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The Krull Dimension of Certain Semiprime Modules Versus Their α-Shortness

S. M. Javdannezhad and N. Shirali

Abstract. We study the R-modules M which are finitely generated, quasi-projective and self-generator (briefly called FQS modules). We extend some basic results from semiprime rings to semiprime FQS modules. In particular, we show that any semiprime FQS module with Krull dimension is a Goldie module. We also show that every FQS module with Krull dimension has only finitely many minimal prime submodules. Consequently, if M is an FQS module with Krull dimension, then $k\text{-dim}\, M$ is equal to $k\text{-dim}\, \frac{M}{P}$ for some prime submodule P of M. Moreover, we observe that an FQS module has the classical Krull dimension if and only if it satisfies ACC on prime submodules. Finally, we prove that a semiprime FQS module M is α-short if and only if $n\text{-dim}\, M = \alpha$, where $\alpha \geq 0$.

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1. Introduction

In 1972, Lemonnier introduced the concept of the deviation of an arbitrary poset $(E, \leq)$ in [23], similarly to the concept of Krull dimension of modules, see also [24]. The Krull dimension of a module M, which is denoted by $k\text{-dim}\, M$ and measures its deviation from being Artinian, was first introduced by Gabriel and Rentschler (for finite ordinals) in 1967. Later, this definition was extended to infinite ordinals by Krause in 1970, see [17, 18, 21]. Lemonnier also defined the concept of the dual Krull dimension of E which he named the codeviation of E, as being the Krull dimension (i.e., the deviation) of E^0, the opposite poset of E, see [18]. We remind the reader that the dual Krull dimension of modules measures the deviation of a module from being Noetherian. We should emphasize, for the sake of the reader, that the dual Krull dimension of a module is also known as the Noetherian dimension and N-dimension of that module, see [19]. Let us denote the dual Krull

Birkhäuser

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dimension of a module M by n-dim M. These dimensions have been investigated by many authors, see for example [3,18,22–24]. On the other hand, the notion of a prime ideal plays an important role in the theory of structures of rings. There are some interesting relations between prime ideals and the Krull dimension of a ring R. In what follows, we list some examples of these relations:

(1) Any semiprime ring with Krull dimension is a Goldie ring.
(2) Any ring with Krull dimension has only finitely many minimal prime ideals.
(3) Any ring with Krull dimension satisfies ACC on prime ideals.
(4) Let R be a semiprime ring with Krull dimension. Then: $k\text{-dim}\, R = \sup\left\{k\text{-dim}\, \frac{R}{E} + 1 : E \leq_e R\right\}.$
(5) If R is a ring with Krull dimension, then k-dim R is equal to $k\text{-dim}\, \frac{R}{P}$, for some prime ideal P of R.
(6) If R is a ring with Krull dimension and P is an ideal of R, maximal with respect to $k\text{-dim}\, \frac{R}{P} = \alpha$, then P is a prime ideal.

Moreover, the classical Krull dimension of a ring R, denoted by $cl.k$-dim R, was originally defined to be the supremum of the lengths of all chains of prime ideals in R. These facts led us to our investigation of prime submodules in a class of modules with Krull dimension. In this paper, we extend the above results and some other useful facts to this class of modules. We should remind the reader that the concept of prime submodules and prime modules is introduced in the literature by various authors. Some of these definitions are slightly different and, therefore, the objects which are introduced by these definition are not necessarily the same, see for example [8,14,25,27]. We should emphasize that we use the notion of a prime submodule in the sense of [27]. Let us give a brief outline of this paper. In Sect. 2, we recall some known facts about prime and semiprime (sub)modules, Goldie modules and non-M-singular modules and give some new facts about these modules. Sanh et al. [27] have shown that if M is a finitely generated, quasi-projective and self-generator with certain properties, then $S = \mathrm{End}_R(M)$ enjoys these properties too. This led us to our definition of FQS modules. Let us recall that an R-module M is called FQS if M is finitely generated, quasi-projective and self-generator. In Sect. 3, we investigate FQS modules. We show that if M is an FQS module, then M is Goldie (critical, dual critical, etc.) if and only if $S = \mathrm{End}_R(M)$ has all the latter properties, respectively. Section 4 is devoted to FQS modules with Krull dimension. In this section, we extend some important facts from prime ideals and semiprime rings to prime submodules and semiprime FQS modules. For example, we show that any semiprime FQS module with Krull dimension is a Goldie module and any FQS module with Krull dimension has only finitely many minimal prime submodules. Consequently, if M is an FQS module with Krull dimension, then k-dim M is equal to $k\text{-dim}\, \frac{M}{P}$, for some prime submodule P of M. We also observe that an FQS module has the classical Krull dimension if and only if it satisfies ACC on prime submodules. In the last section, we study α-short modules and α-DICC modules, see [13,19]. We prove that any semiprime FQS module M is

α-short if and only if n-dim $M = \alpha$, where $\alpha \geq 0$, which is a generalization of [13, Proposition 2.18]. Finally, we show that any semiprime FQS module M is α-DICC if and only if k-dim $M = \alpha$ or n-dim $M = \alpha$.

