On the form and stability of seafloor stratigraphy and shelf profiles: A mathematical model and solution method

S.D. Peckham
INSTAAR, University of Colorado, Boulder 80309-0450, USA
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Abstract

There are a variety of different marine sediment transport processes that affect the evolution of continental shelf profiles. Physically based numerical models make it possible to simulate the evolution of the continental shelf in response to these processes, which may be combined with different magnitudes, frequencies and durations. The basic model that is employed in most numerical simulations is based on conservation of mass and is a moving boundary problem since the river mouth progrades into the receiving basin. The general form of these stratigraphic or seafloor evolution models is a first-order, Exner-type differential equation that is forced by a sediment deposition/erosion function. This latter function depends on which sediment transport processes are modeled and how they are modeled, so it is desirable to have a solution method that works for any such function. This paper presents a general solution method to this type of evolution equation that works for any initial bathymetry and sediment deposition function given as 1D functions of seaward distance. It is based on the Laplace transform.

Numerical results often exhibit “equilibrium profile” solutions for a wide variety of different model scenarios. These profiles may be viewed as travelling waves, such that the rate of progradation gives the speed of this wave as it advances into the receiving basin. When realistic initial and boundary conditions are used, profiles are often seen to evolve toward forms that are independent of the initial bathymetry. Although episodic perturbations such as turbidities disrupt these forms, they still serve as attractors for the system dynamics. An application of the solution method presented in this paper also allows the shape of equilibrium profiles and the progradation rate to be determined from a specification of the initial bathymetry and the sediment deposition function. Several illustrative examples are given in closed form. Results are also presented that make it possible to compute the amount of time required for an equilibrium profile to be re-established after a perturbation.

Keywords: Travelling wave; Laplace transform; Sediment transport; Deposition; Bathymetry; Analytic solutions

1. Introduction

Seafloor stratigraphy has long been of interest to geologists as it provides a detailed record of past events. In recent years seafloor stratigraphy has also started to emerge as an important data source in the study of global climate change, as links between
climate change and basin stratigraphy have been identified (e.g. Morehead et al., 2001). It also plays a key role in various problems that are of interest to the oil industry and to the US Navy. As a result of these applications, there is an increasing interest in process-based numerical models for simulating and studying the evolution of seafloor stratigraphy. One highly developed model of this type is known as SedFlux, a model developed and refined over many years by Syvitski and coworkers (e.g. Syvitski, 1989; Morehead and Syvitski, 1999; Syvitski et al., 1999; Syvitski and Hutton, 2001).

Much of the research in this area has focused on trying to understand how observed features in a stratigraphic sequence vary as the sediment process details and initial conditions (e.g. bathymetric profile and basin depth) are changed (e.g. Schlager and Camber, 1986; Lawrence et al., 1990; Swift and Thorne, 1991; Steckler et al., 1993; Pratson and Haxby, 1996; Pirmez et al., 1998; Adams and Schlager, 2000; O'Grady et al., 2000; O'Grady and Syvitski, 2001; Uličný et al., 2002). This is often done by running a large number of numerical simulations in which input parameters are changed systematically and then examining the impact on various features of interest. It is hoped that a detailed understanding of cause and effect will make it possible to infer the conditions that produced a given stratigraphy. Unfortunately, a purely numerical/computational approach to this problem is tedious, imprecise and necessarily limited in scope.

Numerical simulations by stratigraphic or seafloor evolution models have helped to elucidate many different aspects of stratigraphic evolution and seem to capture the essence of the problem. They have been shown to agree favorably with the Lake Mead data set (Pirmez et al., 1998), core data (Overeem et al., 2005), laboratory experiments (Kubo et al., 2005) and with seismic and acoustic data (Pratson et al., 2007). One of the more interesting features that stands out when examining the results of simulations is that “travelling wave” solutions seem to emerge as attractors for the system dynamics. That is, regardless of the initial bathymetry that is used, a smooth, prograding profile soon emerges as a result of deposition from sediment plumes. The form of this profile is stable in the sense that it is recovered even after perturbations by other sediment transport processes (O'Grady and Syvitski, 2001).

