Confidence distributions: A review

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\begin{abstract}
A review is provided of the concept confidence distributions. Material covered include: fundamentals, extensions, applications of confidence distributions and available computer software. We expect that this review could serve as a source of reference and encourage further research with respect to confidence distributions.
\end{abstract}

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1. Introduction

Point estimators, confidence intervals and p-values have long been fundamental and familiar tools for frequentist statisticians. The notation of confidence distributions is less known to statisticians, although it has a long history.

In recent years, there has been a surge of renewed interest in confidence distributions (see, for example, [113,118,119]); so, we feel it is timely to provide a comprehensive review on the subject. In the more recent developments, “the concept of confidence distribution has emerged as a purely frequentist concept. Conceptually, a confidence distribution is no different from a point estimator or an interval estimator (confidence interval), but it uses a sample-dependent distribution function on the parameter space (instead of a point or an interval) to estimate the parameter of interest” [145, p. 8].

Confidence distributions are a way to represent all possible confidence intervals. The area under the confidence density between any two points gives the confidence that the parameter value will lie between those points. Mau [96] stated: “A confidence distribution function is a tool for flexible statistical analyses. It provides one- and two-sided tests of simple and interval hypotheses for any size, and central and symmetric confidence intervals of any level. Given an interval of equivalent values, it quantifies the strength of evidence for ‘no material difference’ between two populations in a set of data, but is independent of the particular choice of such an interval”.

Just as a Bayesian posterior distribution “contains a wealth of information for any type of Bayesian inference, a confidence distribution contains a wealth of information for constructing almost all types of frequentist inferences, including point estimates, confidence intervals and p-values, among others. Some recent developments have highlighted the promising potentials of the confidence distribution concept, as an effective inferential tool” [145, p. 18].

We now give some simple examples of confidence distributions using the concept of pivots. The readers may recall from basic statistics that a pivot is a function of data whose distribution does not depend on population parameters.

For the first example, suppose $x_1, x_2, \ldots, x_n$ is a random sample from a normal distribution with mean $\mu$. Let $\bar{x}$ denote the sample mean and let $s$ denote the sample standard deviation. It is well known $\sqrt{n}(\bar{x} - \mu)/s$ has the Student’s $t$ distribution with $n - 1$ degrees of freedom. So, $\sqrt{n}(\bar{x} - \mu)/s$ is a pivot and a confidence distribution for $\mu$ is $\bar{x} + (s/\sqrt{n})t_{n-1} = C(\mu)$, say, where $t_v$ denotes a Student’s $t$ random variable with degrees of freedom $v$.

For the second example, suppose $x_1, x_2, \ldots, x_n$ is a random sample from a normal distribution with variance $\sigma^2$. Let $s^2$ denote the sample variance. It is well known $(n - 1)s^2/\sigma^2$ has the chi-square distribution with $n - 1$ degrees of freedom. So, $(n - 1)s^2/\sigma^2$ is a pivot and a confidence distribution for $\sigma^2$ is $s^2(n - 1)/\chi_{n-1}^2 = C(\sigma^2)$, say, where $\chi_v^2$ denotes a chi-square random variable with degrees of freedom $v$.

The third example is due to Fraser [42]. Consider a normal distribution with mean $\mu$ and variance $\sigma_0^2$. The $p$-value function from some data $y$ is

$$p(\mu) = \int_{-\infty}^{y} \phi \left( \frac{u - \mu}{\sigma_0} \right) du = \Phi \left( \frac{y - \mu}{\sigma_0} \right)$$

which has normal distribution function shape dropping from 1 at $-\infty$ to 0 at $+\infty$ [42]; it records the probability position of the data with respect to a possible parameter value $\mu$ [42]. The confidence distribution of $\mu$ is normal with mean $y$ and variance $\sigma_0^2$. Other details including the corresponding one-sided test can be found in [42].

Confidence distributions are often generated using bootstrap methods and they are a frequentist concept. Several authors (see, for example, [24]) have compared confidence distributions to Bayesian posteriors and give formula for the implied prior if a confidence distribution is used as a posterior.
Taraldsen and Lindqvist [125] discuss an example, where the posterior is a confidence distribution and the Bayesian credible sets give exact confidence intervals. Parzen [105] shows how quantile functions can be used to interpret the Bayesian posterior distribution as a confidence distribution.

Lindley [89], however, showed that a confidence distribution in the scalar case could not be a Bayes posterior except in the location model case. This means that accuracy for intervals concerning a parameter cannot in general be obtained by a Bayes calculation from likelihood. Asymptotically, Bayes credible intervals often have the first order correct frequentist coverage when sample size tends to infinity (cf., Bernstein–von Mises theorem; LeCam [75,76]).

Confidence distributions can be viewed as “distribution estimators” and are convenient for constructing point estimators, confidence intervals, p-values and more. The basic idea of confidence distributions dates back to Bayes [3] and the fiducial distribution of Fisher [31]. Indeed, as pointed out in [113] the confidence distribution concept is a “Neymanian interpretation of Fisher’s fiducial distribution” [102]. Its developments have proceeded from [31] through various contributions. We mention: Cox [21], Fraser [40], Efron [24], Efron [25], Singh et al. [117], Schweder and Hjort [113], Schweder and Hjort [114], Singh et al. [118,119], Kim and Lindsay [67–69], Bickel [9,11] and Xie and Singh [145]. We also mention the excellent book Cox [22].

Confidence distributions are closely related to fiducial distributions and generalized fiducial distributions. There has been much research on fiducial distributions. Some important recent papers are Fraser [41], Hampel [51], and Hannig et al. [53]. We also mention the following excellent books: Fisher and Bennett [37], Fisher [36], Fisher and Bennett [38], Garthwaite and Jones [47] and Lehmann [78]. For excellent debates about fiducial inference and generalized fiducial inference, and debates with Bayes inference, we refer the readers to Pederson [107], Zabell [150] and Hannig [52]. There is considerable literature on fiducial distributions before 1970, see [33–35,39,103]. We shall not review pre-1970 literature in this paper.

Another much more recent approach for confidence is based on objective Bayes. Some recent papers on this approach are Berger et al. [6], Fraser et al. [45], and Fraser and Reid [44]. The confidence distribution concept subsumes many objective Bayesian posteriors if the posteriors happen to have proper frequentist coverage. Their relationship (confidence distribution versus objective Bayes posterior) is analogous/similar to that of consistent estimator versus maximum likelihood estimator in point estimation. See [143, Section 2] for details.

According to the excellent review paper by Xie and Singh [145], a major theme underlying recent confidence distribution developments and the emerging new field of distributional inference is: “Any statistical approach, regardless of being frequentist, fiducial or Bayesian, can be unified under the concept of confidence distributions, as long as it can be used to build confidence intervals of all levels, exactly or asymptotically”. With this in mind, we consider all aspects of the confidence distribution notion (classical, fiducial, generalized fiducial, Bayes, etc.) in this paper. We review developments in all aspects of the confidence distribution notion and their applications. Details will be elaborated later.

The contents of this paper are organized as follows. In Section 2, we define what a confidence distribution is and describe some of its fundamentals. Some extensions of this notion are described in Section 3. In Section 4, we review some applications of the notion. In Section 5, we mention available computer software in the public domain for confidence and fiducial distributions. Finally, some conclusions are noted in Section 6. We expect that this paper could serve as a source of reference and encourage further research with respect to confidence distributions.

2. Confidence distributions

Many statistical models “with data summarize their inferences about an unknown β in the form of what can be termed a confidence distribution” [43]. By definition, the quantiles of a confidence distribution “span all the possible confidence intervals of a real β” [20]. Two well-known examples are the Bayesian model, which generates a posterior distribution, and the transformation or structural model, which yields conditional confidence intervals summarized by a structural distribution [43]. The confidence distribution is a reinterpretation of the Fisher fiducial distributions, see [25,31,102].

Singh et al. [119] give formal definitions of a confidence distribution (see Definition 2.1) and the associated asymptotic confidence distribution. They provide methods for constructing confidence
distributions, for comparing confidence distributions for the same parameter and related issues, and for exploring inferential information contained within a confidence distribution. They also establish that the normalized profile likelihood function is an asymptotic confidence distribution and formally define and develop the notion of joint confidence distribution for a parameter vector.

In the one parameter model, Schweder and Hjort [113] and Singh et al. [118,119] define the confidence distribution notion that summarizes a family of confidence intervals. Schweder and Hjort [113] state that the confidence distribution "for the real parameter $\beta$ as a distribution depending on the observations $(y, x)$, whose cumulative distribution function evaluated at the true value of $\beta$ has a uniform distribution whatever the true value of $\beta$" [20]. The confidence distribution can be expressed as Definition 2.1 due to Coudin [19]. This definition and definition in [19] follow from [113,118].

**Definition 2.1.** Any distribution with cumulative $C(\beta)$ and quantile function $C^{-1}(\alpha)$ such that

$$\Pr_{\beta} \left[ \beta \leq C^{-1}(\alpha; y; x) \right] = \Pr_{\beta} \left[ C(\beta; y; x) \leq \alpha \right] = \alpha$$

for all $\alpha \in (0, 1)$ and for all probability distributions in the statistical model is called a confidence distribution of $\beta$. If the last equality in (1) is replaced by $\lim_{n \to \infty}$ as the sample size approaches $\infty$ then the distribution is referred to as the asymptotic confidence distribution. Finally, the derivative of a confidence distribution is referred to as the confidence density.

