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Gorenstein *n*-FP-injective and Gorenstein *n*-flat complexes

C. Selvaraj[1] · R. Saravanan[1]

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Abstract In this paper, we introduce and study the notions of Gorenstein n-FP-injective and Gorenstein n-flat complexes, which are special cases of Gorenstein FP-injective and Gorenstein flat complexes respectively. We prove that over a left n-coherent ring R the class of all Gorenstein n-FP-injective (resp., Gorenstein n-flat) complexes is injectively (resp., projectively) resolving and we discuss the relationship between Gorenstein n-FP-injective and Gorenstein n-flat complexes.

Keywords Gorenstein n-FP-injective complex · Gorenstein n-flat complex

Mathematics Subject Classification 16E05 · 16E30

1 Introduction

Throughout this paper, let R be an associative ring with identity and $\mathscr{C}$ be the abelian category of complexes of R-modules. Unless stated otherwise, a complex and an R-module will be understood to be a complex of left R-modules and a left R-module, respectively.

Relative homological algebra for the category of modules was introduced by Auslander and Bridger and studied by Enochs-Jenda-Garcia Rozas, Foxby, Holm, etc. Further, it has been widely generalized to other categories. Among them, the category of complexes of modules has drawn wide attentions and several results in the category of modules have been generalized to that of complexes of modules. Mao and Ding [10] introduced the notions of Gorenstein FP-injective and Gorenstein flat modules. Further, Xin et al. [18] generalized these notions to Gorenstein FP-injective and Gorenstein flat complexes and obtained some nice characterizations of Gorenstein flat complexes. Selvaraj et al. [11] introduced and studied the notions of Gorenstein n-flat and Gorenstein n-FP-injective (or Gorenstein n-absolutely

✉ C. Selvaraj
selvavlr@yahoo.com

1 Department of Mathematics, Periyar University, Salem, TN 636 011, India

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pure) modules. Motivated by the above works in this article, we intend to study the notions of Gorenstein n-FP-injective and Gorenstein n-flat complexes, which are special types of Gorenstein FP-injective and Gorenstein flat complexes, respectively and investigate their homological properties.

In [15, Theorem 2.10], it is proved that the class of all Gorenstein injective complexes is injectively resolving. Over a left n-coherent ring R, we prove the following.

Theorem. *Let R be left n-coherent ring. Then the class of all Gorenstein n-FP-injective complexes is injectively resolving and closed under direct summands.*

In [18, Proposition 3.14], the connection between Gorenstein FP-injective and Gorenstein flat complexes is proved. However, we prove the following result that represents the relationship between Gorenstein n-FP-injective and Gorenstein n-flat complexes.

Theorem. *Let R be a ring and G be a complex of right R-modules. Then the following hold:*

(1) *If G is Gorenstein n-flat, then G^+ is Gorenstein n-FP-injective.*
(2) *If R is left n-coherent and G^+ is Gorenstein n-FP-injective, then G is Gorenstein n-flat.*

In connection to [18, Proposition 4.2], finally we prove the following main result.

Theorem. *Let R be a left n-coherent ring and $0 \to C' \to C \to C'' \to 0$ be an exact sequence of complexes of right R-modules. Then:*

(1) *If C'' is Gorenstein n-flat, then C is Gorenstein n-flat if and only if C' is Gorenstein n-flat.*
(2) *If C' and C are Gorenstein n-flat complexes, then C'' is Gorenstein n-flat if and only if $Tor_1(C'', E) = 0$ for all n-FP-injective complexes E.*

This paper is organized as follows.

In Sect. 2, we present some basic definitions and terminologies in the category of complexes in order to use them in the following sections.

In Sect. 3, we introduce and study the notion of Gorenstein n-FP-injective complexes and we prove that the modules in a Gorenstein n-FP-injective complex are precisely the Gorenstein n-FP-injective modules. Further, over a left n-coherent ring the class of all Gorenstein n-FP-injective complexes is injectively resolving and closed under direct summands.

In the last Section, we introduce the notion of Gorenstein n-flat complexes and exhibit a relationship between Gorenstein n-flat and Gorenstein n-FP-injective complexes. Also we prove that over a left n-coherent ring the class $\mathscr{GF}_n$ of all Gorenstein n-flat complexes is projectively resolving and closed under direct summands.

2 Preliminaries

In this section, we first recall some known definitions and terminologies which will be needed in the sequel.

In this paper a complex

$$\cdots \to C^{-1} \stackrel{\delta^{-1}}{\to} C^0 \stackrel{\delta^0}{\to} C^1 \stackrel{\delta^1}{\to} \cdots$$

will be denoted by C or (C, δ). We will use subscripts to distinguish complexes. So if $\{C_i\}_{i\in I}$ is a family of complexes, C_i will be

$$\cdots \to C_i^{-1} \stackrel{\delta_i^{-1}}{\to} C_i^0 \stackrel{\delta_i^0}{\to} C_i^1 \stackrel{\delta_i^1}{\to} \cdots .$$

Given an R-module M, we will denote by $\overline{M}$ the complex

$$\cdots 0 \to 0 \to M \xrightarrow{id} M \to 0 \to 0 \cdots$$

with M in the -1st and 0th degrees. Similarly, we denote by $\underline{M}$ the complex with M in the 0th degree and 0 in the other places. Note that an R-module M is flat (resp., projective, injective) if and only if the complex $\overline{M}$ is flat (resp., projective, injective).