In order to better understand the emergence of travelling wave solutions and other features observed in numerical simulations the author was motivated to seek analytic solutions to the seafloor evolution equation. The result is a relatively simple framework based on the Laplace transform that makes it possible to obtain numerous closed-form solutions to this moving boundary problem. These solutions provide considerable insight into the structure of stratigraphic sequences that are observed acoustically and in numerical simulations. In particular, the solutions show exactly how input variables such as basin depth, initial profile shape, sediment transport process parameters (e.g. removal rate constants) and so on determine observed features such as profile shape, maximum profile slope, progradation rate, time for the mouth to reach a given offshore position and many others. These solutions can also be used to check the accuracy of numerical solution methods and for educational purposes.

2. A mathematical model for seafloor stratigraphy

Consider a river entering the sea, with an x-axis extending seaward, a z-axis pointing skyward and with the starting position of the river mouth or shoreline located at the origin. We will assume for the purpose of this paper that sea level is fixed and occurs at z = 0. Let z = f(x, t) ≤ 0 denote the seafloor surface as a function of distance x from the starting position of the river mouth or shoreline; f(0, 0) = 0. Let R(x) be a sediment deposition function that specifies the rate at which sediment is deposited on the seafloor at a distance x from the river mouth. Note that R(x) could represent a single sediment transport process or the net result of several different sediment transport processes. In order to conserve mass, the evolution of the seafloor must then be governed by the following partial differential equation (PDE):

\[
\frac{\partial f}{\partial t} = \begin{cases} \frac{R(x - x_m(t))}{x \geq x_m(t)}, \\ 0 & x < x_m(t), \end{cases}
\tag{1}
\]

where \(x_m(t)\) is the position of the river mouth at time t. The corresponding boundary condition for (1) is that we must always have \(f(x, t) \leq 0\), so the river mouth reaches a position \(x_m\) when the rising seafloor locally reaches sea level. For the case of sedimentation from a river plume we would have \(R = \lambda I(x)\), where \(\lambda [1/day]\) is the removal rate constant and \(I(x)\) [meters] is the inventory, given by integrating the sediment concentration, C(z), from...
be a very important constraint on the problem and the seafloor to the ocean surface at \( z = 0 \). (See the paper by Peckham on sediment plumes, in this issue.) For the case of sediment deposited uniformly at the sea surface, we would have \( R = R_0 \), independent of \( x \).

Integrating Eq. (1) with respect to time, \( t \), gives

\[
f(x, t) = \begin{cases} f_0(x) + \int_{t=0}^{t'} R(x - x_m(t')) \, dt' & \text{if } x \geq x_m(t) , \\ 0 & \text{if } x < x_m(t) . \end{cases}
\]

(2)

Here \( t' \) is the integration variable and the function \( f_0(x) \) is again the initial seafloor bathymetry. Unfortunately, we cannot evaluate (2) unless we know the position of the river mouth, \( x_m(t) \), as a function of time, and this position is itself a by-product of the evolution. Problems in which the position of a boundary is determined by the dynamics itself are known as moving boundary problems and require special solution methods.

As a first step in solving this problem, let us change the integration variable in (2) from \( t \) to \( x_m \), to get

\[
f(x, t) = \begin{cases} f_0(x) + \int_{x_m=0}^{x_m(t)} R(x - x_m) \frac{dx_m'}{c(x_m')} & \text{if } x \geq x_m(t) , \\ 0 & \text{if } x < x_m(t) . \end{cases}
\]

(3)

Here \( c = \frac{dx_m}{dt} \) is the rate at which the river mouth progrades, which we will call the progradation rate, and \( c(x) \) is this rate when the river mouth reaches a position \( x \), relative to the original, fixed coordinate system. The original position of the river mouth is at \( x = 0 \). Note that if we knew \( c(x) \) and \( x_m(t) \), we could express the progradation rate as a function of time as \( c(t) = c(x_m(t)) \). Alternately, if we knew \( c(t) \), we could determine \( x_m(t) \) as

\[
x_m(t) = \int_{t=0}^{t'} c(t') \, dt'.
\]