Throughout this paper, all rings are associated with $1 \neq 0$, and all modules are unital right modules. It is convenient, when we are dealing with the above dimensions, to begin our list of ordinals with -1. Let M be an R-module and $S = \mathrm{End}_R(M)$ its endomorphism ring. We denote $AX = \sum_{f\in A} f(X)$, $I_X = \{f \in S : f(M) \subseteq X\}$ and $\mathrm{Ker}(A) = \cap_{f\in A} \ker(f)$ for any submodule $X \subseteq M$ and $A \subseteq S$. Let us recall that an R-module N is said to be (finitely) generated by M or (finitely) M-generated if there exists an epimorphism $\bigoplus_\Delta M \longrightarrow N$ for some (finite) index set Δ. Moreover, an R-module M is called self-generator if it generates all its submodules. We note that an R-module M is self-generator if and only if for each submodule N of M, there exists $\Delta \subseteq S$ such that $N = \sum_{f\in\Delta} f(M)$. A submodule N of M is called essential (or large) in M, denoted by $N \leq_e M$, if $K \cap N \neq 0$ for every non-zero submodule K of M. The full subcategory of R-Mod, subgenerated by M, is denoted by $\sigma[M]$, see [15,28]. An R-module N is called singular in $\sigma[M]$ or M-singular if $N \cong \frac{L}{K}$ for some $L \in \sigma[M]$ and $K \leq_e L$. We recall that every module $N \in \sigma[M]$ contains the largest M-singular submodule which is denoted by $Z_M(N)$. Then, N is M-singular if $Z_M(N) = N$ and if $Z_M(N) = 0$, N is called non-M-singular. The reader is referred to [7,15,18,26,27] for undefined terms and notations.

2. Preliminaries

Let us briefly recall some basic definitions and results from the literature. An R-module M is called quasi(or, self)-projective if it is M-projective. Note that if R is a semisimple ring, then every finitely generated R-module is quasi-projective, also every finite direct sum of quasi-projective modules is quasi-projective, see [20, Theorem 2.1].

Theorem 2.1. [15, 3.4] *Let M be a quasi-projective R-module and $S = \mathrm{End}_R$ (M).*

1. *For any finitely generated right ideal I of S, $I = \mathrm{Hom}_R(M, IM)$.*
2. *Assume in addition that M is finitely generated. Then, for any right ideal I of S, $I = \mathrm{Hom}_R(M, IM)$.*

Theorem 2.2. [15, 4.1] *Let M be an R-module. Then, the following statements are equivalent.*

1. *M is non-M-singular (i.e., $Z_M(M) = 0$).*
2. *For any $0 \neq N \in \sigma[M]$ and $0 \neq f : N \to M$, $\ker(f) \not\leq_e N$.*

Theorem 2.3. [15, 4.5] *Let M and N be R-modules, N be M-singular and $f \in \mathrm{Hom}_R(M, N)$. If M is quasi-projective and $f(M)$ is finitely generated, then $\ker(f) \leq_e M$.*

A submodule X of M is called fully invariant if $f(X) \subseteq X$ for any $f \in S = \mathrm{End}_R(M)$. According to [27, Definition 1.1], a fully invariant proper submodule P of an R-module M is called prime if for any ideal I of S, and any fully invariant submodule X of M, $IX \subseteq P$ implies $X \subseteq P$ or $IM \subseteq P$. An R-module M is called prime if 0 is a prime submodule of M. Any maximal of the set of all fully invariant submodules of M is prime, see [26, Proposition 1.6].

Theorem 2.4. [27, 1.2] *Let P be a proper fully invariant submodule of M. Then, the following conditions are equivalent.*

(1) *P is a prime submodule.*
(2) *For any $\phi \in S = \mathrm{End}_R(M)$ and fully invariant submodule U of M, if $\phi(U) \subseteq P$, then either $\phi(M) \subseteq P$ or $U \subseteq P$.*

Moreover, if M is quasi-projective, then the above conditions are equivalent to:

(3) *$\frac{M}{P}$ is a prime module.*

The next result is a consequence of [27, Lemmas 2.5, 2.6]. We give a short proof for completeness. It is easy to see that if A is a fully invariant submodule of a module M and $\frac{P}{A}$ is a fully invariant submodule of $\frac{M}{A}$, then P is a fully invariant submodule of M.