Given a complex C and an integer m, $C[m]$ denotes the complex such that $C[m]^n = C^{m+n}$ and whose boundary operators are $(-1)^m \delta^{m+n}$. The nth cycle of a complex C is defined as Ker δ^n and is denoted by $Z^n C$. The nth boundary of C is defined as Im δ^{n-1} and is denoted by $B^n C$.

Let C be a complex of left R-modules (resp., of right R-modules) and let D be a complex of left R-modules. We denote by $\mathrm{Hom}(C, D)$ (resp., $C \otimes D$) the usual homomorphism complex (resp., tensor product) of the complexes C and D. The nth degree term of the complex $\mathrm{Hom}(C, D)$ is given by

$$\mathrm{Hom}(C, D)^n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}(C^t, D^{n+t})$$

and whose boundary operators are

$$(\delta^n f)^m = \delta_D^{n+m} f^m - (-1)^n f^{m+1} \delta_C^m.$$

The nth degree term of $C \otimes D$ is given by

$$(C \otimes D)^n = \bigoplus_{t \in \mathbb{Z}} (C^t \otimes_R D^{n-t})$$

and

$$\delta(x \otimes y) = \delta_C^t(x) \otimes y + (-1)^t x \otimes \delta_D^{n-t}(y),$$

for $x \in C^t$ and $y \in D^{n-t}$.

Let $\underline{Hom}(C, D) = Z(Hom(C, D))$. Then we see that $\underline{Hom}(C, D)$ can be made into a complex with $\underline{Hom}(C, D)^n$ the abelian group of morphisms from C to $D[n]$ and with the boundary operator given by $\delta^n(f) : C \to D[n+1]$ with $\delta^n(f)^m = (-1)^n \delta_D^{n+m} f^m$ for all $m \in \mathbb{Z}$ and $f \in \underline{Hom}(C, D)^n$. Note that the new functor $\underline{Hom}$ will have right derived functors whose values will be complexes. It will be denoted by $\underline{Ext}^i(C, D)$. It is easy to see that $\underline{Ext}^i(C, D)$ is

$$\cdots \to Ext^i(C, D[n-1]) \to Ext^i(C, D[n]) \to Ext^i(C, D[n+1]) \to \cdots$$

with boundary operator induced by the boundary operator of D.

For a complex C of left R-modules, we have a functor $- \otimes \mathscr{C} : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$, where $\mathscr{C}_R$ denotes the category of right R-modules. The functor $- \otimes \mathscr{C} : \mathscr{C}_R \to \mathscr{C}_{\mathbb{Z}}$ being right exact, we can construct the left derived functors which we denote by $Tor_i(-, C)$. Given two complexes C and D of $\mathscr{C}$, we use $Ext^i(C, D)$ for $i \geq 0$ to denote the groups we obtain from the right derived functors of Hom and we use C^+ to denote the complex $\underline{Hom}(C, \overline{\mathbb{Q}/\mathbb{Z}})$.

Recall that a complex C is projective (resp., injective) if C is exact and $Z^n C$ is a projective (resp., an injective) R-module for each $i \in \mathbb{Z}$. A complex C is flat if C is exact and $Z^n C$ is a flat R-module for each $i \in \mathbb{Z}$. Equivalently, a complex C is projective (resp., injective) if and only if $\mathrm{Hom}(C, -)$ (resp., Hom $(-, C)$) is exact. Also a complex C is flat if and only if $- \otimes C$ is exact. A ring R is left n-coherent (for integers $n > 0$) if every finitely generated submodule of a free left R-module whose projective dimension at most $n - 1$ is finitely presented [9]. For unexplained terminologies and notations we refer to [1,3,5,6].

Let $\mathcal{C}$ be a class of complexes in $\mathscr{C}$. We say that $\mathcal{C}$ is projectively resolving if it contains all projective complexes and for any short exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{C}$, the conditions $A \in \mathcal{C}$ and $B \in \mathcal{C}$ are equivalent. Also we say that $\mathcal{C}$ is injectively resolving if it contains all injective complexes and for any short exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{C}$, the conditions $B \in \mathcal{C}$ and $C \in \mathcal{C}$ are equivalent.

Definition 2.1 [12] Let n be a non-negative integer. A complex C of R-modules is called n-FP-injective if $Ext^1(N, C) = 0$, for all finitely presented complexes N with $pd\ N \leq n$. A complex C of right R-modules is called n-flat if $Tor_1(C, N) = 0$ for all finitely presented complexes N with $pd\ N \leq n$.

We see that in general {Injective complexes} $\subset$ {FP-injective complexes} $\subset$ {n-FP-injective complexes} and {Flat complexes} $\subset$ {n-flat complexes}.