(4)

At a second key step in setting up the problem, we note that the dynamics must also honor an important “integral constraint,” such that when sediment at a point is piled high enough to reach sea level \( (z = 0) \), the river mouth must prograde. That is, the bathymetry, \( f(x, t) \), cannot exceed zero and \( f(x_m, t) = 0 \). Observe that the river mouth will reach a position \( x \) offshore only when the cumulative sediment deposited by all previous river mouth positions is equal to the magnitude of the initial bathymetry at that position. This fact turns out to be a very important constraint on the problem and can be expressed mathematically as

\[
\int_{x_m=0}^{x_m} R(x - x_m) \frac{dx_m}{c(x_m)} = - f_0(x).
\]

(5)

This is an integral equation in which \( f_0 \) and \( R \) are given functions of \( x \) and the unknown to be determined is the rate function \( c(x) \). It has the general form of a convolution, which is a special case of a Volterra integral equation of the first kind (Press et al., 1992), with a kernel that depends on \( x \) and \( x_m \) only through their difference. In the standard notation for integral equations we would have \( K = R \) and \( \phi(x) = 1/c(x) \). This type of equation can be solved with Laplace transforms, as will be discussed in a later section.

While the Laplace transform gives us the means to solve Eq. (5) for \( c(x) \), how then can we determine the functions \( x_m(t) \) and \( c(t) \)? The key is to recall that \( c(t) = c(x_m(t)) = \frac{dx_m}{c(x_m)} \). It follows that given \( c(x) \), we can simply integrate

\[
\frac{dx_m}{c(x_m)} = \frac{dx_m}{c(x_m)}
\]

(6)

to find \( t \) as a function of \( x_m \). This will always give us an implicit expression for \( x_m(t) \), and an explicit expression in cases where it can be inverted. Note also that a knowledge of \( t(x_m) \) means that we can compute the time required for the river mouth to reach any given distance from its starting point at time zero.

Note that Eqs. (5) and (6) provide a general solution strategy for Eq. (1). Given \( R(x) \) and \( f_0(x) \), we can solve (5) for \( c(x) \), and given \( c(x) \), we can integrate (6) to get \( x_m(t) \). These can then be inserted into (3) to get the solution, \( f(x, t) \).

3. Travelling wave solutions as attractors

It is not uncommon for PDEs that describe a curve evolving in time to have so-called travelling wave solutions, which are curves that travel at a constant speed without changing their shape. These solutions are of the form \( f(x, t) = H(x - ct) \), and can often be found fairly easily by inserting this form into the PDE and solving the resulting ordinary differential equation (ODE) for the function \( H(s) \), where \( s = x - ct \). Doing this for Eq. (1), and noting that we must have \( x_m(t) = ct \) for a travelling wave solution, we find that

\[
H(s) = \frac{1}{c} \int_{s=0}^{s} R(s') \, ds' = - \frac{1}{c} [G(s) - G(0)].
\]

(7)
This shows that the shape of a travelling wave solution to (1) is given in terms of the indefinite integral, \(G(x)\), of the sediment deposition function, \(R(x)\). So if we take the initial seafloor bathymetry, \(f_0(x) = f(x,0) < 0\), to be \(H(x)\), then all future bathymetries will have exactly the same shape, but will prograde seaward at a constant rate, \(c\). Note that Eq. (1) does not impose any constraint on the value of \(c\), but the vertical dimension of the travelling wave (7) is scaled by \(1/c\). Therefore \(c\) can be determined from a boundary condition such as \(f_0(x) \rightarrow -D\) as \(x \rightarrow \infty\).

