SOME EXPLICIT FORMULAS FOR CERTAIN NEW CLASSES OF BERNOULLI, EULER AND GENOCCHI POLYNOMIALS

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ABSTRACT. In recent years, the subject of Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials have been studied extensively. Recently, the authors have introduced in [27, 28] some new generalized classes of Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. In this paper, with the help of a result involving an explicit formula for the generalized potential polynomials obtained by Cenkci [5], we develop some explicit formulas related with these new classes of polynomials.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Genocchi polynomials $G_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$, each of degree n as well as in α, are defined respectively by the following generating functions (see,[8, vol.3, p.253 et seq.], [14, Section 2.8] and [18]):

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^{\alpha} \cdot \mathrm{e}^{xt} = \sum_{k=0}^{\infty} B_k^{(\alpha)}(x) \frac{t^k}{k!} \qquad (|t| < 2\pi; 1^{\alpha} := 1), \tag{1}$$

$$\left(\frac{2}{\mathrm{e}^t + 1}\right)^{\alpha} \cdot \mathrm{e}^{xt} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(x) \frac{t^k}{k!} \qquad (|t| < \pi; 1^{\alpha} := 1) \tag{2}$$

and

$$\left(\frac{2t}{\mathrm{e}^t + 1}\right)^{\alpha} \cdot \mathrm{e}^{xt} = \sum_{k=0}^{\infty} G_k^{(\alpha)}(x) \frac{t^k}{k!} \qquad (|t| < \pi; 1^{\alpha} := 1). \tag{3}$$

The literature contains a large number of interesting properties and relationships involving these polynomials [1, 4, 7, 8, 10, 25]. These appear in many applications in combinatorics, number theory and numerical analysis. Lately, some interesting analogues of the classical Bernoulli polynomials, the classical Euler polynomials and the classical Genocchi polynomials have been investigated. Q.-M. Luo and H.M. Srivastava [22, 24] introduced the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ (the case $\alpha = 1$ was investigated first by T.M. Apostol [2, Eq.(3.1), p.165]). In 2006, Q.-M. Luo [15] invented the generalized Apostol-Euler polynomials

$\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ and the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ in [18]. Many authors have investigated these polynomials and numerous very interesting papers can be found in the literature. The reader should read ([3, 6, 9, 16, 17, 19, 20, 21, 23, 26]).

Recently, the authors in [27, 28] have investigated some properties of new related classes to these polynomials. Explicitly, they considered the following new classes defined respectively by the next definitions.

Definition 1.1. For arbitrary real or complex parameter α and for b, $c \in \mathbb{R}^+$, the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x,b,c;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t=0$, with $|t\log b + \log\lambda| < 2\pi$ by means of the generating function

$$\left(\frac{t^m}{\lambda b^t - \sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}}\right)^{\alpha} \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x,b,c;\lambda)\frac{t^k}{k!}. \tag{4}$$

Definition 1.2. For arbitrary real or complex parameter α and for b, $c \in \mathbb{R}^+$, the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t=0$, with $|t\log b+\log\lambda| < \pi$ by means of the generating function

$$\left(\frac{2^m}{\lambda b^t + \sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}}\right)^{\alpha} \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x,b,c;\lambda)\frac{t^k}{k!}. \tag{5}$$

Definition 1.3. For arbitrary real or complex parameter α and for b, $c \in \mathbb{R}^+$, the generalized Apostol-Genocchi polynomials $\mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)$, $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$, are defined, in a suitable neighborhood of $t=0$, with $|t\log b + \log\lambda| < \pi$ by means of the generating function

$$\left(\frac{2^m t^m}{\lambda b^t + \sum_{l=0}^{m-1}\frac{(t\log b)^l}{l!}}\right)^{\alpha} \cdot c^{xt} = \sum_{k=0}^{\infty} \mathfrak{G}_n^{[m-1,\alpha]}(x,b,c;\lambda)\frac{t^k}{k!}. \tag{6}$$

If we set $b=c=\mathrm{e}$, $m=1$ and $\lambda=1$ in each of these definitions, we recover the definitions of the classical Bernoulli, Euler and Genocchi polynomials defined respectively by (1), (2) and (3).