In nontechnical terms, “a confidence distribution is a function of both the parameter and the random sample, with two requirements. The first requirement simply requires that a confidence distribution should be a distribution on the parameter space. The second requirement sets a restriction on the function so that inferences (point estimators, confidence intervals, hypothesis testing, etc.) based on the confidence distribution have desired frequentist properties. This is similar to the restrictions in point estimation to ensure certain desired properties, such as unbiasedness, consistency, efficiency, etc.” [145]. Note that confidence distributions are invariant to monotonic transformations.

Unlike the classical fiducial inference, “more than one confidence distributions may be available to estimate a parameter under any specific setting. Also, unlike the classical fiducial inference, optimality is not a part of requirement. Depending on the setting and the criterion used, sometimes there is a unique “best” (in terms of optimality) confidence distribution. But sometimes there is no optimal confidence distribution available or, in some extreme cases, we may not even be able to find a meaningful confidence distribution. This is not different from the practice of point estimation” [145].

In Definition 2.1, $(-\infty, C^{-1}(\alpha))]$ constitutes a one-sided stochastic confidence interval with coverage probability $\alpha$, and the realized confidence $C(\beta_0; y; x)$ is the $p$-value of the one-sided hypothesis $H_0: \beta \leq \beta_0$ versus $H_1: \beta > \beta_0$ when the observed data are $y, x$ [19]. The realized $p$-value when testing $H_0: \beta = \beta_0$ versus $H_1: \beta \neq \beta_0$ is $2 \min\{C(\beta_0), 1 - C(\beta_0)\}$ [19]. These relations are stated in Lemma 2 of Schweder and Hjort [113], which states: “the confidence of the statement $\beta \leq \beta_0$ is the degree of confidence $C(\beta_0)\) for the confidence interval $(-\infty, C^{-1}(\alpha))$, and is equal to the $p$-value of a test of $H_0: \beta \leq \beta_0$ versus $H_1: \beta > \beta_0$”. So, tests and confidence intervals on $\beta$ are contained in the confidence distribution [19]. Moreover, the values associated with the highest $p$-value for testing $H_0: \beta = \beta_0$ may provide estimators of $\beta$ [19]. The mean or the median $p$-value for testing $H_0: \beta = \beta_0$ may also provide estimators of $\beta$ [19].

Schweder and Hjort [113] also note that, since the cumulative function $C(\beta)$ is an invertible function of $\beta$ and is uniformly distributed, $C(\beta)$ constitutes a pivot conditional on $x$ [19]. Reciprocally, whenever a pivot increases with $\beta$ (for example, a continuous statistic $T(\beta)$ with cumulative distribution function $F$ that is independent of $\beta$ and free of any nuisance parameter), $F(T(\beta))$ is uniformly distributed and satisfies conditions for providing a confidence distribution [19]. Let $\widehat{\beta}$ be such a continuous real statistic increasing with $\beta$ with a free of nuisance parameter distribution [19]. A test of $H_0: \beta \leq \beta_0$ is rejected when $\widehat{\beta}_{\text{obs}}$ is large, with $p$-value $\Pr_{\beta_0}[\widehat{\beta} > \beta_{\text{obs}}] = 1 - F_{\beta_0}(\widehat{\beta}_{\text{obs}}) = C(\beta_0)$.

where $F_{\beta_0}(\beta)$ is the sampling distribution of $\widehat{\beta}$ [19]. Consequently, simulated sampling distributions and simulated realized $p$-values yield a way to construct simulated confidence distributions [19]. The sampling distribution and the confidence distribution are fundamentally different theoretical
notations [19]. The sampling distribution is the probability distribution of \( \hat{\beta} \) obtained by repeated samplings whereas the confidence distribution is an ex-post object which contains the confidence statements one can have on the value of \( \beta \) given \( y, x, \hat{\beta}^{\text{obs}} \) [19].

Since the confidence distribution by design is unbiased (the confidence median is a median unbiased point estimator), it is reasonable to judge the confidence distribution by its spread [113]. The smaller its spread the better [113]. Most measures of spread in the distribution \( C \) around \( \beta_0 \) can be represented by a functional

\[
g(C) = \int_{-\infty}^{0} C(\beta + \beta_0) g(d\beta) + \int_{0}^{\infty} \{1 - C(\beta + \beta_0)\} g(d\beta),
\]

where \( g \) is a measure on the right hand side of the equation [113]. If this measure is chosen as the Lebesgue measure, \( g(C) \) is the mean absolute variation around \( \beta_0 \) with respect to the distribution \( C \) [113].

With spread functional as defined by (2), the Neyman–Pearson lemma for confidence distributions due to Schweder and Hjort [113] can be stated as: “If the confidence distribution \( C \) is a function of the data through a one-dimensional statistic that is sufficient and in which the likelihood ratio is increasing, then \( C \) is uniformly most powerful in the sense that for any other confidence distribution \( C' \) and for any choice of spread functional, \( g(C) \) is ex ante stochastically less than \( g(C') \) when the true parameter of the probability distribution is \( \beta_0 \)”.

While addressing a problem of absurd confidence statement, Plante [108] showed that “the existence of a proper-pivotal vector (a pivotal vector \( T(X, \theta) \) such that the effective range of \( T(X, \cdot) \) is independent of \( x \), and the existence of a confidence distribution are mutually equivalent”.

There are “several sophisticated methods for obtaining confidence distributions from parametric or non-parametric bootstrapping” [27,115]. The acceleration and bias correct bootstrap method introduced by Efron [23], Schweder and Hjort [113] and Schweder et al. [115], assuming no acceleration, can be described as follows:

Suppose that on some transformed scale, from \( \psi \) and \( \hat{\psi} \) to \( \beta = h(\psi) \) and \( \hat{\beta} = h(\hat{\psi}) \), respectively (\( h \) is assumed increasing), one has

\[
(\beta - \hat{\beta}) - b \sim N(0, 1)
\]

to a very good approximation, for suitable constant \( b \) for bias on the normal score scale. Then the pivot in (3) is increasing in \( \beta \) and \( C(\beta) = \Phi(\beta - \beta - b) \) is the confidence distribution for \( \beta \), where \( \Phi(\cdot) \) is the standard normal distribution function. So,

\[
C(\psi) = \Phi(h(\psi) - h(\hat{\psi}) - b),
\]

the confidence distribution for \( \psi \). This approximation to the confidence distribution under assumption (3) is good. It however requires both \( h \) and \( b \) to be known. In practice, we use \( \hat{\beta}^* = h(\hat{\beta}^*) \) bootstrapped from the estimated parametric model. If (3) holds uniformly in a neighbourhood of the true parameters, then

\[
(\hat{\beta}^* - \hat{\beta}) - b \sim N(0, 1)
\]

is a good approximation too. So, the bootstrap distribution is

\[
\hat{G}(t) = \Pr(\hat{\psi}^* \leq t) = \Pr(\hat{\beta}^* \leq h(t)) = \Phi(h(t) - \hat{\beta} + b),
\]

giving \( h(t) = \Phi^{-1}(\hat{G}(t)) - b + \hat{\beta} \). The bias parameter \( b \) follows from \( \hat{G}(\hat{\psi}) = \Phi(b) \). Substituting into (4), we obtain

\[
\hat{C}_{abc}(\psi) = \Phi(\Phi^{-1}(\hat{G}(\psi)) - 2b),
\]

an approximation to the confidence distribution. More details can be found in [23,113,115].

The following problem was examined in [15]: “given an independently and identically distributed sample \( x_1, x_2, \ldots, x_n \) from a location family, does limiting normal shape of the likelihood at the maximum imply limiting normality of the confidence distribution? Conditions were given under which it did, together with a general inversion theorem for transferring such properties to the distribution for conditional analyses.”
3. Extensions

3.1. Bayes and likelihood calculation from confidence distributions [24]

Suppose that a data set \( x \) is observed from a parametric family of densities \( g_\mu(x) \), depending on an unknown parameter vector \( \mu \), and that inferences are desired for \( \beta = t(\mu) \), a real-valued function of \( \mu \) [24]. Let \( \beta_\alpha(\alpha) \) be the upper endpoint of an exact or approximate one-sided level-\( \alpha \) confidence interval for \( \beta \) [24]. The standard intervals, for example, have

\[
\beta_\alpha(a) = \hat{\beta} + \hat{\sigma} z_\alpha,
\]

where \( \hat{\beta} \) is the maximum likelihood estimator of \( \beta \), \( \hat{\sigma} \) is the Fisher information estimator of standard error for \( \hat{\beta} \), and \( z_\alpha \) is the 100\( \alpha \)-quantile of a standard normal distribution, \( z_\alpha = \Phi^{-1}(\alpha) \) [24]. We write the inverse function of \( \beta_\alpha(\alpha) \) as \( \alpha_x(\beta) \), meaning the value of \( \alpha \) corresponding to upper endpoint \( \beta \) for the confidence interval, and assume that \( \alpha_x(\beta) \) is smoothly increasing in \( \beta \) [24].