It is observed in numerical simulations that solutions to (1) often approach travelling waves regardless of the initial bathymetry that is used. In other words, travelling wave solutions are often attractors for Eq. (1). In order for this to occur, \(c(x)\) must approach a limit, \(c\), as \(x\) gets large. Given \(R(x)\) and \(f_0(x)\), we can solve Eq. (5) (e.g. using the Laplace transform) and then check to see whether it approaches a limit or not. If it does, then it turns out that the final shape of the travelling wave will be very similar to (7). To see this, we first use (5) to eliminate \(f_0(x)\) from Eq. (3) to get

\[
f(x,t) = \int_{x_m=x_m(t)}^{x} \frac{-R(x-x_m')}{c(x_m')} \, dx_m'.
\]

Now if \(c(x)\) has a limit, \(c\), then if \(x_m\) is sufficiently large, we can rewrite (8) as

\[
f(x,t) \sim \frac{1}{c} \int_{x_m=x_m(t)}^{x} R(x-x_m') \, dx_m'.
\]

Here, the symbol “\(\sim\)” denotes asymptotic equality. Changing the integration variable in (9) from \(x_m'\) to \(s = x - x_m'\) we get

\[
f(x,t) \sim \frac{1}{c} \int_{s=0}^{x-x_m(t)} R(s) \, ds.
\]

Assuming again that \(c(x)\) has a limit, \(c\), we can use Eq. (4) to write \(x_m(t) = ct + L(t)\), where \(L(t) = \int_{t=0}^{t}[c - c(t')] \, dt'\). If \(L(t)\) approaches a limiting value, \(L\), then Eq. (10) becomes

\[
f(x,t) \sim \frac{1}{c} \int_{s=0}^{x-c t-L} R(s) \, ds.
\]

If we now denote the indefinite integral of \(R(x)\) as \(G(x)\), we see that

\[
f(x,t) \sim \frac{1}{c} \left[G(x - ct - L) - G(0)\right] = H(x - c t),
\]

which has the functional form of a travelling wave. The shape of this travelling wave, \(H(x)\), is again given in terms of the indefinite integral of \(R(x)\), as in Eq. (7). Now, however, there is an offset parameter, \(L\), which basically represents the distance from the origin at which the travelling wave solution is achieved.

4. Specific examples

4.1. Example 1: sediment deposition by a river plume

Let us consider the specific and fairly realistic example of the seafloor generated by a river plume. In this case, we have \(R(x) = \lambda I(x) (m/\text{day})\) as mentioned previously. In the case of a river entering a receiving basin with a fixed width, such as a fjord, the inventory function can be shown to have the simple exponential form (Syvitski, 1989)

\[
I(x) = I_0 e^{-bx},
\]

where \(b = \lambda / u_0\) and \(u_0\) is the flow velocity at the river mouth. A similar but more complex exponential form occurs in the case of a river entering the ocean (Syvitski, 1989; Syvitski et al., 1998). With this choice of \(R(x)\), the term \(R(x_m-x)\) in Eq. (5) can be written as the product of a function of \(x_m\) and a function of \(x\). The function of \(x_m\) can then be moved outside of the integral (and over to the right-hand side) and we can simply differentiate both sides and simplify to find \(c(x)\). This provides us with an expression for the progradation rate that is valid for any initial bathymetry, namely

\[
c(x) = \frac{-\lambda I_0}{bf_0(x) + f_0'(x)}. \tag{13}
\]

If the initial bathymetry, \(f_0(x)\), has any shape that decreases to a constant basin depth, \(z = -D\), for large \(x\), then (13) shows that \(c(x)\) will approach the limiting value

\[
c = \frac{u_0 I_0}{D}. \tag{14}
\]