Theorem 2.5. *Let X be a fully invariant submodule of a quasi-projective R-module M. Then, $\frac{P}{X}$ is a prime submodule of $\frac{M}{X}$ if and only if P is a prime submodule of M contains X.*

Proof. By the assumption, we immediately conclude that $\frac{M}{X}$ is quasi-projective. By Theorem 2.4(3), $\frac{P}{X}$ is a prime submodule of $\frac{M}{X}$ if and only if $\frac{M/X}{P/X}$ is a prime module if and only if $\frac{M}{P}$ is a prime module, equivalently P is a prime submodule of M. □

According to [27, Definition 2.1], a fully invariant submodule X of an R-module M is called semiprime if it is an intersection of prime submodules of M. An R-module M is called semiprime if 0 is a semiprime submodule of M. Also, a submodule $U \leq M$ is called M-annihilator if $U = \mathrm{Ker}(A) = \cap_{f\in A}\ker(f)$ for some $A \subseteq S = \mathrm{End}_R(M)$.

Remark 2.6. Let M be an R-module and $S = \mathrm{End}_R(M)$. It is easy to see that $\mathrm{Ker}(A) = \{x \in M : f(x) = 0, \forall f \in A\} = r.\mathrm{ann}_M(A)$ for each $A \subseteq S$. If there is no confusion, we briefly write $r_M(A)$ or $r(A)$ instead of $r.\mathrm{ann}_M(A)$. Note that U is an M-annihilator submodule of M if $U = r(A)$ for some subset A of S. It is well known that $r(A) = rlr(A)$ and so if U is an M-annihilator submodule of M, then $U = \ker\{f \in S : U \subseteq \ker(f)\}$.

Theorem 2.7. *Let M be an R-module and $S = \mathrm{End}_R(M)$.*

(1) *If M satisfies ACC on M-annihilator submodules, then S satisfies ACC on right annihilator ideals.*
(2) *If M is self-generator, then the converse of* (1) *holds.*

Proof. (1) It is proved in [26, Lemma 3.2].
(2) Suppose that M is self-generator and S satisfies ACC on right annihilator ideals. If $U_1 \subsetneq U_2 \subsetneq U_3 \subsetneq \cdots$ is an ascending chain of M-annihilator

submodules of M, then $M_i = \mathrm{Ker}(S_i)$ where $S_i = \{f \in S : U_i \subseteq \ker(f)\}$ by Remark 2.6. This gives $S_1 \supsetneq S_2 \supsetneq S_3 \supsetneq \cdots$ (note, if $f \in S_{n+1}$, then $U_{n+1} \subseteq \mathrm{Ker}(f)$ so that $U_n \subseteq \mathrm{Ker}(f)$ and $f \in S_n$). This implies that $r.\mathrm{ann}_S(S_1) \subseteq r.\mathrm{ann}_S(S_2) \subseteq r.\mathrm{ann}_S(S_3) \subseteq \cdots$. Since $S_n \supsetneq S_{n+1}$, then there exists $g \in S_n - S_{n+1}$ and we conclude that $g(U_n) = 0 \neq g(U_{n+1})$. Note that M is self-generator and hence $U_{n+1} = \sum_{f\in\Delta} f(M)$ for some $\Delta \subseteq S$, thus there exists $f \in \Delta$ such that $gf(M) \neq 0$. Hence, $f \notin r.\mathrm{ann}_S(g)$ and $f \notin r.\mathrm{ann}_S(S_n)$. Now, let $\varphi \in S_{n+1}$. Then, $\varphi f(M) = \varphi(f(M)) \subseteq \varphi(U_{n+1}) = 0$ and $\varphi f = 0$. This implies that $f \in r.\mathrm{ann}_S(S_{n+1})$ and we have $r.\mathrm{ann}_S(S_1) \subsetneq r.\mathrm{ann}_S(S_2) \subsetneq r.annS(S_3) \subsetneq \cdots$, a contradiction. □

3. FQS modules

We begin with a formal definition of the central concept of the article.

Definition 3.1. An R-module M is called FQS if it is finitely generated, quasi-projective and self-generator.

Lemma 3.2. *Let M be an* FQS *module. Then, $\frac{M}{A}$ is an* FQS *module for each fully invariant submodule A of M.*

Proof. By our assumption, it is clear that $\frac{M}{A}$ is finitely generated and quasi-projective; hence, it suffices to prove that $\frac{M}{A}$ is self-generator. Let $\frac{X}{A}$ be a submodule of $\frac{M}{A}$. Hence, we conclude that $X = \sum_{f\in\Delta} f(M)$ for some $\Delta \subseteq S = \mathrm{End}_R(M)$. Now for each f, we define $\bar{f} : \frac{M}{A} \longrightarrow \frac{M}{A}$ by $\bar{f}(m+A) = f(m)+A$. Since A is fully invariant, $\bar{f}$ is well defined and so $\bar{f}$ is an R-homomorphism. Hence, we have $\sum_{f\in\Delta} \bar{f}(\frac{M}{A}) = \sum_{f\in\Delta}(\frac{A+f(M)}{A}) = \frac{A+\sum_{f\in\Delta} f(M)}{A} = \frac{A+X}{A} = \frac{X}{A}$ and, consequently, $\frac{M}{A}$ is self-generator. □

Example 3.3. (1) Every finitely generated semisimple module is an FQS module. Also, if R is a semisimple ring, then every finitely generated R-module is an FQS module.
(2) For every prime number p and every $n \in \mathbb{N}$, the additive group $\mathbb{Z}_{p^n}$ is an FQS module.