As defined in Section 2, the confidence distribution for \( \beta \) is the distribution having density

\[
\pi^+_x(\beta) = d\alpha_x(\beta)/d\beta.
\]

Efron [24] proposed a modification of the confidence distributions that enjoys better Bayesian properties. The modification involves a data-doubling device, similar to that used in bias-reduction techniques such as Richardson extrapolation and the jackknife [24]. We imagine having observed two independent datasets from \( g_\mu(\cdot) \), both of which happen to equal the actual data set \( x \) [24]. The doubled data set \((x, x)\) gives upper endpoint \( \beta_{xx}(\alpha) \), depending on the system of confidence intervals being used, and confidence density

\[
\pi^+_{xx}(\beta) = d\alpha_{xx}(\beta)/d\beta,
\]

see [24]. The implied likelihood for \( \beta \) given \( x \) is defined to be the ratio

\[
L^+_x(\beta) = \frac{\pi^+_{xx}(\beta)}{\pi^+_x(\beta)},
\]

see [24]. Efron [24] obtained this by regarding both confidence densities as Bayesian posteriors based on the same prior. Efron [24] showed that \( L^+_x(\beta) \) has better likelihood properties than \( \pi^+_x(\beta) \) because it removes the effect of initial conditions on the confidence density. He presents many examples and theory to verify that \( L^+_x(\beta) \) is a reasonable and practical tool for making Bayes, empirical Bayes and likelihood inferences about \( \beta \).

3.2. Estimating equations and confidence regions [61]

Hu and Kalbfleisch [61] considered interval estimation of a parameter \( \theta \) when “the estimation of \( \theta \) is defined by a linear estimating equation based on independent observations. The method involves bootstrap resampling of the estimating function that defines the equation with \( \theta \) replaced by its estimated value. By this process, the distribution of the estimating function itself can be approximated, a confidence distribution for \( \theta \) is induced and confidence regions can be simply defined”.

Hu and Kalbfleisch [61] termed the procedure, Estimating Function Bootstrap, and, showed, under fairly general conditions, that it “yields confidence intervals whose coefficients are accurate to first order”.

Hu and Kalbfleisch [61]’s ideas are similar to those of Hu and Zidek [63]. The latter developed related bootstrap methods for estimating equations in the linear model. Parzen et al. [106] is also closely related.

3.3. Confidence characteristic of a distribution [14]

After examining the differentiability properties of the decomposition concentration function of Hengartner and Theodorescu [60, p. 110], Boshnakov [14] defines the confidence characteristic of
a distribution as the measure induced by the decomposition concentration function, and related concepts such as the confidence distribution function and confidence density of the confidence characteristic. The arguments of these functions can be interpreted as the Lebesgue measure of the highest density region of the original distribution. Boshnakov [14] shows that at any confidence level, the “Lebesgue measures of the shortest confidence region of a distribution and its confidence characteristic are the same, and that similar properties hold for entropies and the level sets of the densities”.

More specifically, letting \( \omega \) to be a probability measure and \( \nu \) to be the probability measure corresponding to the decomposition concentration function, \( G \), of \( \omega \), Boshnakov [14] defines

- the map from \( \omega \) to \( \nu \) to be the confidence transformation of \( \omega \),
- \( \nu \) to be the confidence characteristic of \( \omega \),
- \( G \) to be the confidence distribution function of \( \omega \) and
- \( g \) to be the confidence density of \( \omega \).

An attractive feature is that the “decomposition concentration function has very good differentiability properties for any distribution” [14].

### 3.4. Combining information from independent sources [118]

Singh et al. [118] develop “methodology, together with related theories, for combining information from independent studies through confidence distributions. Two general combination methods are developed: the first along the lines of combining \( p \)-values, which dates back to Fisher [32], Littell and Folks [90] and Marden [94], among many others, with some notable differences in regard to optimality of Bahadur type efficiency; the second by multiplying and normalizing confidence densities. The latter approach is inspired by the common approach of multiplying likelihood functions for combining parametric information”. The two approaches are compared in the case of combining asymptotic normality based confidence distributions. The resulting function of combined confidence distributions is required to be a confidence distribution (or an asymptotic confidence distribution) so that it can be used to make inferences, store information or combine information in a sequential way.

Singh et al. [118] also develop “adaptive combining methods, with supporting asymptotic theory which should be of practical interest”. The key point of “the adaptive development is that the methods attempt to combine only the correct information, downweighting or excluding studies containing little or wrong information about the true parameter of interest” [118].

In a follow up paper, Xie et al. [146] use confidence distributions to develop a “unifying framework, as well as robust meta-analysis approaches, for combining studies from independent sources .... The proposed combining framework not only unifies most existing meta-analysis approaches, but also leads to development of new approaches”. Xie et al. [146] also “develop, under the unifying framework, two new robust meta-analysis approaches, with supporting asymptotic theory”.

### 3.5. Confidence nets [112]

The concept of confidence distributions is difficult to define in higher dimensions. For example, Fisher’s fiducial distributions in higher dimensions are known not to satisfy the laws of probability. Furthermore, their marginals need not be fiducial. Schweder [112] introduces the concept of confidence nets to avoid these difficulties. This concept for scalars was known as confidence curves [4,12].

A confidence net represents a family of nested confidence regions indexed by degree of confidence. They are obtained by mapping the deviance function into the unit interval. With \( C(\cdot) \) as defined in Section 2, let

\[
N(\theta) = 1 - 2 \min \{ C(\theta), 1 - C(\theta) \} = |1 - 2C(\theta)|.
\]

Schweder [112] refers to \( N(\theta) \) as the tail-symmetric confidence net for \( \theta \). Confidence net is defined as
Definition 3.1. A stochastic function $N$ from parameter space to the unit interval is a confidence net if for each $\theta$, $N(\theta; X)$ is uniformly distributed on the unit interval when the data $X$ follows some distribution parameterized by $\theta$.

Confidence nets can be defined more generally to including vector $\theta$. Let $L(\theta)$ denote the likelihood function of $\theta$. Let $\hat{\theta}$ denote the maximum likelihood estimator of $\theta$. Let $D(\theta) = 2 \log L (\hat{\theta}) - 2 \log L(\theta)$ denote the deviance. Proposition 1 in [112] shows that

$$\tag{5} N(\theta) = F_\theta (D(\theta)),$$

where $F_\theta (\cdot)$ denotes a continuous cumulative distribution function of the deviance evaluated at its true value. Note that (5) also gives the relationship between confidence nets and likelihood.

Schweder [112] shows how confidence nets can be constructed by bootstrapping and the abc-method of Efron [23]. The author also provides an application to personal incomes in Norway.

3.6. Others

Lawless and Fredette [74] utilize the notion of confidence distribution to provide a unified treatment of frequentist prediction intervals and predictive distributions. They note three novel aspects to the development: a simple but general approach is given that produces prediction intervals and predictive distributions with well-calibrated frequentist probability interpretations; simulation-based methods of Beran [5] and Escobar and Meeker [28] are used to generate predictive distributions efficiently; the predictive distributions defined are shown to possess good properties when considered as estimators of the true distributions of $y$ given $x$.

The notion of confidence distribution is not commonly used in the econometric literature, for two reasons [19]. First, it is commonly defined in the one-parameter case [19]. Second, it requires that the test statistic be a pivot with known exact distribution [19]. Coudin [19] extends the notion to multidimensional parameters using a sign transformation.

Now a remark relating to discrete statistics. Confidence distributions based on discrete statistics cannot lead to a continuous uniform distribution [19]. Approximations must be used. Schweder and Hjort [113] propose half correction. For discrete statistics, they use

$$C (\beta_0) = \Pr_{\hat{\beta}_0} \left[ \hat{\beta} > \hat{\beta}_{\text{obs}} \right] + \frac{1}{2} \Pr_{\hat{\beta}_0} \left[ \hat{\beta} = \hat{\beta}_{\text{obs}} \right],$$

based on ideas presented in [122]. Coudin [19] uses a randomization principle for approximation. The discrete statistic $\hat{\beta}$ is associated with an auxiliary one that is independently, uniformly and continuously distributed over $[0, 1]$. Lexicographical order is used to order ties. For other approaches for corrections, see [25,52].

Balch [2] introduces a related concept known as confidence structure: “A confidence structure represents inferential uncertainty in an unknown parameter by defining a belief function whose output is commensurate with Neyman–Pearson confidence”.

Singh and Xie [116] introduce the concept of confidence distribution posterior, the analogue of the usual Bayes posterior for confidence distributions. It is shown to have the coverage properties of the usual Bayes posterior.

Schweder and Hjort [113, Section 3] give several relationships between likelihoods and confidence distributions. Let $\text{piv}(T; \psi)$ denote a pivot in a one-dimensional statistic $T$, an increasing function in $\psi$. Let $F$ and $f$ denote the cumulative distribution and probability density functions of $\text{piv}(T; \psi)$. Proposition 3 in [113] says that the likelihood of $\psi$ can be expressed as

$$L (\psi; T) = f (\text{piv}(T; \psi)) \left| \frac{\partial \text{piv}(T; \psi)}{\partial \psi} \right|.$$

In the case pivot is additive and normally distributed, Proposition 4 in [113] shows that

$$\log L(\psi) = -(1/2) \left\{ \Phi^{-1} (F (C(\psi))) \right\}^2.$$
4. Examples of confidence, fiducial and generalized fiducial distributions

Here, we describe some particular confidence, fiducial and generalized fiducial distributions and some application areas. Because of space restrictions, only a selection is given. Special confidence, fiducial, generalized fiducial and other distributions not mentioned include those relating to: Pareto parameters [142]; common normal mean [65]; proportion [65]; discrete uniform [58]; binomial parameter [58, 133]; log-normal mean [73, 126]; means of two log-normal distributions [73]; multinomial parameters [59]; ratios of log-normal means [56]; inverse Gaussian parameter [64]; exponential reliability function [29, 141]; truncated exponential [104]; unbalanced two-component normal mixed linear model [87]; Weibull mean [72]; means of several log-normal distributions [79]; binomial and Poisson parameters [70]; Weibull reliability [71]; linear measurement error model [129, 130]; noisy and dependent features [100]; Weibull scale and shape [133]; generalized Pareto parameters [132].