Definition 2.2 [18] A complex C of R-modules (resp., right R-modules) is called Gorenstein FP-injective (resp., Gorenstein flat) if there exists an exact sequence of injective (resp., flat) complexes

$$\cdots \to E_{-2} \to E_{-1} \to E_0 \to E_1 \to E_2 \to \cdots$$

such that Ker $(E_0 \to E_1) \cong G$ and $\mathrm{Hom}(E, -)$ (resp., $- \otimes E$) leaves the sequence exact for any FP-injective (resp., injective) complex E.

Definition 2.3 [11] An R-module M is called Gorenstein n-FP-injective (or Gorenstein n-absolutely pure) if there exists an exact sequence of n-FP-injective R-modules

$$\cdots \to E_{-2} \to E_{-1} \to E_0 \to E^0 \to E^1 \to \cdots$$

such that Ker $(E^0 \to E^1) \cong G$ and $\mathrm{Hom}(E, -)$ leaves the sequence exact for any n-FP-injective R-module E.

Definition 2.4 [11] A right R-module M is called Gorenstein n-flat if there exists an exact sequence of right R-modules

$$\cdots \to F_{-2} \to F_{-1} \to F_0 \to F^0 \to F^1 \to \cdots$$

such that F_i is n-flat for each $i \in \mathbb{Z}$, Ker $(F^0 \to F^1) \cong M$ and $- \otimes E$ leaves the sequence exact for any n-FP-injective R-module E.

3 Gorenstein *n*-FP-injective complexes

In this section, we introduce and study the notion of Gorenstein n-FP-injective complexes and also investigate their homological properties.

Definition 3.1 Let n be a non-negative integer. We say that a complex G is Gorenstein n-FP-injective if there exists an exact sequence of complexes

$$\mathbb{X} : \cdots \to E_{-2} \to E_{-1} \to E_0 \to E_1 \to E_2 \to \cdots$$

such that (i) E_i is injective for all $i \in \mathbb{Z}$;
(ii) Ker $(E_0 \to E_1) = G$;
(iii) $\mathrm{Hom}(E, \mathbb{X})$ is exact for any n-FP-injective complex E.

Remark 3.2 (1) Every injective complex is Gorenstein n-FP-injective. For any injective complex E, there is an exact sequence of complexes of the form

$$\cdots \to 0 \to E \overset{id}{\to} E \to 0 \to \cdots$$

such that $\mathrm{Im}(E \to E) = E$ and $\mathrm{Hom}(E', -)$ leaves the sequence exact for any n-FP-injective complex E'.

(2) If G is a Gorenstein n-FP-injective complex, then $\mathrm{Ext}^i(E, G) = 0$ for all n-FP-injective complexes E and for all $i \geq 1$.

(3) We see that the class {Gorenstein n-FP-injective complexes} is contained in {Gorenstein FP-injective complexes} $\cap$ {Gorenstein injective complexes}.

Proposition 3.3 *The class of all Gorenstein n-FP-injective complexes is closed under direct products.*

Proof It follows from the fact that the direct product of injective complexes is injective. □

Proposition 3.4 *If G is a Gorenstein n-FP-injective complex, then G^m is a Gorenstein n-FP-injective module for all $m \in \mathbb{Z}$.*

Proof Let G be a Gorenstein n-FP-injective complex. Then there exists an exact sequence

$$\mathbb{X} : \cdots \to E_{-2} \to E_{-1} \to E_0 \to E_1 \to E_2 \to \cdots$$

of injective complexes with Ker $(E_0 \to E_1) = G$ such that $\mathrm{Hom}(E, \mathbb{X})$ is exact for any n-FP-injective complex E. Now for any $m \in \mathbb{Z}$, we have the following exact sequence of injective R-modules

$$\mathbb{X}^m : \cdots \to E^m_{-2} \to E^m_{-1} \to E^m_0 \to E^m_1 \to E^m_2 \to \cdots$$

with Ker $(E^m_0 \to E^m_1) = G^m$. Hence $\mathbb{X}^m$ is an exact sequence of n-FP-injective modules with Ker $(E^m_0 \to E^m_1) = G^m$. Let E be any n-FP-injective left R-module. Then $\overline{E}[-m-1]$ is an n-FP-injective complex for any $m \in \mathbb{Z}$. Thus we have $Hom(\overline{E}[-m-1], X)$ is exact by hypothesis. Since $Hom(\overline{E}[-m-1], Y) \cong Hom(E,\ Y^m)$ for any complex Y, then the exactness of the sequence

$$\cdots \to Hom(\overline{E}[-m-1], E_{-1}) \to Hom(\overline{E}[-m-1], E_0) \to Hom(\overline{E}[-m-1], E_1) \to \cdots$$

gives the exactness of the following sequence

$$\cdots \to Hom(E, E^m_{-1}) \to Hom(E, E^m_0) \to Hom(E, E^m_1) \to \cdots .$$

Hence $\mathrm{Hom}(E, \mathbb{X}^m)$ is exact for any n-FP-injective R-module E. Therefore G^m is a Gorenstein n-FP-injective module for all $m \in \mathbb{Z}$. □

Proposition 3.5 *The following are equivalent for any complex G:*

(1) *G is Gorenstein n-FP-injective;*

(2) (a) *$Ext^i(E, G) = 0$ for all n-FP-injective complexes E and for all $i \geq 1$.*

(b) *There exists an exact sequence of complexes*

$$\mathbb{E} : \cdots \to E_2 \to E_1 \to G \to 0$$

with E_i injective for $i \geq 1$, such that $Hom(E, \mathbb{E})$ is exact for any n-FP-injective complex E;

(3) *There exists a short exact sequence of complexes of the form*

$$0 \to C \to E \to G \to 0,$$

where E is injective and C is Gorenstein n-FP-injective.

