Locally robustable gain scheduling in nonlinear systems with uncertain time varying inputs

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In this paper, we propose a gain scheduling control law in nonlinear systems with bounded uncertain time varying inputs. A matching condition is presented to cancel the uncertainty which appears in the term linked directly with the control inputs. If the nonlinear systems are locally robustable, using the proposed control law, the output error can be reduced to the desired bound. Finally, an illustrating example for the regulation problem is provided.

1. Introduction

Recently there has been considerable progress in the theory of gain scheduling (Huang and Rugh 1990, Rugh 1991, Shahruz and Behtash 1990, and Shamma and Athans 1991). First, the control law for the systems with slowly varying parameters has been developed. Huang and Rugh (1990) considered nonlinear systems with both measured and unmeasured disturbance signals, and proposed an output feedback control law for the reference tracking problem. An analytical framework for the state feedback control law was proposed for the regulation problem by Rugh (1991). Moreover, state feedback control laws with the derivative information on the scheduling variables are proposed for somewhat fast time varying inputs (Sureshbabu and Rugh 1995). However, these developed control schemes are restricted to the exactly known exogenous signals.

In practice, there exists some cases where the nonlinear plants are not exactly modelled due to the uncertainties or the additional disturbances in the time varying inputs. The existing controllers are designed and destined only for the systems with the exactly known time varying inputs. Thus, using those without any modification (or additional compensators) may cause undesirable performance or destabilized the overall system. This paper proposes a compensated state feedback controller for the regulation problems. Using the proposed control law, the locally robustable system, whose dynamics are locally independent of the uncertainty, can be controlled to provide the desired output error bound. In addition, we propose a matching condition for gain scheduling control in order to check whether the given system is locally robustable.

2. Preliminaries

Consider the nominal system given by

\[ \dot{x}(t) = f(x(t), w(t)) + G(x(t), w(t))u(t), \]
\[ x(0) = x_0, t \geq 0 \]  
\[ y(t) = h(x(t), w(t)), \]  

where \( x(t) \) is the \((n \times 1)\) state vector, \( u(t) \) is the \((p \times 1)\) control input, \( w(t) \) is the \((m \times 1)\) time varying input, and \( y(t) \) is the \((q \times 1)\) output. The functions \( f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), \( G: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times p} \) and \( h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^q \) are assumed to be twice continuously differentiable. It is assumed that \( w(t) \) and its time derivative are bounded. The control law which achieves the control objective to minimize \( \varepsilon \) while rejecting the exogenous signal \( w(t) \) such that

\[ \lim_{t \to \infty} \| r_d - y(t) \| \leq \varepsilon \]  

is given by

\[ u(t) = k(x(t), w(t)) \]  

where \( r_d \) is the \((q \times 1)\) constant reference input. The following assumptions are proposed for the nominal system.

Assumption 1: There exists an open neighbourhood \( \Gamma \) of the origin in \( \mathbb{R}^m \) and smooth function \( x(\omega) \) and \( u(\omega) \) for each \( \omega \in \Gamma \) such that
\[ 0 = f(x(w), w) + G(x(w), w)u(w) \]
\[ r_d = h(x(w), w) \]
\[ u(w) = k(x(w), w). \]

**Assumption 2:** \( \frac{\partial f_c(x(w), w)}{\partial x} \) is Hurwitz for each \( w \in \Gamma \), where
\[ f_c(x, w) = f(x, w) + G(x, w)k(x, w). \]

Thus, the nominal control law is obtained by (Rugh 1991)
\[ u(t) = k(x, w) = u(w) + K_1(w)(x - x(w)) \]
where \( K_1(w) \) is determined so that the eigenvalues of \( \frac{\partial f_c(x(w), w)}{\partial x} \) with (5) should have specified values with negative real parts for each \( w \).