4.1. Examples of confidence distributions

Example 4.1.1 (Normal Correlation Coefficient [123]). Let \( f(r; \rho) \) be either an exact or an approximate density of \( r \), the sample correlation coefficient. Define

\[
p(\rho) = \int_{-\infty}^{1} f(u; \rho) du
\]

be the confidence distribution function at an observed \( r \) value. Note that \( p(\rho) \) gives the probabilities to the left of the observed \( r \) value. Sun and Wong [123] show that a 100\((1 - \gamma)\) percent confidence interval for \( \rho \) can be obtained from (6) as

\[
\left( \min \left\{ p^{-1} \left( \frac{\gamma}{2} \right), p^{-1} \left( 1 - \frac{\gamma}{2} \right) \right\}, \max \left\{ p^{-1} \left( \frac{\gamma}{2} \right), p^{-1} \left( 1 - \frac{\gamma}{2} \right) \right\} \right),
\]

the level set of \(|2p(\rho) - 1|\) at \( \gamma \).

Example 4.1.2 (Combining Individual Confidence Intervals [127]). Tian et al. [127] present the following generalization of the combination method of confidence distributions studied by Singh et al. [118]. Suppose that we are interested in constructing a 100\((1 - \alpha)\) percent one-sided confidence interval, \((a, \infty)\) for \( \Delta \), a common parameter, based on all data from \( n \) independent studies. For a given \( \eta \), there are \( n \) study-specific one-sided \( \eta \)-level confidence intervals for \( \Delta \). Now, for any fixed value of \( \Delta \), say zero, we examine whether zero is the true value of \( \Delta \). If yes, then on average, zero should belong to at least 100\(n\eta\) percent of the above \( n \) intervals. The decision on whether the interval \((a, \infty)\) should include zero can be made easily via a simple hypothesis testing procedure. To this end, let \( y_i = 1 \), if zero belongs to the observed \( \eta \) interval from the \( i \)th study, and \( y_i = 0 \), otherwise. Then, we include zero in \((a, \infty)\) if

\[
t(\eta) = \sum_{i=1}^{n} w_i (y_i - \eta) \geq c,
\]

where \( w_i \) is a study-specific positive weight, \( c \) is chosen such that \( \Pr(T(\eta) < c) \leq \alpha \),

\[
t(\eta) = \sum_{i=1}^{n} w_i (B_i - \eta)
\]

is the null counterpart of \( t(\eta) \), and \( \{B_i, \ i = 1, 2, \ldots, n\} \) are \( n \) independent Bernoulli random variables with a "success" probability of \( \eta \). We repeat this process with all other possible values for \( \Delta \) and obtain the final confidence interval, \((a, \infty)\). Here, the weight \( w_i \) may be the sample size for the \( i \)th study.

Example 4.1.3 (Highest Confidence Density Region [128]). Let \( \theta_0 \) be a vector of the unknown true values of \( p \) parameters. Suppose that we are interested in constructing a confidence region for \( \theta_0 \) with a
pre-specified confidence level. Often these parameters have certain intrinsic constrains. Conventionally, such a confidence region is obtained via an estimator of a transformation of \( \theta_0 \).

Tian et al. [128], under a general setting, utilize the confidence distribution to construct classical confidence regions for \( \theta_0 \). They show that the \((1 - \alpha)\) highest confidence density region is a bona fide \((1 - \alpha)\) asymptotic confidence region that has the smallest volume among a rather large class of \((1 - \alpha)\) confidence regions for \( \theta_0 \). They also show that the points in the highest confidence density region can be obtained efficiently via a Markov chain Monte Carlo procedure.

4.2. Examples of fiducial/generalized fiducial distributions

Example 4.2.1 (Quantiles of Two Exponential Populations [50]). Let \( X_{11}, \ldots, X_{1n_1} \) and \( X_{21}, \ldots, X_{2n_2} \) be independent random samples from \( \text{Exp}(\lambda_1) \) and \( \text{Exp}(\lambda_2) \), respectively. Let \( \theta_i \) denote the \( p_i \)th quantile of the \( \text{Exp}(\lambda_i) \) distribution, \( i = 1, 2 \). Note that \( \theta_i = -\log(1 - p_i)/\lambda_i \), \( i = 1, 2 \). For the purpose of testing equality of \( \theta_1 \) and \( \theta_2 \), Guo and Krishnamoorthy [50] show the fiducial distribution of \( \theta_1 - \theta_2 \) as that of

\[
y_1 \frac{\log(1 - p_1)}{\lambda_1 Y_1} - y_2 \frac{\log(1 - p_2)}{\lambda_2 Y_2},
\]

where \( Y_i = \sum_{j=1}^{n_i} X_{ij} \) and \( y_i \) is an observed value of \( Y_i \). It is not difficult to show that the distribution is an \( F \) distribution.

Example 4.2.2 (Quantiles of Two Normal Populations [50]). Let \( X_{11}, \ldots, X_{1n_1} \) and \( X_{21}, \ldots, X_{2n_2} \) be independent random samples from \( \mathcal{N}(\mu_1, \sigma_1^2) \) and \( \mathcal{N}(\mu_2, \sigma_2^2) \), respectively. Let \( \bar{X}_i \) and \( S_i \) denote, respectively, the mean and variance based on the \( i \)th sample, \( i = 1, 2 \). Let \( (\bar{X}_1, \bar{X}_2, S_1^2, S_2^2) \) be an observed value of \( (\bar{X}_1, \bar{X}_2, S_1^2, S_2^2) \).

Let \( Q_{i,p_i} \) denote the \( p_i \)th quantile of the \( \mathcal{N}(\mu_1, \sigma_1^2) \) distribution, \( i = 1, 2 \). Note that \( Q_{i,p_i} = \mu_i + z_{p_i} \sigma_i \), \( i = 1, 2 \), where \( z_{p_i} , 0 < \ell < 1 \) denotes the \( 100p_i \)th quantile of the standard normal distribution. For the purpose of testing equality of \( Q_{1,p_1} \) and \( Q_{2,p_2} \), Guo and Krishnamoorthy [50] derive the fiducial distribution of \( Q_{1,p_1} - Q_{2,p_2} \). When the variances are assumed equal, Guo and Krishnamoorthy [50] show that the fiducial distribution of \( Q_{1,p_1} - Q_{2,p_2} \) is the same as that of

\[
\bar{X}_d - t_{n_1+n_2-2}(\delta_1) s_d \sqrt{1/n_1 + 1/n_2},
\]

where \( \bar{X}_d = \bar{X}_1 - \bar{X}_2, \delta_1 = (z_{p_1} - z_{p_2})/\sqrt{1/n_1 + 1/n_2} \) and \( s_d^2 = ((n_1 - 1) s_1^2 + (n_2 - 1) s_2^2)/(n_1 + n_2 - 2) \) is the pooled sample variance. When the variances are assumed not equal, Guo and Krishnamoorthy [50] show that the fiducial distribution of \( Q_{1,p_1} - Q_{2,p_2} \) is the same as that of

\[
\bar{X}_d - Z \sqrt{\frac{v_1^2}{n_1 U_1^2} + \frac{v_2^2}{n_2 U_2^2} + z_{p_1} \frac{v_1}{W_1^2} - z_{p_2} \frac{v_2}{W_2^2}},
\]

where \( v_i^2 = (n_1 - 1) s_i^2, i = 1, 2, Z, U_1^2, U_2^2, W_1 \) and \( W_2 \) are independent random variables with \( Z \sim \mathcal{N}(0, 1) \), \( U_i^2 \sim \chi^2_{n_i-1}, i = 1, 2 \), and \( W_i^2 \sim \chi^2_{n_i-1}, i = 1, 2 \).

Example 4.2.3 (Exponential Quantile [139]). Suppose \( X_1, X_2, \ldots, X_n \) is a random sample from a two parameter exponential distribution given by the probability density function \( f(x) = (1/\sigma) \exp(-(x - \mu)/\sigma) \) for \( x > \mu, -\infty < \mu < \infty \) and \( \sigma > 0 \). The 100th quantile is \( x_p = \mu - \sigma \log(1 - p) \). Wang and Li [139] show that the fiducial distribution of \( x_p \) can be expressed as

\[
\Pr \left( W - \frac{VX}{nY} - \frac{2V}{Y} \log(1 - p) < x_p \right),
\]
where
\[
W = X_{(1)},
\]
\[
V = \sum_{i=1}^{r} X_{(i)} + (n - r)X_{(r)} - nX_{(1)},
\]
\[
X = 2n(W - \mu)/\sigma,
\]
\[
Y = 2V/\sigma,
\]
and \(X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}\) are order statistics in ascending order. Wang and Li [139] also provide a simulation based algorithm for computing the corresponding fiducial interval.