Proof (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) follows from the definition of Gorenstein n-FP-injective complexes.

(3) $\Rightarrow$ (2) Suppose $\mathbb{X} : 0 \to C \to E \to G \to 0$ is an exact sequence of complexes with E injective and C Gorenstein n-FP-injective. Then we have $\mathrm{Ext}^i(E', C) = 0$ for all $i \geq 1$ and for any n-FP-injective complex E' by (1) $\Leftrightarrow$ (2). Thus we have the following exact sequence

$$\cdots \to Ext^i(E', E) \to Ext^i(E', G) \to Ext^{i+1}(E', C) \to \cdots$$

which gives that $Ext^i(E', G) = 0$ for all $i \geq 1$ and for any n-FP-injective complex E'. On the other hand, the complex C being Gorenstein n-FP-injective, we have the following exact sequence

$$\mathbb{E}' : \cdots \to E_2 \to E_1 \to C \to 0$$

such that $\mathrm{Hom}(E', \mathbb{E}')$ is exact for any n-FP-injective complex E'. Combining the exact sequences $\mathbb{E}'$ and $0 \to C \to E \to G \to 0$ yield the following exact sequence

$$\cdots \to E_2 \to E_1 \to E \to G \to 0$$

such that $\mathrm{Hom}(E', -)$ leaves the sequence exact for any n-FP-injective complex E'. $\square$

Corollary 3.6 *Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an exact sequence of complexes. If C_3 is Gorenstein n-FP-injective and C_1 is injective, then C_2 is Gorenstein n-FP-injective.*

Proof Suppose C_3 is Gorenstein n-FP-injective. By Proposition 3.5, there exists a short exact sequence of complexes

$$0 \to C \to E \to C_3 \to 0$$

such that E is injective and C is Gorenstein n-FP-injective. Consider the pullback of $C_2 \to C_3$ and $E \to C_3$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C & = & C & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C_1 & \to & E' & \to & E & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Then it follows from the exact sequence $0 \to C_1 \to E' \to E \to 0$ that E' is injective since E and C_1 are injective. Thus we have a short exact sequence

$$0 \to C \to E' \to C_2 \to 0$$

with E' injective and C Gorenstein n-FP-injective. Therefore C_2 is Gorenstein n-FP-injective by Proposition 3.5. □

As an application of Proposition 3.5, we have the following result.

Proposition 3.7 *Let R be any ring and $0 \to G' \to G \to G'' \to 0$ be an exact sequence of complexes. Then:*

(1) *If G' and G'' are Gorenstein n-FP-injective, then G is Gorenstein n-FP-injective.*
(2) *If G' and G are Gorenstein n-FP-injective, then G'' is Gorenstein n-FP-injective.*

Proof (1) Suppose $0 \to G' \to G \to G'' \to 0$ is an exact sequence of complexes with G' and G'' Gorenstein n-FP-injective. By Proposition 3.5, there exist exact sequences of complexes

$$\mathbb{E} : \cdots \to E_2 \to E_1 \to G' \to 0$$

and

$$\mathbb{K} : \cdots \to K_2 \to K_1 \to G'' \to 0$$

with E_i and K_i are injective for all $i \geq 1$, such that $\mathrm{Hom}(E, \mathbb{E})$ and $\mathrm{Hom}(E, \mathbb{K})$ are exact for any n-FP-injective complex E. Then by the Horseshoe Lemma, we have the following exact sequence of complexes

$$\mathbb{L} : \cdots \to E_2 \oplus K_2 \to E_1 \oplus K_1 \to G \to 0$$

such that $E_i \oplus K_i$ is injective for all $i \geq 1$. Let E be any n-FP-injective complex. Then the exactness of $\mathrm{Hom}(E, \mathbb{L})$ follows from the exactness of $\mathrm{Hom}(E, \mathbb{E})$ and $\mathrm{Hom}(E, \mathbb{K})$. On the other hand, $\mathrm{Ext}^i(E, G) = 0$ since $\mathrm{Ext}^i(E, G') = 0$ and $\mathrm{Ext}^i(E, G'') = 0$ for any n-FP-injective complex E and for all $i \geq 1$. Therefore G is Gorenstein n-FP-injective by Proposition 3.5.