### 3. Analysis of uncertain nonlinear systems

Consider the uncertain system given by
\[ \dot{x}(t) = f(x(t), \tilde{w}(t)) + G(x(t), \tilde{w}(t))\tilde{u}(t), \]
\[ x(0) = x_0, t \geq 0 \]
\[ y(t) = h(x(t), \tilde{w}(t)) \]
where \( \tilde{w}(t) = w(t) + \Delta w(t) \) and \( \tilde{u}(t) = u(t) + \Delta u(t) \). Here, \( w(t) \) is the nominal value and \( \Delta w(t) \) is the unknown but bounded uncertainty. Moreover, \( u(t) \) is the nominal control term defined in (5), and \( \Delta u(t) \) is the additional control term (or compensator output) to cancel the effect of \( \Delta w \) in \( G(x, \tilde{w}) \). Letting \( G = [g_1 \ g_2 \ \cdots \ g_p] \), \( \tilde{u} = [\tilde{u}_1 \ \tilde{u}_2 \ \cdots \ \tilde{u}_p]^T \) and \( k = [k_1 \ k_2 \ \cdots \ k_p]^T \) and expanding \( \dot{x} \) in the second term of the system (6) with (5) as a Taylor series in \( \tilde{w} \) about \( \tilde{w} = w \) gives
\[ \dot{x} = f(x, w) + \frac{\partial f(x, w)}{\partial x} \Delta w + G(x, w)\tilde{u} \]
\[ + \left( \sum_{i=1}^{p} \frac{\partial g_i(x, w)}{\partial w} \tilde{u}_i \right) \Delta w + a_1(\Delta w) + a_2(\Delta w) \]
\[ = f_c(x, w, \Delta w) + \eta(x, w, \Delta u)\Delta w + G(x, w)\Delta u + a_1(\Delta w) + a_2(\Delta w) \]
where
\[ f_c(x, w, \Delta w) = f_c(x, w) + \frac{\partial f(x, w)}{\partial x} \Delta w \]
\[ \eta(x, w, \Delta u) = \sum_{i=1}^{p} \frac{\partial g_i(x, w)}{\partial w} (k_i + \Delta u_i) \]
\[ a_1(\Delta w) = f(x, \tilde{w}) - f(x, w) - \frac{\partial f(x, w)}{\partial \tilde{w}} \Delta w \]
\[ a_2(\Delta w) = G(x, \tilde{w})\tilde{u} - G(x, w)\tilde{u} \]
\[ - \left( \sum_{i=1}^{p} \frac{\partial g_i(x, w)}{\partial \tilde{w}} \tilde{u}_i \right) \Delta w. \]

Letting \( q(t) = x(w(t)) \) in Assumption 1 and \( e(t) = x(t) - q(t) \) in closed loop system (8), gives,
\[ \dot{e} = \dot{x} - \dot{q} = f_c(e + q, w, \Delta w) + \eta(e + q, w, \Delta u)\Delta w + G(e + q, w)\Delta u + a_1(\Delta w) + a_2(\Delta w) \]
\[ - \frac{\partial q}{\partial \tilde{w}}. \]

Note that using Assumption 1, \( f_c(q, w) = 0 \) and
\[ f_c(q, w, \Delta w) = \frac{\partial f(q, w)}{\partial \tilde{w}} \Delta w. \]

Thus, we can easily obtain
\[ \int_0^1 \frac{d}{d\theta} f_c(q + \theta e, w, \Delta w) \ d\theta = f_c(q + e, w, \Delta w) \]
\[ - \frac{\partial f(q, w)}{\partial \tilde{w}} \Delta w \] (10)
and
\[ \int_0^1 \frac{d}{d\theta} f_c(q + \theta e, w, \Delta w) \ d\theta = \dot{A}_c(q, w, \Delta w) e \]
\[ + \dot{R}_c(q, w, \Delta w, e) \] (11)
where
\[ A_c(t, \Delta w) := \dot{A}_c(q, w, \Delta w) = \frac{\partial f_c(x, w, \Delta w)}{\partial x} \bigg|_{x=q} \]
\[ R_c(t, \Delta w) := \dot{R}_c(q, w, \Delta w, e) \]
\[ = \int_0^1 [\dot{A}_c(q + \theta e, w, \Delta w) - \dot{A}_c(q, w, \Delta w)] \ d\theta \]
Here, we notice that \( A_c(t, \Delta w) = A_c(t) + \Delta A(t, \Delta w) \)
where
\[ A_c(t) = \frac{\partial f_c(x, w)}{\partial x} \bigg|_{x=q} \]
and
\[ \Delta A(t, \Delta w) = \frac{\partial}{\partial x} \left( \frac{\partial f(x, w)}{\partial \tilde{w}} \Delta w \right) \bigg|_{x=q} \]
Finally, equating (10) and (11), and applying the result to system (9) gives
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\[ \dot{e} = A_e(t, \Delta w)e + R_e(t, \Delta w)e + \frac{\partial f(q, w)}{\partial w} \Delta w - \frac{\partial q}{\partial w} \hat{w} + \eta(q + e, w, \Delta u) \Delta w + G(q + e, w) \Delta u + o_1(\Delta w) + o_2(\Delta w) \] (12)