In this paper, we establish explicit formulas for all these classes of polynomials in the case where $b=c=\mathrm{e}$ and $\lambda=1$. This is done with the help of a generalization, by Cenkci [5], of a theorem involving the potential polynomials due to Howard [13] and by making use of the multinomial theorem.

2. Preliminaries

In this section, we recall the result given by Howard in [13] involving the potential polynomials and we give the generalization obtained recently by Cenkci [5].

For $r \geq 0$ and $a_r \neq 0$, let $F(t) = \sum_{j=r}^{\infty} a_j \frac{t^j}{j!}$ be a formal power series. For $\alpha \in \mathbb{C}$, the potential polynomials $F_n^{(\alpha)}$ [7, 13] are defined by the exponential generating function

$$\left(\frac{a_r\frac{t^r}{r!}}{F(t)}\right)^{\alpha}=\sum_{n=0}^{\infty}F_n^{(\alpha)}\frac{t^n}{n!} \tag{7}$$

If $r\geq 1$, then the exponential Bell polynomials $B_{n,k}(0,...,a_r,a_{r+1},...)$ in an infinite number of variables $a_r,a_{r+1},...$ can be defined by means of

$$(F(t))^k=k!\sum_{n=0}^{\infty}B_{n,\,k}(0,...,0,a_r,a_{r+1},...)\frac{t^n}{n!}. \tag{8}$$

When k is a positive integer, then

$$F_n^{(-k)}=\left(\frac{r!}{a_r}\right)^k\frac{n!k!}{(n+rk)!}B_{n+rk,\,k}(0,...,0,a_r,a_{r+1},...). \tag{9}$$

In 1982, Howard in [13] obtained the following theorem:

Theorem 2.1. *If $F_n^{(\alpha)}$ is defined by (7) and if $B_{n,k}$ is defined by (8), then*

$$\begin{aligned}F_n^{(\alpha)}=&\sum_{k=0}^{n}(-1)^k\binom{\alpha+k-1}{k}\binom{\alpha+n}{n-k}\left(\frac{r!}{a_r}\right)^k\frac{n!k!}{(n+rk)!}\\ &\times B_{n+rk,\,k}(0,...,0,a_r,a_{r+1},...).\end{aligned} \tag{10}$$

Recently, Cenkci [5] has extended this last theorem by introducing the generalized potential polynomials $F_n^{(\alpha)}(x)$ where $\alpha\in\mathbb{C}$. He thus considered the following definition.

Definition 2.2. For $r\geq 0$ and $a_r\neq 0$, let $F(t)=\sum_{j=r}^{\infty}a_j\frac{t^j}{j!}$ be a formal power series. For an independent variable x, we define the generalized potential polynomials $F_n^{(\alpha)}(x)$ by means of the exponential generating function

$$\left(\frac{a_r\frac{t^r}{r!}}{F(t)}\right)^{\alpha}\mathrm{e}^{xt}=\sum_{n=0}^{\infty}F_n^{(\alpha)}(x)\frac{t^n}{n!}. \tag{11}$$

Obviously, setting $x=0$ in (11), we have $F_n^{(\alpha)}(0)=F_n^{(\alpha)}$.

Theorem 2.3. *If $F_n^{(\alpha)}(x)$ is defined by (11) and if $B_{n,k}$ is defined by (8), then*

$$\begin{aligned}F_n^{(\alpha)}(x)=&\sum_{k=0}^{n}(-1)^k\binom{\alpha+k-1}{k}\left(\frac{r!}{a_r}\right)^k\cdot\sum_{l=0}^{n-k}\binom{n}{l+k}\binom{\alpha+k+l}{l}x^{n-k-l}\\ &\times\frac{(l+k)!k!}{(l+k+rk)!}B_{l+k+rk,\,k}(0,...,0,a_r,a_{r+1},...).\end{aligned} \tag{12}$$