(2) Suppose G' and G are Gorenstein n-FP-injective. By Proposition 3.5, there exists a short exact sequence of complexes

$$0 \to C \to E \to G \to 0$$

with E injective and C Gorenstein n-FP-injective. Consider the pullback of $G' \to G$ and $E \to G$:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & C & = & C & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & E' & \to & E & \to & G'' & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & G' & \to & G & \to & G'' & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0. & & & & \end{array}$$

Since G' and C are Gorenstein n-FP-injective, we obtain from the short exact sequence $0 \to C \to E' \to G' \to 0$ that E' is Gorenstein n-FP-injective by (1). Hence we have the following short exact sequence of complexes

$$0 \to E' \to E \to G'' \to 0$$

with E injective and E' Gorenstein n-FP-injective. Therefore G'' is Gorenstein n-FP-injective by Proposition 3.5. □

In [15, Theorem 2.10], it is proved that the class of all Gorenstein injective complexes is injectively resolving. Using Proposition 3.7 and Remark 3.2, we prove the following main result of this section.

Theorem 3.8 *Let R be a left n-coherent ring. Then the class of all Gorenstein n-FP-injective complexes is injectively resolving and closed under direct summands.*

Proof Since every injective complex is Gorenstein n-FP-injective by Remark 3.2, the class of all Gorenstein n-FP-injective complexes is injectively resolving by Proposition 3.7. On the other hand, it is closed under direct summands by Proposition 3.3 and [7, Proposition 1.4]. □

Proposition 3.9 *Let R be a left n-coherent ring. Then the following are equivalent:*

(1) *G is a Gorenstein n-FP-injective complex.*
(2) *$Ext^i(E, G) = 0$ for all n-FP-injective complexes E and all $i \geq 1$.*

Proof (1) $\Rightarrow$ (2) It follows from Remark 3.2.

(2) $\Rightarrow$ (1) It is similar to that of [18, Theorem 3.3]. □

Corollary 3.10 *Let R be a left n-coherent ring and $0 \to G' \to G \to G'' \to 0$ be an exact sequence of complexes with G'' and G are Gorenstein n-FP-injective complexes. Then G' is Gorenstein n-FP-injective if and only if $Ext^1(E, G') = 0$ for all n-FP-injective complexes E.*

Proof The result follows from Proposition 3.9. □

Proposition 3.11 *Let R be a left n-coherent ring. If G is a Gorenstein n-FP-injective complex, then $G[m]$ is Gorenstein n-FP-injective for any $m \in \mathbb{Z}$.*

Proof Suppose G is a Gorenstein n-FP-injective complex. Then by Proposition 3.9, $Ext^1(E, A) = 0$ for all n-FP-injective complexes E. Let E be an n-FP-injective complex, then $E[m]$ is an n-FP-injective complex for all $m \in \mathbb{Z}$. Now

$$Ext^i(E, G[m]) \cong Ext^1(E[-m], G) = 0$$

for all $i \geq 0$. Hence $G[m]$ is Gorenstein n-FP-injective for any $m \in \mathbb{Z}$ by Proposition 3.9. □

The following result is an extension of [18, Lemma 2.1].

Lemma 3.12 *Let R be a ring. Then the following are equivalent:*

(1) *R is a left Noetherian ring;*
(2) *Every FP-injective complex of R-modules is injective;*
(3) *Every n-FP-injective complex of R-modules is injective.*

Proof (1) $\Leftrightarrow$ (2) It follows from [18, Lemma 2.1].

(2) $\Rightarrow$ (3) Suppose C is n-FP-injective. Then C is exact and C^i is n-FP-injective R-module for all $i \in \mathbb{Z}$ by [12, Theorem 3.11]. Since R is a Noetherian ring, C^i is an FP-injective R-module for all $i \in \mathbb{Z}$ by [14, Proposition 3.4]. Hence C^i is an injective R-module for all $i \in \mathbb{Z}$ by hypothesis. Therefore C is an injective complex.

(3) $\Rightarrow$ (2) It follows from the fact that every FP-injective complex is n-FP-injective. □

Proposition 3.13 *The following are equivalent for a left n-coherent ring R:*

(1) *R is a left Noetherian ring;*
(2) *Every n-FP-injective complex of R-modules is Gorenstein n-FP-injective.*

Proof (1) $\Rightarrow$ (2) It follows from Lemma 3.12.

(2) $\Rightarrow$ (1) Let C be an n-FP-injective complex. Since C is Gorenstein n-FP-injective, there exists an exact sequence of complexes $0 \to C \to E \to K \to 0$ with E injective. Then K is n-FP-injective follows from [12, Corollary 4.6]. Hence $\mathrm{Ext}^1(K, C) = 0$ by hypothesis and Remark 3.2. Thus the above short exact sequence is split and hence C is injective. Therefore R is Noetherian by Lemma 3.12. □

Proposition 3.14 *The following conditions hold for a left Noetherian ring R :*

(1) *Every FP-injective complex of R-modules is injective;*
(2) *Every n-FP-injective complex of R-modules is injective;*
(3) *Every Gorenstein injective complex is Gorenstein FP-injective;*
(4) *Every Gorenstein injective complex is Gorenstein n-FP-injective.*

Proof It follows from Lemma 3.12. □

4 Gorenstein n-flat complexes

In this section we introduce the notion of Gorenstein n-flat complexes and study their homological behaviors. Also we discuss the relationships between Gorenstein n-flat and Gorenstein n-FP-injective complexes.