Two general forms of initial bathymetries that start at \(z = 0\) and decrease to a constant basin depth of \(z = -D\) are \(f_0(x) = -D[1 - e^{-ax}]\) \((a > 0)\) and \(f_0(x) = -D[1 - (ax + 1)^p]\) \((a > 0, \ p < 0)\). Eq. (14) shows that if we increase either the flow velocity, \(u_0\), or the inventory, \(I_0\), at the river mouth then the river mouth will prograde faster. It also shows that if we increase the depth of the receiving basin, \(D\), then the river mouth will prograde at a slower rate, which is intuitive. It might seem that \(c\) should also
depend on the removal rate constant, \( \lambda \), but this is not the case for this choice of \( R(x) \). While increasing \( \lambda \) by itself increases the rate at which sediment is deposited on the seafloor, an increase in \( \lambda \) also changes the shape of the exponential curve, \( I(x) \), causing it to fall off faster. These two effects happen to exactly cancel one another. Since \( c \) has a limiting value given by (14), Eq. (12) shows that the seafloor bathymetry will tend toward a travelling wave with an exponential shape,

\[
f(x, t) = -D[1 - e^{-\lambda/\lambda_0}(x-ct-L)].
\]

(15)

In this case, the exponential form of \( R(x) \) resulted in an exponential form for the travelling wave, \( H(x) \), since the indefinite integral of an exponential curve is again exponential. Figs. 1 and 2 illustrate the cases where the initial bathymetry is given by (1) a ramp that descends to a receiving basin of uniform depth and (2) the travelling wave given by (15) with \( L = 0 \).

As an example of how the river mouth position can be computed as a function of time, let us assume that \( f_0(x) = -D[1 - e^{-ax}] \), where \( D \) is the depth of the receiving basin. Putting this form for \( f_0(x) \) in (13) and simplifying, we find that

\[
c(x) = \frac{\lambda I_0}{D[b + (a-b)e^{-ax}]}.
\]

(16)

As \( x \) becomes large, this approaches the constant value given by (14). (Recall that \( u_0 = \lambda/b \).)

However, if we take \( a = b \), then (16) immediately reduces to (14). This is the case where the initial bathymetry is given by the travelling wave solution. Inserting (16) into (6) and integrating, we obtain

\[
t(x_m) = \left( \frac{Db}{\lambda I_0} \right) \left[ x_m + \frac{(a-b)}{ab} (1 - e^{-ax_m}) \right].
\]

(17)

This gives the time for the river mouth to reach any given position, \( x_m \). Solving (17) for \( x_m \), we find that

\[
x_m(t) = \left( \frac{\lambda I_0}{b D} \right) t + \frac{c_1}{b} + \left( \frac{1}{a} \right) F_L \left[ \frac{(a-b)}{b} \right] \exp \left[ -a \left( \frac{\lambda I_0 t}{b D} + \frac{c_1}{b} \right) \right].
\]

(18)

Here, \( F_L \) denotes the Lambert W-function; \( F_L(z) \) gives the principal solution to the equation \( z = we^w \). (This function is available in Mathematica as the ProductLog function.) In order to satisfy the requirement that \( x_m(0) = 0 \), we must have

\[
c_1 = \left( \frac{b-a}{a} \right).
\]

(19)

Using the fact that \( F_L(0) = 0 \), we see from (18) that if \( t \) is sufficiently large, then \( x_m(t) \sim c t + L \), with \( c \) given by Eq. (14), and \( L = c_1/b \). (See the discussion just prior to Eq. (11).) We also see that if \( a = b \), then \( x_m(t) = c t \) for all times; this is the case where the initial bathymetry has the form of the travelling wave

\[
f_0(x) = -D[1 - e^{-ax}].
\]
wave solution. An explicit expression for $c(t)$ can be found by taking the derivative of (18) with respect to time, or by inserting (18) into (16).