**Example 4.2.4** *(Quantiles of Several Normal Populations [82])* Suppose \(X_{i1}, \ldots, X_{in_i}\) is an independent random sample from the ith normal population \(N(\mu_i, \sigma_i^2)\), \(i = 1, \ldots, k\). Let \(\bar{X}_i\) and \(S_i^2\) denote the ith sample mean and ith sample variance. For any \(\ell \in (0, 1)\), the \(\ell\)-quantile for the ith normal population \(\theta_i\) satisfies \(\Pr(X_i \leq \theta_i) = \ell\), so that \(\theta_i = \mu_i + \sigma_i z_\ell\), \(i = 1, \ldots, k\), where \(z_\ell\) is the 100\(\ell\) quantile of the standard normal distribution. For the purpose of testing \(H_0 : \theta_1 = \cdots = \theta_k\) versus \(H_1 : \theta_i \neq \theta_j\) for some \(i \neq j\), Li et al. [82] show that the fiducial distribution of \(\theta_i\) is the same as that of
\[
\bar{X}_i - \frac{\sqrt{n_i - 1} z_\ell}{\sqrt{n_i} S_{n_i-1}} + \frac{\sqrt{n_i - 1} S_i}{X_{n_i-1}} z_\ell,
\]
where \(z_\ell\) is a standard normal random variable and \((\bar{X}_i, S_i)\) are observed values of \((\bar{X}_i, S_i)\).

**Example 4.2.5** *(Ratio of Means [65])* Ratio of means is an old problem considered by Fieller [30]. Suppose \((X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)\) is a random sample from a bivariate normal distribution with means \(\mu = (\mu_x, \mu_y)^T\) and variance–covariance matrix
\[
\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}.
\]
Let
\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i,
\]
\[
V = \frac{1}{2(n - 1)} \left[ \sum_{i=1}^{n} (X_i - \bar{X})^2 + \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right],
\]
and let \(\bar{X}, \bar{Y}, \nu\) denote the corresponding observed values. Suppose the interest is the ratio of means \(\theta = \mu_x/\mu_y\). Iyer and Patterson [65] show that the fiducial distribution of \(\theta\) is the same as that of
\[
R = \frac{\bar{X} - (\bar{X} - \mu_x) \sqrt{\bar{Y}}}{\bar{Y} - (\bar{Y} - \mu_y) \sqrt{\bar{X}}}. \sqrt{\nu}.
\]
Iyer and Patterson [65] also note that the distribution of \(R\) may be obtained by observing that, conditional on \(V\), \(R\) is distributed as the ratio of two independent normal random variables, with the numerator being equal to \(\bar{X}\), the denominator being equal to \(\bar{Y}\), and both the numerator and the denominator having a common variance equal to \(\sigma^2 \nu/(nV)\).

**Example 4.2.6** *(Difference in Normal Means [83,147])* Inference on the difference in normal means is popularly known as the Behrens–Fisher problem. Suppose \(X_{11}, \ldots, X_{1n_1}\) and \(X_{21}, \ldots, X_{2n_2}\) are independent random samples from \(N(\mu_1, \sigma_1^2)\) and \(N(\mu_2, \sigma_2^2)\), respectively. Suppose we wish to test
H_0 : \mu_1 = \mu_2 \text{ versus } H_1 : \mu_1 \neq \mu_2. \text{ Li and Xu [83] show that the fiducial distribution of } \mu_i \text{ is that of}

\[ T_i^* = \frac{\mu_i - \bar{X}_i}{\sqrt{S_i^2/n_i}} \sim t_{n_i-1} \]

and that the fiducial distribution of \( \mu_1 - \mu_2 \) is that of

\[ \bar{X}_1 - \bar{X}_2 - \left( T_1^* \sqrt{S_1^2/n_1} - T_2^* \sqrt{S_2^2/n_2} \right). \]

where \( \bar{X}_i \) is the sample mean \( S_i^2 \) is the sample variance. Xiong and Mu [147] show that the fiducial distribution of \( \sigma_i^2 \) is that of

\[ U_i^* = \frac{(n_i - 1) S_i^2}{C_i^2} \]

and that the fiducial distribution of \( \sigma_1^2/n_1 + \sigma_2^2/n_2 \) is that of

\[ \frac{U_1^*}{n_1} + \frac{U_2^*}{n_2}, \]

where \( C_i^2 \sim \chi^2_{n_i-1}, i = 1, 2 \) are independent random variables. Li and Xu [83] also derive fiducial distributions for \( \beta_1 - \beta_2 \) under the regression models \( Y_i = X_i \beta_i + \epsilon_i, i = 1, 2 \). Hu et al. [62] and Li et al. [84] derive similar fiducial distributions for partially linear models. Li et al. [85] derive similar fiducial distributions for varying-coefficient regression models.

**Example 4.2.7** (Multiple Comparisons [131]). In a multiple comparison problem, we have \( k \) populations with means \( \mu = (\mu_1, \ldots, \mu_k) \). Data, which follows an independent normal distribution, is of the form \( X_i = (X_{i1}, \ldots, X_{i|X_{i1}}) \) for all \( i = 1, \ldots, k \), where \( X_i \) is independent of \( X_j \) for all \( i \) and \( j \). We assume \( X_{i1}, \ldots, X_{i|X_{i1}} \) is an independent random sample from \( N(\mu_i, \eta_i) \) for \( i = 1, \ldots, k \). We are interested in the \( k \) treatment means. We would like to make some judgement on the equality or inequality of the means within competing models.

For the problem of multiple comparisons, Wandler and Hannig [131] derive the joint fiducial distribution of population means. They give two sets of expressions: the joint fiducial density of \( (\mu_1, \ldots, \mu_k, \eta) \) for the case of constant variance; and, the joint fiducial density of \( (\mu_1, \ldots, \mu_k, \eta_1, \ldots, \eta_k) \) for the case of non-constant variance.

**Example 4.2.8** (Normal Distribution with Several Variance Components [86]). Liao and Iyer [86] seek tolerance intervals for random variables \( W \) having a normal distribution with mean \( \theta \) and variance \( \tau = \sum_{i=1}^{q} h_i \sigma_i^2 \). This involves deriving fiducial distribution based intervals for \( \theta \) and \( \tau \).

Suppose mutually independent statistics \( \hat{\theta}, S_1^2, \ldots, S_q^2 \) are available, where \( \hat{\theta} \) is normally distributed with mean \( \theta \) and variance \( \sigma^2 = \sum_{i=1}^{q} c_i \sigma_i^2 \), \( h_i \) and \( c_i \) are known constants, and \( U_i = n_i S_i^2 / \sigma_i^2 \) are independent chi-squared random variables with \( n_i \) degrees of freedom, for \( i = 1, 2, \ldots, q \). Define

\[ R_{\theta} = t - \left( \frac{T - \theta}{\sigma} \right) \sqrt{\sum_{i=1}^{q} c_i \sigma_i^2 S_i^2 / S_i^2}, \]

and

\[ R_{\tau} = \sqrt{\sum_{i=1}^{q} h_i \sigma_i^2 S_i^2 / S_i^2}, \]

where \( t \) denotes an observable random variable \( \hat{\theta} \), \( T \) denotes its observed value, and \( S_i^2 \) denotes observed value of \( S_i^2 \).
With the above notation, Liao and Iyer [86] show that the fiducial intervals for \( \theta \) and \( \tau \) are given by 
\[
\{ \theta \mid R_\theta, (1-\alpha)/2 \leq \theta \leq R_\theta, (1+\alpha)/2 \} \quad \text{and} \quad \{ \tau \mid r_\tau \leq R_{r, \alpha} \},
\]
respectively, where \( R_\theta \) and \( r_\tau \) are observed values of \( R_\theta \) and \( R_{r, \alpha} \), respectively, and \( R_{r, \alpha} \) and \( R_{r, \alpha} \) denote percentiles. Liao and Iyer [86] also describe a Monte Carlo algorithm for calculating \( R_{r, \alpha} \) and \( R_{r, \alpha} \).

**Example 4.2.9 (Sum of Variance Components with Unbalanced Designs [80])**. Consider the one-way variance component model \( Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \), where \( i = 1, \ldots, a \), \( j = 1, \ldots, b_i \), and the total sample size is \( n = b_1 + \cdots + b_a \). Let \( \theta_1 \) and \( \theta_2 \) be independent, where \( \theta_1 \) is an identity matrix of dimension \( a \times a \) and \( \Delta \) is a diagonal matrix. Let \( W_1 = (\sigma_a^2 \mathbf{I}_a + \sigma_e^2 \Delta)^{-1/2} \mathbf{T}_1 \) and \( W_2 = T_2^2/\sigma_e^2 \). For observed values \((t_1, t_2) \) of \((\mathbf{T}_1, \mathbf{T}_2) \), Li and Li [80] show that the fiducial distribution of \( \theta \) is

\[
\Pr \left( \frac{\mathbf{t}_1^T \mathbf{t}_1}{\mathbf{W}_1^T \mathbf{W}_1} + \left( 1 - \frac{\mathbf{W}_1^T \Delta \mathbf{W}_1}{\mathbf{W}_1^T \mathbf{W}_1} \right) \frac{T_2}{W_2} < \theta \right).
\]

This can be used to generate fiducial intervals for \( \theta \). Li and Li [81] show that the resulting fiducial intervals are superior to several known ones.