Definition 4.1 Let n be a non-negative integer and let G be a complex of right R-modules. We say that the complex G is Gorenstein n-flat if there exists an exact sequence of complexes of right R-modules

$$\mathbb{F} : \cdots \to F_{-2} \to F_{-1} \to F_0 \to F_1 \to F_2 \to \cdots$$

such that (i) F_i is flat for all $i \in \mathbb{Z}$;
(ii) Ker $(F_0 \to F_1) = G$;
(iii) $\mathbb{F} \otimes E$ is exact for any n-FP-injective complex E.

We denote by $\mathcal{GF}_n$ the class of all Gorenstein n-flat complexes.

Proposition 4.2 *The class $\mathcal{GF}_n$ is closed under direct sums.*

Proof It follows from the fact that the direct sum of flat complexes is flat and since the tensor product of complexes commutes with direct sums. □

Remark 4.3 (1) Every flat complex is Gorenstein n-flat. For any flat complex F, there is an exact sequence of complexes of the form

$$\cdots \to 0 \to F \stackrel{id}{\to} F \to 0 \to \cdots$$

such that $\mathrm{Im}(F \to F) = F$ and $- \otimes E$ leaves the sequence exact for any n-FP-injective complex E.

(2) If G is a Gorenstein n-flat complex of right R-modules, then we have $Tor_i(G, E) = 0$ for all n-FP-injective complexes E and for all $i \geq 1$.

Proposition 4.4 *If G is a Gorenstein n-flat complex of right R-modules, then G^i is a Gorenstein n-flat right R-module for all $i \in \mathbb{Z}$.*

Proof Suppose G is a Gorenstein n-flat complex. By definition, there exists an exact sequence of complexes of right R-modules

$$\mathbb{F} : \cdots \to F_{-2} \to F_{-1} \to F_0 \to F_1 \to F_2 \to \cdots$$

such that F_i is flat for all $i \in \mathbb{Z}$ and $\mathrm{Ker}(F_0 \to F_1) = G$. For each $i \in \mathbb{Z}$, there is an exact sequence of right R-modules

$$\mathbb{F}^i : \cdots \to F^i_{-2} \to F^i_{-1} \to F^i_0 \to F^i_1 \to F^i_2 \to \cdots$$

such that F^i_j is flat for all $j \in \mathbb{Z}$ and $\mathrm{Ker}(F^i_0 \to F^i_1) = G^i$. Let E be any n-FP-injective R-module. Then $\overline{E}$ is an n-FP-injective complex by [12, Corollary 3.12]. Thus $\mathbb{F}^i \otimes E \cong \mathbb{F}^i \otimes \overline{E}$ is exact by hypothesis. Therefore G^i is a Gorenstein n-flat right R-module for all $i \in \mathbb{Z}$. □

Lemma 4.5 *Let R be a left n-coherent ring. Then the following are equivalent:*

(1) *G is a Gorenstein n-flat complex of right R-modules;*
(2) *$Tor_i(G, E) = 0$ for all n-FP-injective complexes E and for all $i \geq 1$.*

Proof (1) ⇒ (2) It follows from Remark 4.3.
(2) ⇒ (1) It similar to that of [18, Theorem 3.3]. □

In [18, Proposition 3.14], the connection between Gorenstein FP-injective and Gorenstein flat complexes is proved. The following result represents the relationship between Gorenstein n-FP-injective and Gorenstein n-flat complexes.

Theorem 4.6 *Let R be a ring and G be a complex of right R-modules. Then the following hold:*

(1) *If G is Gorenstein n-flat, then G^+ is Gorenstein n-FP-injective.*
(2) *If R is left n-coherent and G^+ is Gorenstein n-FP-injective, then G is Gorenstein n-flat.*

Proof (1) $\Rightarrow$ (2) Suppose G is Gorenstein n-flat. Then there is an exact sequence of flat complexes of right R-modules

$$\mathbb{F} : \cdots \to F_{-1} \to F_0 \to F_1 \to \cdots$$

with G= Ker$(F_0 \to F_1)$ such that $\mathbb{F} \otimes E$ is exact for any n-FP-injective complex E. So we have the following exact sequence of injective complexes

$$\mathbb{F}^+ : \cdots \to F_1^+ \to F_0^+ \to F_{-1}^+ \to \cdots$$

with G^+= Ker$(F_0^+ \to F_{-1}^+)$. Let C be any n-FP-injective complex. Then the exact sequence

$$\cdots \to F_{-1} \otimes C \to F_0 \otimes C \to F_1 \otimes C \to \cdots$$

induces the following exact sequence

$$\cdots \to (F_1 \otimes C)^+ \to (F_0 \otimes C)^+ \to (F_{-1} \otimes C)^+ \to \cdots .$$

Thus we get the exact sequence

$$\cdots \to \underline{Hom}(C, F_1^+) \to \underline{Hom}(C, F_0^+) \to \underline{Hom}(C, F_{-1}^+) \to \cdots .$$

This yields the following exact sequence

$$\cdots \to Hom(C, F_1^+) \to Hom(C, F_0^+) \to Hom(C, F_{-1}^+) \to \cdots .$$

Therefore G^+ is Gorenstein n-FP-injective.