Making use of this theorem, Cenkci [5, p. 1502, Equation 3.1] obtain the following relation for the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$. Let $F(t)=\mathrm{e}^t-1=t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots$. Thus $r=1$ and $a_1=1$. From (8), we have

$$k!\sum_{n=0}^{\infty}B_{n,\,k}(1,1,1,...)\frac{t^n}{n!}=(\mathrm{e}^t-1)^k=k!\sum_{n=0}^{\infty}S(n,k)\frac{t^n}{n!} \tag{13}$$

and then we obtain

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k \binom{\alpha+k-1}{k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \times \binom{\alpha+k+l}{l} \frac{(l+k)!\, k!}{(l+2k)!} S(l+2k,k). \tag{14}$$

3. Main results

Our main objective in this section is to apply the multinomial identity and Theorem 2 to all the classes of generalized Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials defined respectively by (4), (5) and (6) in the case where $b = c = \mathrm{e}$ and $\lambda = 1$.

Lemma 3.1. *(Multinomial identity* [7, p. 28, Theorem B]*) If $x_1, x_2, \ldots, x_r$ are commuting elements of a ring, then for all $n \in \mathbb{N}_0$, we have*

$$(x_1 + \cdots + x_m)^n = \sum_{\substack{\nu_1,\cdots,\nu_m \geq 0\\ \nu_1+\nu_2+\cdots+\nu_m=n}} \binom{n}{\nu_1,\cdots,\nu_m} x_1^{\nu_1}\cdots x_m^{\nu_m} \tag{15}$$

where summation takes place over all integers $\nu_i \geq 0$ and

$$\binom{n}{\nu_1,\cdots,\nu_m} := \frac{n!}{\nu_1!\,\nu_2!\ldots\nu_m!} \tag{16}$$

which are called the multinomial coefficients.

Theorem 3.2. *The following explicit representation holds for the generalized Bernoulli polynomials $B_n^{[r-1,\,\alpha]}(x)$ defined by (4) where $b = c = e$ and $\lambda = 1$*

$$B_n^{[r-1,\alpha]}(x) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k} (r!)^{\alpha+k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)! \times \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0\\ \nu_1+\nu_2+\cdots+\nu_r=k}} \binom{k}{\nu_1,\cdots,\nu_r} \frac{(-1)^{\mu+k}\,(\nu_1!)\, S(l+k+rk-\mu,\nu_1)}{(l+k+rk-\mu)!} \tag{17}$$

where $\mu = \nu_2 + 2\nu_3 + \cdots + (r-1)\nu_r$.

Proof. Considering $F(t) = \mathrm{e}^t - \sum_{h=0}^{r-1} \frac{t^h}{h!} = \mathrm{e}^t - 1 - \sum_{h=1}^{r-1} \frac{t^h}{h!}$. Thus, we have

(18)
$$\begin{aligned}\left(\mathrm{e}^t - \sum_{h=0}^{r-1} \frac{t^h}{h!}\right)^k &= \left[(\mathrm{e}^t - 1) - \sum_{h=1}^{r-1} \frac{t^h}{h!}\right]^k \\ &= \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_r = k}} \binom{k}{\nu_1,\cdots,\nu_r} (-t)^\mu \left(\sum_{l=1}^{\infty} \frac{t^l}{l!}\right)^{\nu_1} \\ &= \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_r = k}} \binom{k}{\nu_1,\cdots,\nu_r} (-t)^\mu (\nu_1!) \sum_{n=0}^{\infty} S(n,\nu_1) \frac{t^n}{n!}\end{aligned}$$

where $\mu = \nu_2 + 2\nu_3 + \cdots + (r-1)\nu_r$. We know from (8) that

$$(F(t))^k = k! \sum_{n=0}^{\infty} B_{n,\,k}(0, ..., 0, a_r, a_{r+1}, ...) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_r = k}} \binom{k}{\nu_1,\cdots,\nu_r} (-1)^\mu\ (\nu_1!)\ S(n,\nu_1)\ \frac{t^{n+\mu}}{n!}. \tag{19}$$

We thus have

$$B_{n,\,k}(0, ..., 0, 1, 1, 1, ...) = \frac{n!}{k!} \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_r = k}} \binom{k}{\nu_1,\cdots,\nu_r} \frac{(-1)^\mu\ (\nu_1!)\ S(n-\mu,\nu_1)}{(n-\mu)!}. \tag{20}$$