One can easily prove the following result.

Lemma 3.4. *Let M be an R-module and $S = \mathrm{End}_R(M)$. Then, for any submodule X of M and each right ideal A of S, we have the following statements.*

(1) *I_X is a right ideal of S and if X is a fully invariant submodule of M, then I_X is an ideal of S. Moreover, $I_X M \subseteq X$ and in case M is self-generator, $I_X M = X$.*
(2) *AM is a submodule of M and if A is an ideal of S, then AM is a fully invariant submodule of M. Moreover, $A \subseteq I_{AM}$ and in case M is a finitely generated and quasi-projective module, $I_{AM} = A$.*

Theorem 3.5. *Let M be an* FQS *module and $S = \mathrm{End}_R(M)$. Then, for any submodule X of M, the following statements are hold.*

(1) *X is a maximal (minimal) submodule of M if and only if I_X is a maximal (minimal) right ideal of S.*
(2) *X is an essential submodule of M if and only if I_X is an essential right ideal of S.*

Proof. (1) It is a consequence of Lemma 3.4 (note., $X \subsetneq Y$ if and only if $I_X \subsetneq I_Y$ where X and Y are two submodules of M).
(2) The sufficiency comes from [26, Lemma 3.6]. Conversely, assume that $I_X \leq_e S_S$ and $X \cap N = 0$, for some submodule N of M. By Theorem 2.1 and Lemma 3.4, we have $0 = \mathrm{Hom}(M, X \cap N) = \mathrm{Hom}(M, X) \cap \mathrm{Hom}(M, N) = \mathrm{Hom}(M, I_X M) \cap \mathrm{Hom}(M, N) = I_X \cap \mathrm{Hom}(M, N)$. Hence, we conclude that $\mathrm{Hom}(M, N) = 0$. This implies that $N = 0$ and hence $X \leq_e M$. □

Remark 3.6. For each $f \in S$, $r.\mathrm{ann}_S(f) = I_{\ker(f)}$.

The next result is also in [16, Proposition 4.4] by different statement.

Theorem 3.7. *Let M be an FQS module and $S = \mathrm{End}_R(M)$. Then*

$$Z(S) = \{f \in S : \ker(f) \leq_e M\}.$$

Proof. As noted above, for each $f \in S$, $r.\mathrm{ann}_S(f) = I_{\ker(f)}$ so $f \in Z(S_S)$ if and only if $r.\mathrm{ann}_S(f) \leq_e S_S$ if and only if $I_{\ker(f)} \leq_e S_S$ and this is equivalent to $\ker(f) \leq_e M_R$ by Theorem 3.5(2). □

Theorem 3.8. *Let M be an FQS module and $S = \mathrm{End}_R(M)$. Then, M is a non-M-singular module if and only if S is a nonsingular ring.*

Proof. First, suppose that $0 \neq f \in Z(S)$. Thus, $\ker(f) \leq_e M$ and we conclude that $Z_M(M) \neq 0$ by Theorem 2.2. Conversely, suppose that $Z_M(M) \neq 0$. Since M is self-generator, there exists $0 \neq f : M \to M$ such that $f(M) \subseteq Z_M(M)$. This implies $f(M)$ is M-singular. Since M is an FQS module, M is quasi-projective, also M and so $f(M)$ is finitely generated. Thus, we have $\ker(f) \leq_e M$ by Theorem 2.3. This implies that $f \in Z(S)$ by Theorem 2.2 and so $Z(S) \neq 0$. □

Lemma 3.9. *Let X be a fully invariant submodule of M and A be an ideal of $S = \mathrm{End}_R(M)$. Then, we have the following.*
(1) *If X is a prime submodule of M, then I_X is a prime ideal of S. Moreover, if M is self-generator, then the converse is true.*
(2) *If M is an FQS module and A is a prime ideal of S, then AM is a prime submodule of M.*

Proof. (1) It is proved in [26, Theorem 1.10].
(2) Since M is finitely generated and quasi-projective, we have $A = I_{AM}$. Moreover, M is self-generator and $A = I_{AM}$ is a prime ideal of S. Hence, according to part (1), AM is a prime submodule of M. □

The next result is a consequence of Lemma 3.9 and Theorem 3.5.

