing\} = \bigcup\{U \colon U \text{ is open and } n(B,U) = \varnothing\} = X \backslash \overline{f^{+}(B)}$$

(by virtue of Proposition 9.1 and the fact that $X \backslash \overline{f^{+}(A)}$ is open).

The following statement contains several useful facts from the general topology:

Proposition 10.4. *(i) If U and V are disjoint open sets, then $\operatorname{int} \overline{U} \bigcap \overline{V} = \varnothing$.*

(ii) If U is an open set, then, for any $A \subset X$, $\overline{A \bigcap U} = \overline{\overline{A} \bigcap U}$.

(iii) If U is an open set, then, for any $A \subset X$, $\operatorname{int}\left(\overline{A} \bigcap \overline{U}\right) = \operatorname{int} \overline{A \bigcap U}$.

(iv) If U is an open set, then $\operatorname{int}\left(\overline{A} \bigcap \overline{U}\right) \neq \varnothing$ implies that $A \bigcap U \neq \varnothing$. In particular, if $\operatorname{int} \overline{A} \neq \varnothing$, then $A \bigcap \operatorname{int} \overline{A} \neq \varnothing$.

(v) If U is an open set, $A \subset \overline{\operatorname{int} A}$, and $U \bigcap \overline{A} \neq \varnothing$, then $U \bigcap \operatorname{int} A = \operatorname{int}(A \bigcap U) \neq \varnothing$.

Proof. (i) If $U \bigcap V = \varnothing$, then $\overline{U} \bigcap V = \varnothing$. This yields $\operatorname{int} \overline{U} \bigcap V = \varnothing$. Thus, $\operatorname{int} \overline{U} \bigcap \overline{V} = \varnothing$.

(ii) Since $F := X \backslash U$ is a closed set, we get $\overline{A \backslash F} \subset \overline{\overline{A} \backslash F}$. Moreover, $\overline{\overline{A} \backslash F} = \overline{\overline{A} \backslash \overline{F}} \subset \overline{A \backslash F}$. Thus, $\overline{A \backslash F} = \overline{\overline{A} \backslash F}$ and, hence, $\overline{A \bigcap U} = \overline{\overline{A} \bigcap U}$.

(iii) By using item (ii), we obtain

$$\begin{aligned} \operatorname{int} \overline{A \bigcap U} = \operatorname{int} \overline{\overline{A} \bigcap U} \supset \operatorname{int} \overline{\operatorname{int} \overline{A} \bigcap U} = \operatorname{int} \overline{\operatorname{int} \overline{A} \bigcap \overline{U}} \\ \supset \operatorname{int}\left(\operatorname{int} \overline{A} \bigcap \overline{U}\right) = \operatorname{int} \overline{A} \bigcap \operatorname{int} \overline{U} = \operatorname{int}\left(\overline{A} \bigcap \overline{U}\right). \end{aligned} \tag{10.1}$$

The inclusion in the opposite direction is obvious. Thus, item (iv) follows from item (iii).

(v) Let $V := \operatorname{int} A$. If $U \bigcap V = \varnothing$, then $U \bigcap \overline{A} = U \bigcap \overline{V} = \varnothing$, which is a contradiction.

Finally, we note that the referee drew our attention to the recent works [3, 5]. In the first of these works, among other results it is shown that, generally speaking, $\mathbb{N}$-transitivity and $\mathbb{N}_0$-transitivity are not equivalent and suggested certain conditions for their equivalence. In [3], the main results of our paper are proved for the case where the phase space is Hausdorff. Under this assumption, only isolated points can be opene atoms, and this simplifies the construction in some extent. Due to that work, conditions (iv) and (v) in Proposition 2.3 were added. It should be noted that both these works contain numerous examples, unlike our paper.

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