(2) $\Rightarrow$ (1) Suppose G^+ is Gorenstein n-FP-injective. Then $Ext^i(E, G^+) = 0$ for all n-FP-injective complexes E and all $i \geq 1$. Let E be any n-FP-injective complex. Then $E[m]$ is n-FP-injective for any integer m. Now let m be any integer. Then

$$Ext^i(E, G^+[m]) \cong Ext^i(E[-m], G^+) = 0,$$

which yields that $\underline{Ext}^i(E, G^+) = 0$ for all $i \geq 1$. Thus

$$Tor_i(E, G)^+ \cong \underline{Ext}^i(E, G^+) = 0.$$

Hence $Tor_i(E, G) = 0$ for all $i \geq 1$. Therefore G is Gorenstein n-flat by Lemma 4.5. □

Recall from [16] that a ring R is n-FC if it is left and right coherent and the FP-injective dimensions of R_R and ${}_RR$ are at most n.

Proposition 4.7 *Let R be an n-FC-ring. Then the class $\mathscr{GF}_n$ is closed under pure subcomplexes.*

Proof Suppose A is a pure subcomplex of a Gorenstein n-flat complex C. Then we have the pure exact sequence of complexes of right R-modules

$$0 \to A \to C \to C/A \to 0.$$

This induces a split exact sequence

$$0 \to (C/A)^+ \to C^+ \to A^+ \to 0.$$

Since C is Gorenstein n-flat, we have that C^+ is Gorenstein n-FP-injective by Proposition 4.6. Then it follows that A^+ is Gorenstein n-FP-injective. Therefore A is Gorenstein n-flat by Proposition 4.6. □

Proposition 4.8 *The following are equivalent for any complex G of right R-modules:*

(1) *G is Gorenstein n-flat;*
(2) (a) *$Tor_i(G, E) = 0$ for all n-FP-injective complexes E and for all $i \geq 1$.*
 (b) *There exists an exact sequence of complexes of right R-modules*

$$\mathbb{F} : 0 \to G \to F_1 \to F_2 \to \cdots$$

with F_i flat for $i \geq 1$ such that $\mathbb{F} \otimes E$ is exact for any n-FP-injective complex E;
(3) *There exists a short exact sequence of complexes of right R-modules*

$$0 \to G \to F \to C \to 0,$$

where F is flat and C is Gorenstein n-flat.

Proof (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) follows from the definition of Gorenstein n-flat complexes.

(3) $\Rightarrow$ (2) Suppose $\mathbb{X} : 0 \to G \to F \to C \to 0$ is an exact sequence of complexes with F flat and C Gorenstein n-flat. Then we have $Tor_i(C, E) = 0$ for all $i \geq 1$ and for any n-FP-injective complex E by (1) $\Leftrightarrow$ (2). Thus we have the following exact sequence

$$\cdots \to Tor_{i+1}(C, E) \to Tor_i(G, E) \to Tor_i(F, E) \to \cdots$$

which gives that $Tor_i(G, E) = 0$ for all $i \geq 1$ and for any n-FP-injective complex E. On the other hand, the complex C being Gorenstein n-flat, we have the following exact sequence

$$\mathbb{F}' : 0 \to C \to F_1 \to F_2 \to \cdots$$

such that $\mathbb{F}' \otimes E$ is exact for any n-FP-injective complex E. Combining the exact sequences $\mathbb{X}$ and $\mathbb{F}'$ yields the following exact sequence

$$0 \to G \to F \to F_1 \to F_2 \to \cdots$$

such that $- \otimes E$ leaves the sequence exact for any n-FP-injective complex E. $\square$

Corollary 4.9 *Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an exact sequence of complexes. If C_1 is Gorenstein n-flat and C_3 is flat, then C_2 is Gorenstein n-flat.*

Proof Suppose C_1 is Gorenstein n-flat. By Proposition 4.8, there exists a short exact sequence of complexes

$$0 \to C_1 \to F \to C \to 0$$

such that F is flat and C is Gorenstein n-flat. Consider the pushout of $C_1 \to C_2$ and $C_1 \to F$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & C_3 & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & C & = & C & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Then it follows from the exact sequence $0 \to F \to F' \to C_3 \to 0$ that F' is flat since F and C_3 are flat. Thus we have a short exact sequence

$$0 \to C_2 \to F' \to C \to 0$$

with F' injective and C Gorenstein n-flat. Therefore C_2 is Gorenstein n-flat by Proposition 4.8. □

In connection to [18, Proposition 4.2] and Theorem 4.6, we prove the following main result of this section.