4.2. Example 2: uniform sediment deposition

One can imagine a situation where sediment is either produced within the water column as the result of some process (e.g. biological) or “rains down” uniformly on the ocean surface, as could happen with ash after a volcanic eruption. It is also possible for the net deposition from two different sediment processes, such as plume sedimentation and slope failures, to produce more or less uniform net sedimentation (E. Hutton, personal communication). While this case may be less typical than our first example, it provides another good illustration of how the limiting form of the travelling wave is determined as the integral of $R(x)$. Setting $R = R_0$ in (5) and then taking the derivative of both sides with respect to $x$, we find that

$$c(x) = \frac{R_0}{f_0(x)}.$$  \hfill (20)

This shows that if the receiving basin eventually has a flat bottom, then $f_0(x)$ will approach a constant value of $-D$, $f_0'(x)$ will approach zero and $c(x)$ will tend to infinity. In fact, in the case where $f_0(x) = -D$ for all $x > x_c$, every point with $x > x_c$ will reach sea level at the same time, causing the progradation rate, $c$, to be infinite. If, however, the initial bathymetry is given by $f_0(x) = -Sx$ (a descending ramp with slope $-S$), then we will have $c = R_0/S$, a constant. The seafloor bathymetry in this case will be

$$f(x, t) = -S(x - ct),$$  \hfill (21)

which represents a linear travelling wave. See Fig. 3. In general, when $R = R_0$ we have $t(x_m) = -f_0(x_m)/R_0$, $x_m(t) = f_0^{-1}(-R_0 t)$ and

$$f(x, t) = \begin{cases} f_0(x) + R_0 t & \text{if } x \geq x_m(t), \\ 0 & \text{if } x < x_m(t). \end{cases}$$  \hfill (22)

4.3. Example 3: sediment deposition from a plume onto a ramp

As in Example 1, let us again assume that $R(x) = \lambda I_0 e^{-bx}$, where $b = \lambda/u_0$. Suppose again that the initial bathymetry does not tend to a finite depth, $-D$, but is instead a descending ramp with slope $-S$, such that $f_0(x) = -Sx$. Inserting this $f_0(x)$ into Eq. (13) gives

$$c(x) = \frac{\lambda I_0}{S} \left( \frac{1}{1 + bx} \right),$$  \hfill (23)

which shows that the progradation rate, $c$, approaches zero for large values of $x$. Note that a steeper slope gives rise to a slower progradation.
rate. Inserting (23) into (6) and integrating give

\[ t(x_m) = \left( \frac{x_m S}{2 I_0} \right) \left[ 1 + (b/2)x_m \right]. \]  

(24)

Solving this quadratic for \( x_m(t) \) we have

\[ x_m(t) = \frac{1}{b} \left[ -1 + \sqrt{1 + 2b x_m(t) I_0 t} \right]. \]  

(25)

Inserting (23) and (25) into (3) and simplifying give

\[ f(x, t) = \begin{cases} 
 f_0(x) + S \text{e}^{-b x} [x_m(t) \text{e}^{b x_m(t)}], & \text{if } x \geq x_m(t), \\
 0, & \text{if } x < x_m(t).
\end{cases} \]  

(26)

This example might be likened to a standing wave except that \( c \) never actually reaches 0. See Fig. 4. It is essentially realized in practice for smaller rivers and whenever the depth, \( D \), of the receiving basin is sufficiently large or the ramp down to the bottom of the basin is sufficiently long. In these cases it will take an exceedingly long time for the delta to build out to the basin depth and in reality, other sediment transport processes (such as reworking by large storm waves) will dominate the physics before this can occur.

4.4. Example 4: effect of waves on sediment deposition from a plume

As pointed out by Pirmez et al. (1998) and others, the action of surface waves can inhibit the accumulation of sediment near the shore, despite the large amount of sediment that would otherwise be deposited there by a river plume. The result is a modified version of the sediment deposition function for river plumes, such that \( R(x) \) attains its maximum value at some distance offshore. This is thought to be one of the main causes for the classic sigmoidal shape of many clinoforms. As a simple model of this situation, let us assume that

\[ R(x) = \frac{bcD \text{sech}^2[b(x - x_p)]}{1 + \tanh(bx_p)}. \]  

(27)

Here \( x_p \) is an offset parameter that corresponds to the offshore \( x \)-value where \( R(x) \) attains its maximum value, \( R_p = R(x_p) = bcD/[1 + \tanh(x_p)] \). Note that \( R(0) \) always lies between 0 and \( R_p \). Using (7) it can then be shown that the shape of the corresponding travelling wave is

\[ H(x) = -D \left[ \frac{\tanh[b(x - x_p)] + \tanh(bx_p)}{1 + \tanh(bx_p)} \right]. \]  