**Example 4.2.10 (Variance Components in an Unbalanced One-Way Random Effects Model [16])**. Consider the one-way random effects model \( Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \), where \( i = 1, \ldots, a \), \( j = 1, \ldots, b_i \), and the total sample size is \( n = b_1 + \cdots + b_a \). Let \( \mathbf{Z} \) be an \( n \times a \) matrix whose elements in the \( i \)-th column are ones for the \( b_i \) observations in class \( A_i \). Let \( \mathbf{H} \) be an \( n \times (n-1) \) matrix whose columns span the space orthogonal to the space spanned by the column vector of ones, and satisfies \( \mathbf{H}^T \mathbf{H} = \mathbf{I} \), where \( \mathbf{I} \) is an \((n-1) \times (n-1)\) identity matrix. Let \( 0 = \Delta_1 < \cdots < \Delta_d \) be the distinct eigenvalues of \( \mathbf{H}^T \mathbf{ZZ}^T \mathbf{H} \) having multiplicities \( r_1, \ldots, r_d \). With this notation, Burch [16] shows that the fiducial distribution of \( \theta \) is that of

\[
\frac{1}{d} \sum_{j=2}^{d} \Delta_j U_j \left( \frac{U_1}{Q_1} \sum_{j=2}^{d} Q_j - \sum_{j=2}^{d} U_j \right),
\]

where \( Q_j = \sigma_e^2 (1 + \Delta_j \theta) U_j \) and \( U_j \sim \chi^2_{r_j}, \) \( j = 1, \ldots, d \) are independent random variables.

**Example 4.2.11 (Balanced One-Way Random Effects Model [65])**. Consider the model \( X_{ij} = \mu + \alpha_i + \epsilon_{ij} \) for \( i = 1, \ldots, a \) and \( j = 1, \ldots, n \), where \( X_{ij} \) is the \( j \)-th observation corresponding to the \( i \)-th random effects \( \alpha_i \). Assume \( \alpha_i \sim N(0, \sigma^2_{\alpha}) \), \( \epsilon_{ij} \sim N(0, \sigma^2_e) \), and \( \alpha_i \) and \( \epsilon_{ij} \) are all jointly independent. Let

\[
\bar{X}_{ia} = \frac{1}{n} \sum_{j=1}^{n} X_{ij}, \quad \bar{X}_{aa} = \frac{1}{a} \sum_{i=1}^{a} \bar{X}_{ia},
\]
\[ SS_w = \sum_{i=1}^{a} \sum_{j=1}^{n} (X_{ij} - \bar{X}_{iw})^2, \quad SS_b = n \sum_{i=1}^{a} (\bar{X}_{iw} - \bar{X}_{w})^2, \]

and let \( ss_w, ss_b \) denote the corresponding observed values. Suppose the interest is in the variance of random effects \( \theta = \sigma^2_a \). Iyer and Patterson [65] show that the fiducial distribution of \( \theta \) is the same as that of

\[ R = \frac{SS_b}{nSS_b} (\sigma^2_e + n\sigma^2_a) - \frac{SS_w}{nSS_w} \sigma^2_e. \]

Iyer and Patterson [65] suggest that the fiducial distribution of \( R \) can be determined numerically. A similar fiducial distribution for \( \theta \) is derived in [149].

Arendacká [1] derives the fiducial distribution for the variance of random effects terms for a model with heteroscedastic errors: \( X_{ij} = \mu + A_i + \epsilon_{ij}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n_i \), where \( \mu \) is the unknown common mean, \( A = (A_1, \ldots, A_k)^T \sim N(\mathbf{0}, \sigma^2_A \mathbf{I}_k) \) is a vector of random effects and \( \epsilon_{ij} \sim N(0, \sigma^2_e) \), \( i = 1, \ldots, k, \quad j = 1, \ldots, n_i \) are random errors, which are mutually independent.

**Example 4.2.12 (General Linear Balanced Model [88])** A general linear random model with balanced data can be described as:

\[ Y = \mathbf{1}_n \mu + \sum_{i=1}^{r} \mathbf{X}_i \beta_i + \epsilon, \]

where \( Y \) is an \( n \times 1 \) observable random vector, \( \mu \) is the overall mean, \( \mathbf{1}_n \) is the vector of length \( n \) with all entries equal to one, \( \beta_i \) is a \( q_i \times 1 \) random vector, and \( \mathbf{X}_i \) is a Kronecker product of identity matrices and the vectors with all entries equal to one. Moreover, it is assumed that the random error \( \epsilon \), the random effects \( \beta_1, \beta_2, \ldots, \beta_r \) are pairwise uncorrelated, and \( \beta_i \) follows a normal distribution with zero means and covariance matrix \( \sigma^2_A \mathbf{I}_q, \quad i = 1, \ldots, r \). For convenience, let \( \sigma^2_{r+1} \) denote \( \sigma^2_e \). Then, the observable total sum of squares for the model can always be partitioned into quadratic forms \( \mathbf{Y}^T \mathbf{A}_i \mathbf{Y} \) such that \( U_i = S^2_i/\psi^2_i = \mathbf{Y}^T \mathbf{A}_i \mathbf{Y}/\psi^2_i \sim \chi^2_i, \quad i = 1, \ldots, r + 1 \), where \( \psi^2_i \) are linear combinations of \( \sigma^2_1, \ldots, \sigma^2_{r+1} \) such that \( \sigma^2_i = \sum_{j=1}^{r+1} c_{ij} \psi^2_j \) for some \( c_{ij} \), and the matrices \( \mathbf{A}_1, \ldots, \mathbf{A}_{r+1} \) are non-negative definite.

For this model, Lin and Liao [88] derive the fiducial distributions of the variance components. They show that the fiducial distribution of \( \psi^2_i \) is the same as that of \( \psi^2_i S^2_i/S^2_i \), where \( S^2_i \) is the observed value of \( S^2_i \) for \( i = 1, \ldots, r + 1 \). It follows that the fiducial distribution of \( \sigma^2_i \) is the same as that of

\[ \sum_{j=1}^{r+1} c_{ij} \psi^2_j S^2_j/S^2_j \]

for \( i = 1, \ldots, r + 1 \).

**Example 4.2.13 (Fixed Effects in Two-Way ANOVA [101])** Mu et al. [101] consider the two-way ANOVA model: \( X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \) for \( k = 1, \ldots, n_{ij}, \quad i = 1, \ldots, I \), and \( j = 1, \ldots, J \), where \( e_{ijk} \) are independent normal random variables with zero means and variances \( \sigma^2_e \), \( \mu \) is the general mean, \( \alpha_i \) is an effect due to the \( i \)th level of the factor \( \alpha \), \( \beta_j \) is an effect due to the \( j \)th level of the factor \( \beta \), and \( \gamma_{ij} \) is an effect due to the interaction of the factor level \( \alpha_i \) and the factor level \( \beta_j \). Let \( n_i = \sum_{j=1}^{J} n_{ij}, \quad n_j = \sum_{i=1}^{I} n_{ij}, \) and \( n = \sum_{i=1}^{I} \sum_{j=1}^{J} n_{ij} \). Let \( \alpha_i, \beta_j \) and \( \gamma_{ij} \) satisfy

\[ \sum_{i=1}^{I} n_i \alpha_i = 0, \quad \sum_{j=1}^{J} n_j \beta_j = 0, \]

\[ \sum_{i=1}^{I} n_i \gamma_{ij} = 0, \quad j = 1, \ldots, J, \]
\[ \sum_{j=1}^{I} n_j \gamma_j = 0, \quad i = 1, \ldots, I. \]

Let \( \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_j \) and

\[
\begin{align*}
X &= (X_{111}, \ldots, X_{1n_1}, \ldots, X_{ij}, \ldots, X_{ijn})^T, \\
A &= \text{diag} (1_{n_1}, \ldots, 1_{n_j}, \ldots, 1_{n_1}, \ldots, 1_{n_j})^T, \\
\theta &= (\mu_{11}, \ldots, \mu_{n_j}, \ldots, \mu_{11}, \ldots, \mu_{n_j})^T, \\
V_{ij} &= \text{diag} (0_{n_1 \times n_1}, \ldots, 1_{n_j}, \ldots, 0_{n_1 \times n_1})^T, \quad i = 1, \ldots, I, \quad j = 1, \ldots, J,
\end{align*}
\]

where \( 1_m, 1_m \) and \( 0_{m \times m} \) denote the \( m \)-vector of ones, the \( m \times m \) identity matrix and the \( m \times m \) zero matrix, respectively.

Under this model, the least squares estimator of \( \theta \) is \( \hat{\theta} = (A^TA)^{-1}A^TX \). Let \( S_{ij}^2 = (X - A\hat{\theta})^T V_{ij} (X - A\hat{\theta}) / (n_j - 1) \). Then Mu et al. [101] show that the fiducial distribution of \( (\theta^T, \sigma_{\theta 1}^2, \ldots, \sigma_{\theta 2}^2)^T \) is that of

\[
\left( \hat{\theta}, \text{diag} \left( \frac{S_{11}}{S_{ij}}, \ldots, \frac{S_{ij}}{S_{ij}} \right) \left( \hat{\theta}^2 - \theta^2 \right) \right)^T, \quad \left( \frac{S_{ij}^2 \sigma_{\theta 1}^2}{S_{ii}^2}, \ldots, \frac{S_{ij}^2 \sigma_{\theta 2}^2}{S_{ii}^2} \right),
\]

where \( (\hat{\theta}^2, S_{ii}^1, \ldots, S_{ij}^*) \) is an independent copy of \( (\theta, S_{ii}^1, \ldots, S_{ij}^*) \).