4. Construction of control laws

The following assumptions are proposed for the robust control design with a matching condition.

**Assumption 3:** There exists a function \( \tilde{\eta}: R^d \times R^m \times R^p \rightarrow R^{p \times m} \) such that

\[ \eta(x, w, \Delta u) = G(x, w) \tilde{\eta}(x, w, \Delta u) \]

**Assumption 4:** There exist a function \( \rho: R^d \times R^m \rightarrow R^+ \) and a constant \( K \geq 0 \) such that

\[ \| \tilde{\eta}(x, w, \Delta u) \| \leq \rho(x, w) + K \| \Delta u \| \]

**Assumption 5:** There exists a constant \( \gamma \geq 0 \) such that

\[ \| \Delta w(t) \| \leq \gamma, \ t \geq 0. \]

Define a Lyapunov function candidate \( V(t, e(t)) = e^T(t)Q(t)e(t) \). We notice that \( Q(t) \) is the well defined, continuously differentiable \((n \times n)\) unique positive-definite solution of

\[ A_e^T(t)Q(t) + Q(t)A_e(t) = -I \] (13)

Differentiating \( V(t, e(t)) \) with respect to \( t \) and applying (12) to the result gives

\[ \dot{V}(t, e(t)) = W + 2e^T(t)Q(t)\eta(t)\Delta w \\
+ 2e^T(t)Q(t)G(q + e, w) \Delta u \\
+ 2e^T(t)Q(t)o_2(\Delta w) \] (14)

where

\[ W = -e^T(t)e(t) + e^T(t)\dot{Q}(t)e(t) \\
+ 2e^T(t)Q(t)R_e(t, \Delta w)e(t) \\
+ 2e^T(t)Q(t)\Delta A(t, \Delta w)e(t) \\
+ 2e^T(t)Q(t)\frac{\partial f(q, w)}{\partial w} \Delta w - \frac{\partial q}{\partial w} \hat{w} + o_1(\Delta w) \]

**Definition 1—Locally robustable:** The system (6) is locally robustable if and only if there exists \( \Delta u(t) \) such that

\[ \dot{V} \leq W + 2e^T(t)Q(t)o_2(\Delta w) \] (15)

Now, we propose an additional control input given by

\[ \Delta u = \Theta(x, w) = -\gamma \rho(x, w) \psi \]

\[ \frac{1}{1 - \gamma K} \| \psi \| , \]

where \( \psi^T = e^T Q G(x, w) \) (16)

Then, the overall structure is shown in Fig. 1.

**Theorem 1:** Under Assumptions 1–5, the system (6) with (5) and (16), is locally robustable under the following condition

\[ \gamma < \frac{1}{K} \]

**Proof:** Using Assumptions 3–5, (14) can be rewritten as follows

\[ \dot{V}(t, e) = W + 2e^T(\tilde{\eta})\Delta w + 2e^T \Delta u + 2e^T Q o_2(\Delta w) \leq W + 2\| \psi \| (\rho(x, w) + K \| \Delta u \|) \| \Delta w \| + 2e^T \Delta u + 2e^T Q o_2(\Delta w) \leq W + 2\gamma \rho(x, w) \| \psi \| + 2 \frac{\gamma^2 K \rho(x, w)}{|1 - \gamma K|} \| \psi \| \]