(28)

This travelling wave produces clinoforms that have the classic sigmoidal shape (see Fig. 5). It also has \( H(0) = 0 \) and descends to a basin depth of \( D \) for every choice of \( b \) and \( x_p \). The parameter \( b \) determines the steepness of the travelling wave. If \( b \) and \( R_p \) were specified by a physical process, then \( c \) could be computed as in Example 1. Unfortunately, the Laplace transform of (27) is quite complicated,
so for most choices of $f_0(x)$ it is not possible to give $c(x)$, $t(x_m)$ and $x_m(t)$ in terms of simple functions.

5. Using the Laplace transform to find $c(x)$

As mentioned previously, Eq. (5) can be solved for $c(x)$ using the Laplace transform. The Laplace transform approach immediately reproduces the results from our four previous examples, but can solve a much larger class of problems. Recall that the Laplace transform of a function $f(x)$ is given by

$$\mathcal{L}\{f(x)\}(s) = \int_{0}^{\infty} f(x) e^{-sx} dx.$$  \hfill (29)
Laplace transforms have the well-known property (e.g. Kaplan, 1981, p. 198) that the transform of a convolution, such as Eq. (5), is given as the product of the transforms of the two functions in the convolution. This property is often referred to as the convolution theorem. It follows that
\[
\mathcal{L}[R \ast \phi] = -\mathcal{L}[f_0]
\]
and
\[
\phi(x) = \frac{1}{c(x)} = -\mathcal{L}^{-1} \left[ \frac{\mathcal{L}[f_0]}{\mathcal{L}[R]} \right],
\]
where \(\mathcal{L}^{-1}\) denotes the inverse Laplace transform. An extensive table of Laplace transforms and inverses is given by Spiegel (1968, pp. 161–173). Symbolic math programs like Mathematica can also be used to rapidly find the Laplace transform and its inverse. The following Mathematica code shows how \(c(x), t(x_m)\) and \(x_m(t)\) can be found, given the functions \(f_0(x)\) and \(R(x)\):

```mathematica
Clear[f0, R, c, tm, xm, x, s]
c[f0_, R_] := Simplify[-1/
InverseLaplaceTransform[
  LaplaceTransform[f0, x, s] / 
  LaplaceTransform[R, x, s], s, x] ]
tm[f0_, R_, x2_] := Integrate[1/c[f0, R], {x, 0, x2}]
xm[f0_, R_, t_] := Simplify[Solve[t = = tm[f0, R, x], x]]
```

The results of Example 3 can then be obtained by evaluating
\[
c[ -S x, lam I0 Exp[-bx]]
tm[ -S x, lam I0 Exp[-bx], x]
xm[ -S x, lam I0 Exp[-bx], t]
\]

Another well-known property of the Laplace transform is that
\[
\mathcal{L}[f'] = s \mathcal{L}[f] - f(0).
\]
If \(f(0) = 0\), as is the case with the initial bathymetry function \(f_0(x)\), then this can be iterated to show that
\[
\mathcal{L}[f^{(n)}] = s^n \mathcal{L}[f],
\]
where \(f^{(n)}\) denotes the \(n\)th derivative of \(f\). It follows from (33) that if \(R\) has the form of a power-law, \(R(x) = ax^{n-1}\), where \(n\) is a positive integer, and \(a > 0\), then we have
\[
c(x) = -a(n - 1)! / f_0^{(n)}(x).
\]

This example is not very realistic, however, since it requires the sedimentation to increase in the seaward direction. A property of the Laplace transform similar to (33) holds for integrals as opposed to derivatives.