**Example 4.2.14** (Regressions [20,133]). Consider the regression model \( y_i = \beta x_i + u_i \) for \( i = 1, 2, \ldots, n \), where \( u_i \) and \( x_i \) are independent standard normal random variables. The total sum of squares statistic

\[
T(\beta) = \sum_{i=1}^{n} \left( y_i - x_i \beta \right) x_i \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}^{1/2}
\]

is a pivotal function and decreases with \( \beta \). Coudin and Dufour [20] note that the fiducial distribution function of \( \beta \) given the observations \( y_i, x_i \) is

\[ 1 - F_{\hat{\beta}_0}(T(\beta_0)), \quad (7) \]

where \( F_{\hat{\beta}_0} \) is a Monte Carlo estimate of the sampling distribution of \( T \) under the hypothesis that \( \beta = \beta_0 \) for some \( \beta_0 \). Coudin and Dufour [20] show a simulated version of (7), given two hundred observations of \( (u_i, x_i) \) based on \( T \).

Wang et al. [133] derive fiducial distributions for the simple linear regression model \( Y_i = \beta_0 + \beta_1 x_i + E_i \) for \( i = 1, 2, \ldots, n \), where \( E_i \) are independent normal random variables with zero means and variance \( \sigma^2 \). Let \( B = (B_0, B_1)^T \) denote the least squares estimator of \( \beta = (\beta_0, \beta_1)^T \).

\[
T = \frac{1}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}} \left( \begin{array}{cc}
\frac{\sum_{i=1}^{n} x_i^2}{n} & 0 \\
-\bar{x} / \sqrt{\sum_{i=1}^{n} x_i^2 / n} & \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 / \sum_{i=1}^{n} x_i^2}
\end{array} \right).
\]
and
\[ S^2 = \frac{1}{n-2} \sum_{i=1}^{n} (Y_i - B_0 - B_1 x_i)^2, \quad V = \frac{(n-2)S^2}{\sigma^2} \sim \chi_{n-2}^2. \]

Wang et al. [133] show that the fiducial distributions of \( \sigma \) and \( \beta \) are those of \( \tilde{\sigma} = s / \sqrt{V/(n-2)} \) and \( \tilde{\beta} = \tilde{\sigma} T \Phi \), where \( (s, \tilde{\beta}) \) are observed values of \((S, B)\) and \( \Phi \) is a bivariate standard normal random vector.

**Example 4.2.15** (Assessing Agreement between Two Instruments [136]). Wang and Iyer [136] present an approach for making inferences about the intercept and slope of a linear regression model when both variables are subject to measurement errors. Suppose \((X_i, Y_i), i = 1, \ldots, n\) are pairs of random variables representing the measurements of the \( n \) artefacts using two methods. The lowercase versions, \((x_i, y_i)\), represent the corresponding realized values. Since both \( X_i \) and \( Y_i \) are affected by measurement errors, we write
\[ X_i = \theta_i + e_i, \quad Y_i = \beta_0 + \beta_1 \theta_i + \delta_i, \quad i = 1, \ldots, n, \]
where \( e_i \) and \( \delta_i \) are independent errors assumed to be \( N(0, \sigma^2_x) \) and \( N(0, \sigma^2_y) \), respectively, and at least two \( \theta_i \)’s are distinct. The total number of artefacts available is \( n \), and each artefact is measured by both methods under study.

Let \( S^2_x \) denote an estimator of \( \sigma^2_x \) with degree of freedom \( \nu_x \). Let \( \beta = (\beta_0, \beta_1)^T \), let \( \Phi \) denote a standard bivariate normal random vector, let \( T \) denote the Cholesky factor of \( V = \frac{1}{n} \sum_{i=1}^{n} \theta_i^2 - \left( \sum_{i=1}^{n} \theta_i \right)^2 \), and let \( B = \beta + \sigma_y T \Phi \). Using lowercase letters to denote observed values, Wang and Iyer [136] show that the fiducial distribution of \( \sigma_x \) is that of
\[ \tilde{\sigma}_x = \frac{\sqrt{\nu_x S_x}}{X^2_{\nu_x}} \]
and that the fiducial distribution of \( \theta_i \) is that of
\[ \tilde{\theta}_i = x_i - \frac{X_{\nu_x} S_x}{X^2_{\nu_x}} Z_i, \]
where \( Z_i \) are independent standard normal random variables. Wang and Iyer [136] also show that the fiducial distribution of \( \beta \) is that of \( \tilde{b} = \tilde{\sigma}_y T \Phi \) calculated at \( \theta_i = \tilde{\theta}_i, i = 1, \ldots, n \).

Some other recent applications of fiducial distributions considered by the same authors are: construction of uncertainty intervals and regions for measurements in the presence of both type-A and type-B errors [134,135]; inference on the parameters \( \mu \) and \( \sigma \) of a Gaussian distribution in the presence of resolution errors [54]; construction of uncertainty intervals for the distance between \( k \) normal means and the origin [137]; and, a measure for assessing the equivalence between the results of two laboratories [138].

**Example 4.2.16** (Normal Tolerance Bound [65]). Suppose \( X_1, \ldots, X_n \) is a random sample from \( N(\mu, \sigma^2) \). Let
\[ \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad SS_x = \sum_{i=1}^{n} (X_i - \bar{X})^2, \]
\[ V = \frac{(n-2)S^2_x}{\sigma^2} \sim \chi_{n-2}^2. \]
and let \( \bar{x} \), SS, denote the corresponding observed values. Suppose the interest is the \( \gamma \)-confidence upper tolerance bound \( \theta = \mu + z_\gamma \sigma \). Iyer and Patterson [65] show that the fiducial distribution of \( \theta \) is the same as that of

\[
R = \bar{x} - (\bar{X} - \mu) \sqrt{\frac{\text{SS}_x}{\text{SS}_x} + z_\gamma \sigma} \sqrt{\frac{\text{SS}_x}{\text{SS}_x}},
\]

where \( z_\gamma \) denotes the \( 100\gamma \)th percentile of a standard normal variable. It is easy to show that \( R \) has a non-central \( t \) distribution.

**Example 4.2.17 (Normal Correlation Coefficient [7]).** Let \( f(r; \rho) \) be either an exact or an approximate density of \( r \), the sample correlation coefficient. Berger and Sun [7] suggest the following simulation based method for constructing a \( 100(1 - \gamma) \) percent fiducial interval for \( \rho \):

- draw independent \( Z \sim N(0, 1) \), \( \chi^2_{n-1} \), and \( \chi^2_{n-2} \);
- set \( \rho = Y / \sqrt{1 + Y^2} \), where \( Y = -\left(Z / \sqrt{\chi^2_{n-1}}\right) + \left(r / \sqrt{1 - r^2}\right) \sqrt{\chi^2_{n-2} / \chi^2_{n-1}} \);
- repeat the process ten thousand times;
- use the \( 50\gamma \) upper and lower quantiles of these generated \( \rho \) to form the desired fiducial limits.

This method is based on the fact that fiducial distribution for \( \rho \) can be given by

\[
\psi \left( -\left(Z / \sqrt{\chi^2_{n-1}}\right) + \left(r / \sqrt{1 - r^2}\right) \sqrt{\chi^2_{n-2} / \chi^2_{n-1}} \right),
\]

where \( \psi(x) = x / \sqrt{1 + x^2} \).

**Example 4.2.18 (Comparing Several Coefficients of Variation [92]).** Let \( X_{i1}, X_{i2}, \ldots, X_{in_i}, i = 1, 2, \ldots, k \) be \( k \) independent samples from normal distributions with mean \( \mu_i \) and variance \( \sigma_i^2 \). Let \( \bar{X}_i = (1/n_i) \sum_{j=1}^{n_i} X_{ij}, S_i^2 = (1/n_i) \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \), and let \( \bar{x}_i \) and \( s_i^2 \) represent the observed values of \( \bar{X}_i \) and \( S_i^2 \), respectively. Suppose the interest is in testing \( H_0 : \mu_1 / \sigma_1 = \cdots = \mu_k / \sigma_k \) versus \( H_1 : \) at least one \( \mu_i / \sigma_i \) is different. Let \( C = (\mu_1 / \sigma_1, \ldots, \mu_k / \sigma_k)^T \) and \( A \) be a \((k - 1) \times k\) matrix defined by

\[
A = \begin{pmatrix}
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -1
\end{pmatrix}.
\]

Then testing \( H_0 : \mu_1 / \sigma_1 = \cdots = \mu_k / \sigma_k \) versus \( H_1 : \) at least one \( \mu_i / \sigma_i \) is different is equivalent to testing \( H_0 : AC = 0 \) versus \( H_1 : AC \neq 0 \). Liu et al. [92] show that the fiducial distribution of \( AC \) is that of

\[
A = \left( R_{\mu_1}, \ldots, R_{\mu_k} \right)^T,
\]

where \( R_{\mu_i} = \bar{x}_i - (s_i / S_i) (\bar{X}_i - \mu_i) \) and \( R_{\sigma_i} = (s_i / S_i) \sigma_i \). Liu et al. [92] also provide an expression for the \( p \)-value for testing \( H_0 : AC = 0 \).