![Figure 1. Overall state feedback system with uncertain time varying inputs.](image-url)
\[ -2 \frac{\gamma_p(x, w)}{1 - \gamma K} \| \psi \| + 2 e^T Q_2(\Delta w) \]
\[ = W - 2 \gamma_p(x, w) \left( 1 + \frac{\gamma K}{1 - \gamma K} \right) \| \psi \| + 2 e^T Q_2(\Delta w) \]
\[ = W + 2 e^T Q_2(\Delta w) \]  
(17)

where \( 1 - \gamma K > 0 \). □

**Remark 1:** If \( W + 2 e^T(t)Q(t)Q_2(\Delta w) < 0 \) could be satisfied, \( \dot{V} < 0 \) will be guaranteed. As shown in (17), using \( \Delta w \) we remove the term \( \frac{1}{\gamma K} \). In other words, we cancel the effect of the uncertainty \( \Delta w \) in the term \( G(x, \Delta w) \) with some error \( Q_2(\Delta w) \).

**Theorem 2:** Under Assumptions 1-5, given the locally robustable system defined by (6), (7), (5) and (16), then for \( \|e(0)\| < e_o, \|e(t)\| \) is bounded by \( e_b \) as \( t \to \infty \) under the following conditions

\[ \begin{align*}
1 - L_{d_q} - 2 \alpha L_q & > 0 \\
\beta^2 (1 - L_{d_q} - 2 \alpha L_q)^2 & - 16 L_q^2 L_r \beta > 0
\end{align*} \]

where

\[ \alpha = \max_i \| \delta A(t, \Delta w(t)) \| \\
\beta = \max_i \left\| \frac{\partial f(q, w)}{\partial w}(t) \Delta w(t) - \frac{\partial q}{\partial w}(t) \dot{w}(t) \right\| + o_1(\Delta w(t)) + o_2(\Delta w(t)) \]
\[ e_o = [\theta(1 - L_{d_q} - 2 \alpha L_q) \]
\[ + (\theta^2 (1 - L_{d_q} - 2 \alpha L_q)^2 - 16 L_q^2 L_r \beta)^{1/2}] / 4 L_q L_r \]
\[ e_b = [\theta(1 - L_{d_q} - 2 \alpha L_q) \]
\[ - (\theta^2 (1 - L_{d_q} - 2 \alpha L_q)^2 - 16 L_q^2 L_r \beta)^{1/2}] / 4 L_q L_r \]

with \( 0 < \theta < 1 \) and the constant coefficients are appropriately defined in the proof.

**Proof:** From the analysis results of Lee and Lim (1995) and Lim and Cho (1996), we can obtain the following conditions with each modification

\[ \| R_z \| \leq L_z \| e \|, \quad \| Q \| \leq L_q, \quad \text{and} \quad \| \dot{Q} \| \leq L_{d_q} \]

where \( L_z = L_R / 2, \quad L_q = \sqrt{n} L_H \) and \( L_{d_q} = 2 \sqrt{n} L_H L_w \) (see the Appendix for details). Then, from (17)

\[ \dot{V} \leq W + 2 e^T Q_2(\Delta w) \]
\[ \leq -\|e\|^2 + \| \dot{Q} \| \| e \|^2 + 2 \| Q \| \| R_z \| \| e \|^2 + 2 \| Q \| \| \delta A(t, \Delta w) \| \| e \|^2 \]
\[ + 2 \| Q \| \left\| \frac{\partial f(q, w)}{\partial w}(t) \Delta w(t) - \frac{\partial q}{\partial w}(t) \dot{w}(t) \right\| + o_1(\Delta w(t)) + o_2(\Delta w(t)) \| e \| \]
\[ \leq -N \| e \|^2 + 2 L_q L_r \| e \|^3 - \theta (1 - L_{d_q} - 2 \alpha L_q) \| e \|^2 \]
\[ + 2 L_q \| e \| \]
\[ = -N \| e \|^2 + 2 L_q L_r \| e \| (\| e \| - e_o)(\| e \| - e_b) \]