Corollary 3.10. *Let M be an FQS module, $S = \mathrm{End}_R(M)$ and P be a fully invariant submodule of M. Then, P is a minimal prime submodule of M if and only if I_P is a minimal prime ideal of S.*

Let us recall that any nonzero ideal of each prime ring R is an essential right ideal. The following is a counterpart of this fact.

Lemma 3.11. *Let M be a prime* FQS *module. Then, every nonzero fully invariant submodule of M is an essential R-submodule.*

Proof. Let $0 \neq X$ be a fully invariant submodule of M. Suppose that Y is a submodule of M such that $X \cap Y = 0$. Note that M is self-generator and hence we have $Y = \sum_{g \in \Delta} g(M)$ for some $\Delta \subseteq S = \mathrm{End}_R(M)$. On the other hand, X is fully invariant so we have $g(X) \subseteq X \cap Y = 0$ for each $g \in \Delta$. Since M is prime, we conclude that $g(M) = 0$ and $Y = 0$. □

It is well known that every prime ideal is either a minimal prime ideal or an essential right ideal. The following is a counterpart of this fact.

Theorem 3.12. *Let M be an* FQS *module. Then, every prime submodule P of M is a minimal prime or an essential R-submodule.*

Proof. Since P is a prime submodule, I_P is a prime ideal of S by Lemma 3.9. If P is not essential, then I_P is not an essential right ideal of S by Theorem 3.5. Hence, I_P is a minimal prime ideal of S. Thus, P is a minimal prime submodule of M by Corollary 3.10. □

Theorem 3.13. *Let M be an FQS module and $S = \mathrm{End}_R(M)$. Then, M is semiprime if and only if S is a semiprime ring.*

Proof. It is clear by Theorem 2.9 in [26] and Proposition 2.3 in [27]. □

4. FQS Modules with Krull Dimension

We begin this section with the following result which is a consequence of Theorems 1.1 and 1.3 and related remarks in [16]. We recall that if M and N are right R-modules and $S = \mathrm{End}_R(M)$, then $\mathrm{Hom}_R(M, N)$ is a right S-module.

Theorem 4.1. *Let M be a finitely generated quasi-projective R-module, $S = \mathrm{End}_R(M)$ and N be an R-module. Then, the lattice of M-generated submodules of N and the lattice of S-submodules of $\mathrm{Hom}_R(M, N)$ are isomorphic.*

The next useful theorem is a natural consequence of Theorem 4.1 and the routine proofs for these facts will be left to the reader.

Theorem 4.2. *If M is an* FQS *module and $S = \mathrm{End}_R(M)$, Then:*

(1) *$\mathcal{L}(M)$, the lattice of R-submodules of M and $\mathcal{L}(S)$, the lattice of right ideals of S are isomorphic. More generally,*
(2) *$\mathcal{L}(X)$, the lattice of R-submodules of X and $\mathcal{L}(I_X)$, the lattice of right ideals of S which is contained in I_X, are isomorphic for every R-submodule X of M.*
(3) *$\mathcal{L}(\frac{M}{X})$, the lattice of R-submodules of $\frac{M}{X}$ and $\mathcal{L}(\mathrm{Hom}_R(M, \frac{M}{X}))$, the lattice of S-submodules of $\mathrm{Hom}_R(M, \frac{M}{X})$ are isomorphic, for every fully invariant submodule X of M.*

The above theorem gives us a good motivation to investigate and compare results concerning lattice theory between $\mathcal{L}(M)$, the lattice of submodules of M and $\mathcal{L}(S)$, the lattice of right ideals of S. In the next theorem, we give some of these facts. We should remind the reader that the first item is proved in [26, Theorem 3.1].

Theorem 4.3. *Let M be an* FQS *module, $S = \mathrm{End}_R(M)$ and X be a submodule of M. Then, we have*

1. $G - \dim M_R = G - \dim S_S$.
2. $k\text{-}\dim M_R = k\text{-}\dim S_S$, *if either side exists.*
3. $k\text{-}\dim X = k\text{-}\dim I_X$.
4. $n\text{-}\dim M_R = n\text{-}\dim S_S$, *if either side exists.*
5. $n\text{-}\dim X = n\text{-}\dim I_X$.
6. $k\text{-}\dim \frac{M}{X} = k\text{-}\dim \frac{S}{I_X}$, *if X is fully invariant.*
7. $n\text{-}\dim \frac{M}{X} = n\text{-}\dim \frac{S}{I_X}$, *if X is fully invariant.*