By making use of Theorem 2 with $a_r = 1$, we obtain

$$\begin{aligned} F_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k} (r!)^k \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)! \\ \times \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_r = k}} \binom{k}{\nu_1,\cdots,\nu_r} \frac{(-1)^{\mu+k}\ (\nu_1!)\ S(l+k+rk-\mu,\nu_1)}{(l+k+rk-\mu)!}. \end{aligned} \tag{21}$$

Finally, since the generalized Bernoulli polynomials $B_n^{[r-1,\alpha]}(x)$ are defined by means of the generating function (4) where $b = c = \mathrm{e}$ and $\lambda = 1$, it is easy to observe that

$$B_n^{[r-1,\alpha]}(x) = (r!)^{\ \alpha}\ F_n^{(\alpha)}(x) \tag{22}$$

and consequently, we have

$$
\begin{aligned}
&B_n^{[r-1,\alpha]}(x) = \\
(23)\quad &\sum_{k=0}^{n} \binom{\alpha+k-1}{k} (r!)^{\,\alpha+k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)! \\
&\times \sum_{\substack{\nu_1,\cdots,\nu_r \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_r = k}} \binom{k}{\nu_1,\cdots,\nu_r} \frac{(-1)^{\mu+k}\,(\nu_1!)\, S(l+k+rk-\mu,\nu_1)}{(l+k+rk-\mu)!}
\end{aligned}
$$

where $\mu = \nu_2 + 2\nu_3 + \cdots + (r-1)\nu_r$. □

As a special case of (23) if we set $r = 2$ then $\mu = \nu_2 = k - \nu_1$ and we divide by 2^α, we get

(24)

$$
\begin{aligned}
\frac{B_n^{[1,\alpha]}(x)}{2^\alpha} = &\sum_{k=0}^{n} \binom{\alpha+k-1}{k} 2^{\,k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)! \\
&\times \sum_{\nu_1=0}^{k} \binom{k}{\nu_1} \frac{(-1)^{\nu_1}\,(\nu_1!)\, S(l+2k+\nu_1,\nu_1)}{(l+2k+\nu_1)!}.
\end{aligned}
$$

This result has been obtained in [5, p. 1505, Eq. 3.15] for the polynomials $A_n^{(\alpha)}(x)$ defined and studied by Howard [11, 12].

Theorem 3.3. *The following explicit representation holds for the generalized Euler polynomials* $E_n^{[r-1,\,\alpha]}(x)$ *defined by (5) where* $b = c = e$ *and* $\lambda = 1$

$$
\begin{aligned}
&E_n^{[r-1,\,\alpha]}(x) = \\
(25)\quad &\sum_{k=0}^{n} (-1)^k \binom{\alpha+k-1}{k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)! \\
&\times \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1} = k}} \binom{k}{\nu_1,\cdots,\nu_{r+1}} \frac{2^{\nu_2-k+(r-1)\alpha}\,(\nu_1!)\, S(l+k-\mu,\nu_1)}{(l+k-\mu)!}
\end{aligned}
$$

where $\mu = \nu_3 + 2\nu_4 + \cdots + (r-1)\nu_{r+1}$.

Proof. Setting $F(t) = \mathrm{e}^t + \sum_{h=0}^{r-1} \frac{t^h}{h!} = 2 + 2t + 2\frac{t^2}{2!} + \cdots + 2\frac{t^{r-1}}{(r-1)!} + \sum_{k=r}^{\infty} \frac{t^k}{k!}$. This implies that $a_0 = 2$ and we have