where \( N = (1 - \theta)(1 - L_{d_q} - 2 \alpha L_q) \). It can be easily proven that, in order to guarantee the existence of the positive range \( [e_b, e_o] \) in which \( \dot{V} < 0 \) is satisfied, (18) should hold. Therefore, we conclude that \( \| e(t) \| \) will be ultimately bounded by \( e_b \). It guarantees \( V(t, e(t)) \leq -N \| e(t) \|^2 \) for \( \| e(t) \| \in [e_b, e_o] \). By simple algebraic manipulations, we obtain

\[ V(t, e(t)) \leq V(0, e(0)) e^{-N/M t} \]
\[ \leq \mu \| e(t) \|^2 \leq V(t, e(t)) \leq M \| e(0) \|^2 e^{-N/M t} \]

where \( \mu \) and \( M \) are defined in the Appendix. Generally, we write the norm bound of \( e(t) \) as

\[ \| e(t) \| \leq \| e(0) \| e^{-N/2 M t} + e_b, \quad t \geq 0. \]

Moreover, \( e_b \) is related to the eigenvalues of \( \partial f_i(q, w) / \partial x \), and determined according to the feedback gain \( K_1(w) \) under the conditions (18). □

**Corollary 1:** If \( h(x, \dot{w}) \) satisfies the Lipschitz condition for \( x \in D \subseteq R^n \), then \( \| r_d - y \| \) can be obtained as follows

\[ \| r_d - y \| \leq L_y \| e \| + \left\| \frac{\partial h(e + q, w)}{\partial w}(t) \Delta w(t) \right\| + \| o(\Delta w) \| \]

where \( L_y \) is a Lipschitz constant.

**Proof:** Using Assumption 1,

\[ \| r_d - y \| = \| h(q, w) - h(x, w + \Delta w) \| \]
\[ = \left\| h(q, w) - h(e + q, w) \right\| \]
\[ - \frac{\partial h(e + q, w)}{\partial \dot{w}} \Delta w(t) + o(\Delta w) \| \]
\[ \leq L_y \| e \| + \left\| \frac{\partial h(e + q, w)}{\partial \dot{w}} \Delta w(t) \right\| + \| o(\Delta w) \|. \]

□

**Remark 2:** For the system which is locally robustable, Theorem 2 and Corollary 1 provide implicitly the maximum tolerable \( \Delta w \) under which the control objective in (3) is satisfied while rejecting the uncertain time varying inputs. □

**Example:** Consider the uncertain plant given by
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\[ \dot{x}_1(t) = -x_1(t) - x_2(t) + \tilde{w}(t) \]
\[ \dot{x}_2(t) = x_2(t) - e^{-x_2(t)} + e^{-\tilde{w}(t)}u(t) \]
\[ y(t) = x_1(t) \]

where \( r_d = 0 \). The nominal plant’s family of operating points (or equilibrium trajectory) with \( \tilde{w} = w \) is given by \( q(t) = x(w) = [0 \ w]^T \) and \( u(w) = 1 - we^w \). In order to place the linearized closed-loop system \( (\partial \delta \xi(q, w)/\partial x) \) eigenvalues both at \(-\lambda\), we obtain
\[ K_1(w) = [(\lambda - 1)^2 e^w, -(1 + 2\lambda e^w)]. \]

Moreover, from Assumptions 3 and 4
\[ \eta = \begin{bmatrix} 0 \\ -e^w(u + \Delta u) \end{bmatrix}, \quad \rho = |u(t)|, \quad K = 1. \]

Therefore, the overall control input is
\[ \tilde{u}(t) = (1 - we^w) + (\lambda - 1)^2 e^w x - (1 + 2\lambda e^w)(x - w) \]
\[ - \frac{\gamma}{1 - \gamma} ((1 - we^w) + (\lambda - 1)^2 e^w x \]
\[ - (1 + 2\lambda e^w)(x - w) | \frac{\psi}{|w|} \]

Simulation is performed under \( \tilde{w} = w + \Delta w \), for the case using the compensator \( \Delta u \) and the case using no compensator. In Figs 2(a) and (b), in order to guarantee \( \epsilon = 0.05 \) in the closed loop system with the designed compensator \( \Delta u(t) \), \( \lambda = 11 \) and \( \lambda = 18 \) are chosen respectively. From Fig. 2, we conclude that the compensated controller depresses the output trajectory below \( \epsilon \) and provides a smaller error than the uncompensated one.