Proof. By applying Theorem 4.2, we only need to show the validity of the facts in parts (6) and (7). To see (6), $k\text{-}\dim \frac{M}{X} = k\text{-}\dim \mathrm{Hom}_R(M, \frac{M}{X})$ by Theorem 4.2(3). Now, it is suffices to show that $\mathrm{Hom}_R(M, \frac{M}{X}) \cong \frac{S}{I_X}$. We define $\varphi\colon S \longrightarrow \mathrm{Hom}_R(M, \frac{M}{X})$, by $\varphi(f) = \pi f$, where π is the canonical projection. It is clear

$$\ker(\varphi) = \{f \in S : \pi f = 0\} = \{f \in S : f(M) \subseteq X\} = I_X.$$

Thus, $\frac{S}{I_X}$ can be embedded in $\mathrm{Hom}_R(M, \frac{M}{X})$. Moreover, M is quasi-projective so for each $g \in \mathrm{Hom}(M, \frac{M}{X})$, there exists $f \in S$ such that $g = \pi f$. This means that φ is an epimorphism. Now by the first isomorphism theorem, $\mathrm{Hom}_R(M, \frac{M}{X}) \cong \frac{S}{I_X}$ and we are done. The proof of (7) is similar. □

The following result is now immediate.

Corollary 4.4. *Let M be an* FQS *module and $S = \mathrm{End}_R(M)$. Then, M is a uniform module if and only if S is a right uniform ring.*

We recall that an R-module M is called Goldie if it has finite Goldie dimension and satisfies ACC on M-annihilator submodules, see [26].

Theorem 4.5. *Let M be an* FQS *module and $S = \mathrm{End}_R(M)$. Then, M is a Goldie module if and only if S is a right Goldie ring.*

Proof. The necessity is proved in [26, Theorem 3.3]. To prove the sufficiency, we assume that S is a Goldie ring. Hence, $G - \dim S_S$ is finite and S satisfies ACC on right annihilator ideals. Now, we conclude that $G - \dim M_R$ is finite and M satisfies ACC on M-annihilator submodules by Theorems 4.3(1) and 2.7. This means that M is a Goldie module. □

It is well known that a semiprime ring R is right Goldie if and only if $Z_r(R) = 0$ and $G - \dim R_R < \infty$, see [24, Theorem 2.3.6]. The following is a generalization of this fact to FQS modules.

Theorem 4.6. *Let M be an* FQS *module. Then, the following statement is equivalent.*

(1) *M is a semiprime Goldie module.*
(2) *S is a semiprime right Goldie ring.*
(3) *S is semiprime, $G-\dim S_S < \infty$ and $Z_r(S) = 0$.*
(4) *M is semiprime, $G-\dim M < \infty$ and $Z_M(M) = 0$.*

Proof. (1) $\Leftrightarrow$ (2) This is a consequence of Theorems 3.13 and 4.5.
(2) $\Leftrightarrow$ (3) It is proved in [24, Theorem 2.3.6].
(3) $\Leftrightarrow$ (4) This is a consequence of Theorems 3.13, 4.3(1) and 3.8. $\square$

We recall that if R is a right nonsingular ring and R_R has finite rank and $x \in R$ then x is regular if and only if $r.\mathrm{ann}_R(x) = 0$, see [17, Lemma 5.7].

Corollary 4.7. *Let M be a semiprime Goldie* FQS *module and $f \in S = \mathrm{End}_R(M)$. Then, f is regular if and only if f is monomorphism.*

Proof. $f \in S$ is regular if and only if $r.\mathrm{ann}_S(f) = I_{\ker(f)} = 0$. This is equivalent to $\ker(f) = 0$, that is f is a monomorphism. $\square$

The following theorem is a generalization of a result first proved by Lemonnier for semiprime rings, see [18, Corollary 3.4].

Theorem 4.8. *Any semiprime* FQS *module with Krull dimension is a Goldie module.*

Proof. By Theorems 3.13 and 4.3(2), we conclude that S is a semiprime ring with Krull dimension and, hence, it is a right Goldie ring. Thus, M is a Goldie module by Theorem 4.5. $\square$

We recall that if α is an ordinal number, an R-module M is called α-critical if $k\text{-dim}\, M = \alpha$ and $k\text{-dim}\, \frac{M}{N} < \alpha$ for every non-zero submodule N of M. A critical module is one which is α-critical for some α. For example, a module if 0-critical if and only if it is simple. Also, an R-module M is called α-dual critical if $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$ for every proper submodule N of M. A dual critical is one which is α-dual critical for some α (note, dual critical modules are also known as conotable, N-critical, and atomic).