Fig. 6. Case shown in Fig. 2, but with a single, Gaussian bump added to initial bathymetry.
6. Perturbations by other sediment transport processes

Figs. 6 and 7 illustrate cases where the travelling wave profile from Example 1 is perturbed by one and two Gaussian-shaped deposits of sediment. These are meant to be a simple representation of the sediment deposited by a turbidite, but also capture the essence of a perturbation by other sediment transport processes. The time required for the travelling wave profile to be fully restored after such a disturbance is determined by the time it takes for the river mouth to move beyond the extent of the disturbance. In order to compute this time, we define \( t = 0 \) to be the time that the perturbation occurs, and \( f_0(x) \) to be the initial bathymetry, including the perturbation. Given \( R(x) \), we can then solve (5) for \( c(x) \), and then integrate (6) to get \( t(x_m) \).

To get the recovery time, say \( t_r \), we simply evaluate \( t(x_m) \) at \( x_m = x_p \), where \( x_p \) is the distance from the river mouth to the most seaward extent of the perturbation. If perturbing events are separated by this amount of time or more, on average, then the travelling wave profile will be almost fully restored between events.

7. Conclusions

A mathematical framework has been developed that helps to explain how various features of the seafloor stratigraphy depend on the sediment deposition function, \( R(x) \), and the initial bathymetry, \( f_0(x) \), and their associated parameters. It also explains the “travelling wave” phenomenon that has been observed in numerical models for simulating the evolution of seafloor stratigraphy, and identifies the key factors that determine the relative stability of the shelf. For many cases of interest, the solution \( f(x, t) \) can be given explicitly, which makes it possible to provide a quantitative answer to many questions (within the model framework) about how the seafloor changes with time or how the solution depends on the parameters in the problem. When closed-form solutions are known, they are equivalent to an infinite number of numerical solutions and provide much greater insight.

Given a sediment deposition function, \( R(x) \), and an initial bathymetry, \( f_0(x) \), the Laplace transform can be used to solve Eq. (5) for the progradation rate, \( c(x) \), which can then be used to determine if the seafloor will evolve to a travelling wave. In cases where it does, the limiting shape of the wave is given in terms of the indefinite integral of the sediment deposition function, \( R(x) \). The solution for \( c(x) \) can also be used to find expressions for \( t(x_m) \), \( x_m(t) \) and \( c(t) \).

The results presented here can be extended to cases where the sediment deposition function and/or sea level is allowed to vary over time and this will be pursued in future work. Note that the travelling wave phenomenon has also been observed in computer simulations where the deposition function varies in time.
Notation

\( a \) arbitrary constant, see section after (14)
\( b \) shorthand for \( \lambda/u_0 \)
\( c(t) \) progradation rate as a function of time
\( c(x) \) progradation rate as a function of distance
\( D \) depth of a receiving basin, as \( x \to \infty \)
\( f(x, t) \) the seafloor surface, a curve that changes in time
\( f_0(x) \) initial bathymetry, \( f(x, 0) \)
\( G(x) \) shorthand for the indefinite integral of \( R(x) \)
\( H(s) \) general travelling wave solution
\( I(x) \) sediment inventory (depth-integrated concentration)
\( I_0 \) value of \( I \) at river mouth, \( I(0) \)
\( L \) offset parameter, see Eq. (11)
\( p \) arbitrary constant, see section after (14)
\( R(x) \) sediment deposition function
\( R_0 \) shorthand for \( R(0) \)
\( s \) shorthand for \( x - ct \)
\( S \) constant slope of a linear ramp
\( t \) time
\( t(x_m) \) time for mouth to reach a position \( x = x_m \)
\( u_0 \) flow velocity at river mouth, from plume solution
\( x \) seaward distance from initial position of river mouth
\( x_c \) hypothetical \( x \)-value, such that \( f_0(x) = -D \)
\( x_m(t) \) river mouth position as a function of time
\( x_p \) distance from river mouth to outer edge of perturbation
\( z \) vertical distance above sea level
\( \lambda \) removal rate constant in a river plume model
\( \phi(x) \) shorthand for \( 1/c(x) \)
\( \mathcal{L}[f] \) the Laplace transform of a function, \( f \)

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References