(26)

$$\left(\mathrm{e}^t + \sum_{h=0}^{r-1} \frac{t^h}{h!}\right)^k = \left[(\mathrm{e}^t - 1) + 2 + \sum_{h=1}^{r-1} \frac{t^h}{h!}\right]^k$$

$$= \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1}= k}} \binom{k}{\nu_1, \cdots, \nu_{r+1}} 2^{\nu_2} t^{\mu} \left(\sum_{l=1}^{\infty} \frac{t^l}{l!}\right)^{\nu_1}$$

$$= \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1}= k}} \binom{k}{\nu_1, \cdots, \nu_{r+1}} 2^{\nu_2} t^{\mu} \, (\nu_1!) \sum_{n=0}^{\infty} S(n, \nu_1) \frac{t^n}{n!}$$

where $\mu = \nu_3 + 2\nu_4 + \cdots + (r-1)\nu_{r+1}$. From (8), simple calculations give

$$B_{n,\,k}(2, 2, 2, ..., 1, 1, 1, 1...) =$$

$$\frac{n!}{k!} \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1}= k}} \binom{k}{\nu_1, \cdots, \nu_{r+1}} \frac{2^{\nu_2} \, (\nu_1!) \, S(n-\mu, \nu_1)}{(n-\mu)!}. \tag{27}$$

We thus obtain from Theorem 2 with $a_0 = 2$ that

(28)

$$F_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{\alpha+k-1}{k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)!$$

$$\times \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1}= k}} \binom{k}{\nu_1, \cdots, \nu_{r+1}} \frac{2^{\nu_2-k} \, (\nu_1!) \, S(l+k-\mu, \nu_1)}{(l+k-\mu)!}$$

and using the fact that the generalized Euler polynomials $E_n^{[r-1,\alpha]}(x)$ are defined by means of the generating function (5) where $b = c = \mathrm{e}$ and $\lambda = 1$, we find that

$$E_n^{[r-1,\,\alpha]}(x) = 2^{(r-1)\alpha} \, F_n^{(\alpha)}(x) \tag{29}$$

and we finally obtain

(30)

$$E_n^{[r-1,\,\alpha]}(x) =$$

$$\sum_{k=0}^{n} (-1)^k \binom{\alpha+k-1}{k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l} (l+k)!$$

$$\times \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1}= k}} \binom{k}{\nu_1, \cdots, \nu_{r+1}} \frac{2^{\nu_2-k+(r-1)\alpha} \, (\nu_1!) \, S(l+k-\mu, \nu_1)}{(l+k-\mu)!}.$$

□

If we set $r = 1$ in (30) then $\mu = 0$ and $\nu_2 = k - \nu_1$ and we recover a result given by Cenkci in [5, p. 1503, Eq. 3.6], that is,

$$E_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{\alpha+k-1}{k} \sum_{l=0}^{n-k} \binom{n}{l+k} x^{n-k-l} \binom{\alpha+k+l}{l}$$

$$\times \sum_{\nu_1=0}^{k} \binom{k}{\nu_1} 2^{-\nu_1} (\nu_1!)\, S(l+k, \nu_1). \tag{31}$$

Now, by making use of the generating functions (5) and (6) where $b = c = \mathrm{e}$ and $\lambda = 1$ in both cases, it is easy to obtain the following relationship between the generalized Euler polynomials $E_n^{[r-1,\, j]}(x)$ of integer order j and the generalized Genocchi polynomials $G_n^{[r-1,\, j]}(x)$ of order j, namely

$$G_n^{[r-1,\, j]}(x) = \frac{n!}{(n-rj)!}\, E_{n-rj}^{[r-1,\, j]}(x) \qquad (n \geq rj). \tag{32}$$

Theorem 3.4. *The following explicit representation holds for the generalized Genocchi polynomials $G_n^{[r-1,\, j]}(x)$ of integer order j defined by (6) where $b = c = e$ and $\lambda = 1$*

(33)

$$G_n^{[r-1,\, j]}(x) =$$

$$\frac{n!}{(n-rj)!} \sum_{k=0}^{n-rj} (-1)^k \binom{\alpha+k-1}{k} \sum_{l=0}^{n-rj-k} \binom{n-rj}{l+k} x^{n-rj-k-l} \binom{\alpha+k+l}{l}$$

$$\times (l+k)! \sum_{\substack{\nu_1,\cdots,\nu_{r+1} \geq 0 \\ \nu_1+\nu_2+\cdots+\nu_{r+1} = k}} \binom{k}{\nu_1, \cdots, \nu_{r+1}} \frac{2^{\nu_2 - k + (r-1)\alpha} (\nu_1!) S(l+k-\mu, \nu_1)}{(l+k-\mu)!}$$

where $n \geq rj$.

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