Theorem 4.9. *Let M be an* FQS *module, $S = \mathrm{End}_R(M)$ and X be a submodule of M. Then:*
(1) *M is α-critical if and only if S is an α-critical ring.*
(2) *M is α-dual critical if and only if S is an α-dual critical ring.*

Proof. (1) If M is α-critical then $k\text{-dim}\, M = \alpha$ and so $k\text{-dim}\, S = \alpha$ by Theorem 4.3(2). Now we assume that $0 \neq A$ is a right ideal of S, then AM is a nonzero submodule of M and so $k\text{-dim}\, \frac{M}{AM} < \alpha$. This implies that $k\text{-dim}\, \frac{S}{I_{AM}} < \alpha$ by Theorem 4.3(6). But $A = I_{AM}$ by Lemma 3.4 and so $k\text{-dim}\, \frac{S}{A} < \alpha$. Hence, we conclude that S is α-critical. Conversely, suppose that S is α-critical then $k\text{-dim}\, S = \alpha$ and so $k\text{-dim}\, M = \alpha$ by Theorem 4.3(2). If $0 \neq X$ is a submodule of M, then I_X is a nonzero right ideal of S and so $k\text{-dim}\, \frac{S}{I_X} < \alpha$. Again, Theorem 4.3(6) implies that $k\text{-dim}\, \frac{M}{X} < \alpha$ and so M is an α-critical module.
(2) If M is α-dual critical, then $n\text{-dim}\, M = \alpha$ and so $n\text{-dim}\, S = \alpha$ by Theorem 4.3(4). We suppose that A be a proper right ideal of S, then AM is a

proper submodule of M and so n-dim $AM < \alpha$. Therefore, n-dim $I_{AM} < \alpha$ by Theorem 4.3(5). Since $I_{AM} = A$ we have n-dim $A < \alpha$ and hence S is α-dual critical too. Conversely, if S is an α-dual critical ring, then n-dim $S = \alpha$ so n-dim $M = \alpha$ by Theorem 4.3(4). If X is a proper submodule of M, then I_X is a proper right ideal of S, thus n-dim $I_X < \alpha$ and so k-dim $X < \alpha$ by Theorem 4.3(5). This means that M is an α-dual critical module. □

Next, we study prime submodules of FQS modules. We start with the following result which is a generalization of [18, Proposition 7.3].

Theorem 4.10. *Let M be an* FQS *module with Krull dimension. Then, M has only finitely many minimal prime submodules.*

Proof. The ring $S = \mathrm{End}_R(M)$ has Krull dimension by Theorem 4.3(2). Hence, we conclude that S has only finitely many minimal prime ideals. Suppose that $P_1, P_2, \ldots, P_n$ are all minimal prime ideals of S. By Corollary 3.10, $P_1M,\ P_2M, \ldots, P_nM$ are minimal prime submodules of M. Now suppose that Q is a minimal prime submodule of M, then I_Q is a minimal prime ideal of S by Corollary 3.10 and so $I_Q = P_i$ for some i. This implies that $Q = I_QM = P_iM$, and hence $\{P_iM : 1 \le i \le n\}$ is the set of all minimal prime submodules of M. □

Corollary 4.11. *Let M be an* FQS *module with Krull dimension and X be a fully invariant submodule of M. Then, the set of all minimal prime submodules of M containing X is finite.*

Proof. It is clear that $\frac{M}{X}$ is an FQS module by Lemma 3.2. Thus, $\frac{M}{X}$ has only finitely many minimal prime ideals. We know that there is one-to-one correspondence between the set of all minimal prime submodules of $\frac{M}{X}$ and the set of all minimal prime submodules of M containing X. Therefore, the set of all minimal prime submodules of M containing X is finite. □

It is well known that if R is a ring with Krull dimension, then k-dim $R =$ k-dim $\frac{R}{P}$ for some prime ideals of R, see [18, Corollary 7.5]. We now generalize this fact for the class of FQS modules with Krull dimension.

Theorem 4.12. *Let M be an* FQS *module with Krull dimension. Then, k-dim $M =$ k-dim $\frac{M}{P}$ for some prime submodule P of M.*

Proof. The ring $S = \mathrm{End}_R(M)$ has Krull dimension by Theorem 4.3(2), and so k-dim $S = k$-dim $\frac{S}{Q}$ for some prime ideal Q of S. In this case, $P = QM$ is a prime submodule of M and $I_P = Q$, again by Theorem 4.3(2) we conclude that

$$k\text{-dim}\, M = k\text{-dim}\, S = k\text{-dim}\, \frac{S}{Q} = k\text{-dim}\, \frac{S}{I_P} = k\text{-dim}\, \frac{M}{P}.$$

□

Lemma 4.13. [17, 13.6] *Let M be a right R-module with Krull dimension and $f \in S = \mathrm{End}_R(M)$ be a monomorphism, then k-dim $\frac{M}{f(M)} < k$-dim M.*

The following result is a generalization of [24, Proposition 3.10].

Theorem 4.14. *Let M be a semiprime* FQS *module with Krull dimension. Then,* $k\text{-dim}\, M = \sup\{k\text{-dim}\, \frac{M}{E} + 1 : E \leq_e M\}$.