Theorem 4.10 *Let R be a left n-coherent ring and $0 \to C' \to C \to C'' \to 0$ be an exact sequence of complexes of right R-modules. Then:*

(1) *If C'' is Gorenstein n-flat, then C is Gorenstein n-flat if and only if C' is Gorenstein n-flat.*
(2) *If C' and C are Gorenstein n-flat complexes, then C'' is Gorenstein n-flat if and only if $Tor_1(C'', E) = 0$ for all n-FP-injective complexes E.*

Proof (1) It follows from Proposition 3.7 and Theorem 4.6.

(2) Suppose C'' is Gorenstein n-flat. By Proposition 4.8, we have $\mathrm{Tor}_i(C'', E) = 0$ for all n-FP-injective complexes E and for all $i \geq 1$. Hence $\mathrm{Tor}_1(C'', E) = 0$. Conversely, suppose C and C' are Gorenstein n-flat and $\mathrm{Tor}_1(C'', E) = 0$ for all n-FP-injective complexes E. By Proposition 4.8, there exists a short exact sequence of complexes of right R-modules $0 \to C' \to F \to D \to 0$ with F flat and D Gorenstein n-flat. Consider the pushout of $C' \to F$ and $C' \to C$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & C'' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & D & = & D & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where G is Gorenstein n-flat by (1) since C and D are Gorenstein n-flat. Thus we have the following exact sequence of complexes of right R-modules

$$0 \to G \to F' \to D' \to 0$$

with F' flat and D' Gorenstein n-flat. Consider the pushout of $G \to F'$ and $G \to C''$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & G & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & F' & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & D' & = & D' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Thus we get an exact sequence of complexes of right R-modules

$$0 \to C'' \to F'' \to D' \to 0,$$

which induces a long exact sequence

$$\cdots \to Tor_1(C'', E) \to Tor_1(F'', E) \to Tor_1(D', E) \to 0 \cdots,$$

where E is any n-FP-injective complex. Since D' is Gorenstein n-flat, we have $\mathrm{Tor}_1(F'', E) = 0$ by Proposition 4.8. On the other hand, the exactness of the sequence

$$0 \to F \to F' \to F'' \to 0$$

implies the exactness of the sequence of complexes

$$0 \to (F'')^+ \to (F')^+ \to F^+ \to 0,$$

where F^+ and $(F')^+$ are n-FP-injective by [12, Proposition 3.3]. Since $\mathrm{Tor}_1(F'', F^+) = 0$ and $\underline{Ext}^1(F^+, (F'')^+) \cong Tor_1(F'', F^+)^+$, we have $\underline{Ext}^1(F^+, (F'')^+) = 0$ and so $Ext^1(F^+, (F'')^+) = 0$. This implies that the sequence

$$0 \to (F'')^+ \to (F')^+ \to F^+ \to 0$$

splits. Consequently, $(F'')^+$ is an injective complex. Hence F'' is a flat complex by [6, Theorem 4.1.3]. Therefore C'' is Gorenstein n-flat by Proposition 4.8. □

Using Theorem 4.10 and Remark 4.3, we have the following.

Theorem 4.11 *Let R be a left n-coherent ring. Then the class $\mathcal{GF}_n$ is projectively resolving.*

Proposition 4.12 *Let R be a left coherent ring. Then the class $\mathcal{GF}_n$ is closed under direct products and direct summands.*

Proof Since the class of flat complexes of R-modules is closed under direct products, the class $\mathcal{GF}_n$ is closed under direct products. On the other hand, it is closed under direct summands by Theorem 4.11 and [7, Proposition 1.4]. □

Proposition 4.13 *Let R be a commutative ring and F be a flat complex. If G is a Gorenstein n-flat complex, then $G \otimes F$ is also a Gorenstein n-flat complex.*

Proof Suppose G is Gorenstein n-flat. Then there exists an exact sequence of flat complexes

$$\cdots \to F_{-1} \to F_0 \to F_1 \to \cdots$$

with G=Ker $(F_0 \to F_1)$ such that $- \otimes E$ leaves the sequence exact for any n-FP-injective complex E. Since F is a flat complex and since the tensor product of flat complexes is flat, we have the following exact sequence of complexes

$$\cdots \to F \otimes F_{-1} \to F \otimes F_0 \to F \otimes F_1 \to \cdots$$

with $F \otimes F_i$ flat for all $i \in \mathbb{Z}$ and $F \otimes G$=Ker $(F \otimes F_0 \to F \otimes F_1)$. Let E be any n-FP-injective complex. Then by [6, Proposition 4.2.1 (3)], we have the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & (F \otimes F_{-1}) \otimes E & \longrightarrow & (F \otimes F_0) \otimes E & \longrightarrow & (F \otimes F_1) \otimes E & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & F \otimes (F_{-1} \otimes E) & \longrightarrow & F \otimes (F_0 \otimes E) & \longrightarrow & F \otimes (F_1 \otimes E) & \longrightarrow & \cdots . \end{array}$$

Since the lower sequence is exact by hypothesis, the upper sequence is also exact. Therefore $G \otimes F$ is a Gorenstein n-flat complex. □

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