4.3. Other examples

Here, we mention some other applications of confidence, fiducial and generalized fiducial distributions.

**Example 4.3.1 (Clinical Equivalence).** Mau [97] proposed an observed confidence distribution as a measure of strength of evidence for practically equivalent efficacies of two treatments. The concept is independent of prior opinions about relevant sizes of a difference in efficacy. It also avoids
retrospective power calculations for trials with missed recruitment goals and allows combination of the confidence from independent experiments. For other applications to clinical trials, see [66], Example 4.2 in [86,144].

**Example 4.3.2 (Particle Physics).** The interpretation of results of searches for new particles and phenomena near the sensitivity limit of an experiment is a common problem in particle physics. The loss of sensitivity may be due to a combination of small signal rates, the presence of background comparable to the expected signal, and the loss of discrimination between two models due to insufficient experimental resolution. The search for Higgs bosons at the Large Electron–Positron Collider is such an experiment.

Up until 2000, no significant evidence of Higgs production at the Large Electron–Positron Collider has been observed and no lower bounds on Higgs masses have been reported. Read [110] shows how the confidence distribution method can be used to derive a lower bound. This method has since been successfully used in general scans over the many-parameter space of the minimal super symmetric standard model in searches for the $h$ and $A$ neutral bosons, in two-parameter searches for charged Higgs bosons of a general two-doublet Higgs model, and in combined searches for particles by the Large Electron–Positron Collider working group for super symmetry particle searches.

**Example 4.3.3 (Wavelet Regression).** Hannig and Lee [55] use the confidence distribution idea for statistical inference for wavelet regression. They construct pointwise confidence intervals and address issues of estimation. The development raises two main challenges: incorporating a statistical model selection procedure into the confidence distribution framework; and applying the confidence distribution idea to nonparametric curve estimation problems.

The first challenge is addressed under a general methodology that integrates the confidence distribution idea with common statistical model selection methods in a development analogous to the extension of maximum likelihood to penalized maximum likelihood. With this new methodology, the second challenge is resolved by determining the structural equation that relates the wavelet coefficients and the noisy observations.

**Example 4.3.4 (Others).** Modelling of data concerning bowhead whales off Alaska [111,115]; modelling the abundance of Minke whales in the Northeast Atlantic [120]; medical decision making [4]; estimating proportions in microbial risk assessment [99]; minimum volume confidence regions for a multivariate normal mean vector [26]; population dynamic models [98]; accounting for error due to instrument resolution [54]; cosmic shear photometric redshifts [93]; identical readings under low instrument resolution [46]; mixtures of distributions for centrally censored data with partial identification [17]; confidence intervals for two-stage sample-size-flexible design [140]; genome–scale screening [8]; generalized data augmentation [49]; analysis of rounded exponential data [124]; gene expression [10]; measurement of the asymmetry of Poisson flows [13]; link to the Dempster–Shafer theory of recombination of information from different sources [57]; estimation of false discovery rates [77]; multivariate meta-analysis of heterogeneous studies [91]; efficient network meta-analysis [148].

5. Computer software

Computer software for implementing confidence and fiducial distributions are available. We mention available software available from the R package [109]. Most of these packages appear to cater for fiducial distributions. There appears to be only one R package for confidence distributions, the gmeta package due to Guang Yang, Pixu Shi and Minge Xie of Rutgers University, USA. This begs the need for more software development.

The package PropCIs: computes “confidence intervals for single proportions as well as for differences in dependent and independent proportions, the odds-ratio and the relative risk in a $2 \times 2$ table. Intervals are available for independent samples and matched pairs”. This package was written by Ralph Scherer of the Hannover Medical School, Germany and Alan Agresti of the University of Florida, Gainesville, USA. This package is partly based on the fiducial confidence limits developed in [18].
The package diffdepprop: includes functions “to calculate confidence intervals for the difference of dependent proportions. There are two functions implemented to edit the data (dichotomizing with the help of cutpoints, counting accordance and discordance of two tests or situations). For the calculation of the confidence intervals entries of the fourfold table are needed”. This package was written by Daniela Wenzel and Antonia Zapf. This package is also partly based on the fiducial confidence limits developed in [18].

The package GenBinomApps: computes the “density, distribution function, quantile function and random generation for the Generalized Binomial Distribution. It includes functions to compute the Clopper–Pearson confidence interval and the required sample size. It includes an enhanced model for burn-in studies, where failures are tackled by countermeasures”. This package was written by Horst Lewitschnig and David Lenzi. This package is also partly based on the fiducial confidence limits developed in [18].

The package OptimalCutpoints: enables users to compute “one or more optimal cutpoints for diagnostic tests or continuous markers. Various approaches for selecting optimal cutoffs have been implemented, including methods based on cost-benefit analysis and diagnostic test accuracy measures (Sensitivity/Specificity, Predictive Values and Diagnostic Likelihood Ratios). Numerical and graphical output for all methods is easily obtained”. This package was written by Monica Lopez-Raton and Maria Xose Rodriguez-Alvarez of the Universidade de Santiago de Compostela, Spain. This package is also partly based on the fiducial confidence limits developed in [18].

The package PowerTOST: contains “functions to calculate power and sample size for various study designs used for bioequivalence studies. Moreover, the package contains functions for power and sample size based on ‘expected’ power in case of uncertain (estimated) variability. The package contains functions for the power and sample size for the ratio of two means with normally distributed data on the original scale (based on Fieller’s confidence (‘fiducial’) interval). The package contains further functions for power and sample size calculations based on non-inferiority t-test. This is not a TOST procedure but eventually useful if the question of ‘non-superiority’ must be evaluated. The power and sample size calculations based on non-inferiority test may also performed via ‘expected’ power in case of uncertain (estimated) variability. The package contains functions power.scABEL() and sampleN.scABEL() to calculate power and sample size for the BE decision via scaled (widened) BE acceptance limits based on simulations. The package contains functions power.RSABE() and sampleN.RSABE() to calculate power and sample size for the BE decision via reference scaled ABE criterion according to the FDA procedure based on simulations. The package contains further functions power.NTIDFDA() and sampleN.NTIDFDA() to calculate power and sample size for the BE decision via the FDA procedure for NTID’s based on simulations”. This package was written by Detlew Labes. This package is partly based on the fiducial confidence limits developed in [30].

The package exactci: calculates “exact tests and confidence intervals for one-sample binomial and one- or two-sample Poisson cases”. This package was written by M.P. Fay of the National Institute of Health, USA. This package is partly based on the fiducial confidence limits developed in [48,121].

The package epiR: analyzes “epidemiological data. It contains functions for directly and indirectly adjusting measures of disease frequency, quantifying measures of association on the basis of single or multiple strata of count data presented in a contingency table, and computing confidence intervals around incidence risk and incidence rate estimates. It also contains miscellaneous functions for use in meta-analysis, diagnostic test interpretation, and sample size calculations”. This package was written by Mark Stevenson from Massey University, New Zealand with contributions from Telmo Nunes, Cord Heuer, Jonathon Marshall, Javier Sanchez, Ron Thornton, Jeno Reiczigel, Jim Robison-Cox, Paola Sebastiani, Peter Solymos and Kazuki Yoshida. This package is partly based on the fiducial confidence limits developed in [18,121].

The package tolerance: provides “the limits between which we can expect to find a specified proportion of a sampled population with a given level of confidence. This package provides functions for estimating tolerance limits for various distributions. Plotting is also available for tolerance limits of continuous random variables”. This package was written by Derek S. Young of the Penn State University, USA. This package is partly based on the fiducial confidence limits developed in [70,95].

The package gmeta: provides “an all-in-one solution for meta-analysis problems. It offers a single function that can conduct all current prevalence meta-analysis approaches. Specifically, the function
gmeta provided in package unifies meta-analysis methods, including the combining of $p$-values, fitting meta-analytic fixed-effect and random-effects models, and synthesizing $2 \times 2$ tables evidence, under the same framework of combining confidence distributions (CDs). Furthermore, the gmeta covers two recent developed exact meta-analysis approaches for the combination of $2 \times 2$ tables. The gmeta in addition contributes several robust meta-analysis methods that are able to achieve a level of resistance to model-misspecification and protect the combined results from the impact of outlying studies. The gmeta package also implements a plot function to examine the individual and combined CDs. This package was written by Guang Yang, Pixu Shi and Minge Xie of Rutgers University, USA. This package is partly based on the confidence distribution developments due to Xie et al. [146].

6. Conclusion

The notion of confidence distribution, an entirely frequentist concept, is in essence a Neymannian interpretation of Fisher's fiducial distribution. It contains information related to every kind of frequentist inference. The confidence distribution is a direct generalization of the confidence interval, and is a useful format of presenting statistical inference.

The following quotation from [25] on Fisher's contribution of the fiducial distribution seems quite relevant: "...here is a safe prediction for the 21st century: statisticians will be asked to solve bigger and more complicated problems. I believe there is a good chance that objective Bayes methods will be developed for such problem, and that something like fiducial inference will play an important role in this development. Maybe Fisher's biggest blunder will become a big hit in the 21st century!". We hope that the review provided could be helpful to materialize the prediction by Efron.

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References