5. Conclusions

In this paper, we propose a control law to reduce the output error in the nonlinear system with uncertain time varying inputs. In particular, a matching condition is presented so as to design the additional compensator. Moreover, it is shown that if \( \Delta w \) is less than the maximum tolerable uncertainty in the time varying input, the control objective is achieved.

Appendix

Assumption 2 implies that there exist finite constants \( \sigma_2 \geq \sigma_1 > 0 \), such that
\[ \sigma_1 \leq |\lambda_i(\partial \delta \xi(x(w), w)/\partial x)| \leq \sigma_2, \quad w \in \Gamma, \quad i = 1, 2, \ldots, n. \]

Lemma A.1: If \( \hat{A}_c(x, w, \Delta w) \) satisfies the Lipschitz condition for \( x \in D \subset \mathbb{R}^n \), then there exists a finite constant \( L_R \) such that
\[ \| R_c(t, \Delta w(t)) \| \leq \frac{LR}{2} \| t(t) \|, \quad t \geq 0 \]

Proof: As follows from the definition of \( R_c(t, \Delta w) \), using \( \| \hat{A}_c(q + \theta e, w, \Delta w) - \hat{A}_c(q, w, \Delta w) \| \leq L_R \| \theta e \| \), we obtain \( \| R_c(q, w, \Delta w, e) \| \leq L_R/2 \| e \|. \)

We consider \( P_1(t) \) which transforms \( \hat{A}_c(t) \) into a diagonal matrix \( \hat{A}_c(t) \), i.e. \( \hat{A}_c(t)P_1(t) = P_1(t)\hat{A}_c(t) \). Assume that there exists a constant \( h_1 > 0 \) such that \( \| \hat{A}_c(t) \| \leq h_1 \| \hat{A}_c(t) \| \). Since \( f_c(x, w) \) is twice continuously differentiable and \( w(t) \) is bounded, \( \| \hat{A}_c(t) \| \) can be bound by a finite constant.

Lemma A.2: There exists a finite constant \( L_A \) such that
\[ \| \hat{A}_c(t) \| \leq L_A, \quad t \geq 0 \]

where
\[ L_A = \begin{cases} A_1 \sigma_2, & n_1 = 1 \\ A_1(1 + \sigma_2), & n_1 \geq 2 \end{cases} \]
and $n_J$ is the dimension of the largest Jordan block of $A_c(t)$.

**Proof:**

(i) For $n_J = 1$, $\tilde{A}_c(t) = \text{diag} \{ \lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t) \}$ where $\lambda_i(t), i = 1, \ldots, n$ are the eigenvalues of $A_c(t)$ in the sense of the frozen time approach (Zhu 1993, Wu 1980). Since $P(t)$ conserves the eigenvalues of $A_c(t)$, using Assumption 2 gives $\sigma_1 \leq |\lambda_i(t)| \leq \sigma_2$, $i = 1, \ldots, n, t \geq 0$. Therefore, we obtain $\|A_c(t)\| \leq h_1 \sigma_2$.

(ii) For $n_J \geq 2$, $\tilde{A}_c(t) = \text{diag} \{ \tilde{A}_{c1}(t), \tilde{A}_{c2}(t), \ldots, \tilde{A}_{cp}(t) \}$ where $\tilde{A}_{ci}(t), 1 \leq i \leq p < n$ are Jordan blocks of $A_c(t)$. Let $\lambda_i(t)$ be the eigenvalue of the maximum Jordan block $\tilde{A}_{ci}(t)$ which satisfies $\sigma_1 \leq |\lambda_i(t)| \leq \sigma_2$. We can obtain

$$\|A_c(t)\| \leq h_1 \left( \|\tilde{A}_{ci}(t)\| \|\tilde{A}_{ci}(t)\|_{\infty} \right)^{1/2} \leq h_1 (1 + \sigma_2).$$

Now, we consider the following Kronecker sum of two matrices. Let $H(t) = A_c(t) \otimes \Theta(t)$. In the same way, $H(t)$ can be transformed into $\tilde{H}(t) = P_2(t) H(t) P_2(t)$ with $P_2(t)$, and it is followed by $H^{-1}(t) P_2(t) = P_2(t) \tilde{H}^{-1}(t)$. Assume that there exists a constant $h_2 > 0$ such that $\|H^{-1}(t)\| \leq h_2 \|\tilde{H}^{-1}(t)\|$.

**Lemma A.3:** There exists a finite constant $L_H$ such that

$$\|H^{-1}(t)\| \leq L_H$$

where

$$L_H = h_2 \sum_{i=1}^{n_J} \left( \frac{1}{2 \sigma_1} \right)^i.$$

**Proof:**

(i) For $n_J = 1$,

$$\tilde{H}^{-1}(t) = \text{diag} \left\{ \frac{1}{\mu_1(t)}, \frac{1}{\mu_2(t)}, \ldots, \frac{1}{\mu_n(t)} \right\}$$

where $\mu_i(t), i = 1, \ldots, n$ are the eigenvalues of $H(t)$. Since

$$\frac{1}{2 \sigma_2} \leq \frac{1}{|\mu_i(t)|} \leq \frac{1}{2 \sigma_1},$$

we obtain

$$\|H^{-1}(t)\| \leq h_2 \frac{1}{2 \sigma_1}.$$

(ii) For $n_J \geq 2$, let $\mu_i(t)$ be the eigenvalues of the maximum Jordan block $\tilde{H}_i(t)$ of $H(t)$. Since $\tilde{H}_i^{-1}(t)$ is an upper triangular matrix, using

$$\frac{1}{2 \sigma_2} \leq \frac{1}{|\mu_i(t)|} \leq \frac{1}{2 \sigma_1}$$

$$\|H^{-1}(t)\| \leq h_2 \left( \|\tilde{H}_i^{-1}(t)\| \|\tilde{H}_i^{-1}(t)\|_{\infty} \right)^{1/2} \leq h_2 \sum_{i=1}^{n_J} \left( \frac{1}{2 \sigma_1} \right)^i.$$

Using Kronecker properties, the solution of the Lyapunov equation (13) is obtained as

$$\text{vec} [Q(t)] = -(H^T(t))^{-1} \text{vec} [I]$$

where the $n^2$-vector $\text{vec} [Q(t)]$ is the vector composed of the columns of matrix $Q(t)$ taken in order.

**Lemma A.4:** There exist finite constants $M \geq \mu > 0$ such that

$$\mu \|e(t)\|^2 \leq V(t, e(t)) \leq M \|e(t)\|^2, \quad t \geq 0$$

where $\mu = 1/(2L_A)$ and $M = \sqrt{nL_H}$.

**Proof:**

(i) $\mu = 1/(2L_A)$ (Amato et al. 1993, Lawrence and Rugh 1990).

(ii) Using Lemma 3, $\|\text{vec} [Q(t)]\| \leq \sqrt{nL_H}$ is obtained. Since $\|S\| \leq \|S\|_F = \|\text{vec} [S]\|_2$, for any given matrix $S$,

$$V(t, e(t)) \leq \|Q(t)\| \|e(t)\|^2 = \sqrt{nL_H} \|e(t)\|^2.$$

We notice that (A 2) implies $\|Q(t)\| \leq \sqrt{nL_H}$. Since $\dot{w}(t)$ is bounded, there exists a constant $L_W \leq 0$ such that $\|\dot{A}_c(t)\| \leq L_W$, $t \geq 0$. Therefore,

$$\|Q(t)\| \leq 2\|\dot{A}_c(t) + A_c(t)\|_F \leq 2\|\dot{A}_c(t)\|_F \leq 2\sqrt{nL_H} L_W.$$

**References**