Proof. Consider $\alpha = \sup\{k\text{-dim}\, \frac{M}{E} + 1 : E \leq_e M\}$. Hence, we have $k\text{-dim}\, M \leq \alpha$ by [18, Proposition 1.5]. Conversely, suppose that $E \leq_e M$, then we conclude that I_E is an essential right ideal of S. Since S is a semiprime Goldie ring, there exists a regular element $f \in I_E$ that is a monomorphism by Corollary 4.7. Hence, $k\text{-dim}\, \frac{M}{f(M)} < k\text{-dim}\, M$ by Lemma 4.13. Note that $f(M) \subseteq E$ and so $k\text{-dim}\, \frac{M}{E} \leq k\text{-dim}\, \frac{M}{f(M)} < k\text{-dim}\, M$. This implies that $\alpha \leq k\text{-dim}\, M$ and hence $\alpha = k\text{-dim}\, M$. □

Corollary 4.15. *Let M be a semiprime* FQS *module with Krull dimension and $E \leq_e M$, then $k\text{-dim}\, M = k\text{-dim}\, E$.*

The next result is a generalization of [18, Theorem 7.1] to FQS modules.

Theorem 4.16. *Every* FQS *module with Krull dimension satisfies the ACC on prime submodules.*

Proof. Let M be an FQS module and $P_1 \subsetneq P_2$ be prime submodules of M. It is clear $0 \neq \frac{P_2}{P_1} < \frac{M}{P_1}$. Hence, $\frac{M}{P_1}$ is a prime FQS module by Lemma 3.2 and Theorem 2.4. Also, we conclude that $\frac{P_2}{P_1} \leq_e \frac{M}{P_1}$ and $k\text{-dim}\, \frac{M/P_1}{P_2/P_1} < k\text{-dim}\, \frac{M}{P_1}$ by Lemma 3.11 and Theorem 4.14. Thus, any strictly ascending chain $P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \cdots$ of prime submodules of M gives a strictly descending sequence $k\text{-dim}\, \frac{M}{P_1} > k\text{-dim}\, \frac{M}{P_2} > k\text{-dim}\, \frac{M}{P_3} > \cdots$ of ordinal numbers, a contradiction. □

Theorem 4.17. *Let M be an* FQS *module with Krull dimension and P be a fully invariant submodule of M, maximal with respect to the condition that $k\text{-dim}\, \frac{M}{P} = \alpha$. Then, P is a prime submodule.*

Proof. By Lemma 3.9, it suffices to prove that I_P is a prime ideal of $S = \text{End}_R(M)$. We have $k\text{-dim}\, \frac{S}{I_P} = \alpha$ by Theorem 4.3(6). Now let Q be an ideal of S such that $I_P \subsetneq Q$. This implies that $I_P M = P \subsetneq QM$ and so $k\text{-dim}\, \frac{M}{QM} < \alpha$ by our assumption. Clearly, $Q = I_{QM}$ and $k\text{-dim}\, \frac{S}{Q} = k\text{-dim}\, \frac{M}{QM} < \alpha$ by Theorem 4.3(6). Hence, we conclude that I_P is maximal with respect to $k\text{-dim}\, \frac{S}{I_P} = \alpha$. Consequently, I_p is a prime ideal and we are done. □

Next, we introduce an analogue of the classical Krull dimension for modules.

Definition 4.18. Let M be an R-module. We denote the set of all prime submodules of M by $\text{Spec}(M)$. Let $X(M) = \text{Spec}(M)$ and $X_0(M)$ denote the set of all maximal fully invariant submodules of M. For an ordinal number $\alpha > 0$, $X_\alpha(M)$ denote the set of all prime submodules P of M such that each prime submodule Q properly containing P belongs to X_β for some $\beta < \alpha$. Hence, we have $X_0(M) \subseteq X_1(M) \subseteq X_2(M) \subseteq \cdots$. The smallest ordinal α for which $X_\alpha(M) = X(M)$ is called the classical Krull dimension of M and is denoted by $cl.k\text{-dim}\, M$.

If M is an R-module and $S = \mathrm{End}_R(M)$ then in view of Lemma 3.9, one can easily see that $X(S) = \{I_P : P \in X(M)\}$ and $X(M) = \{AM : A \in X(S)\}$. The following is an important and useful fact.

Lemma 4.19. *Let M be an* FQS *module and $S = \mathrm{End}_R(M)$. Then, for each ordinal number $\alpha \geq 0$:*

(1) $X_\alpha(S) = \{I_P : P \in X_\alpha(M)\}$.
(2) $X_\alpha(M) = \{AM : A \in X_\alpha(S)\}$.

Proof. (1) We proceed by transfinite induction on α. First, let $A \in X_0(S)$, i.e., A is a maximal ideal of S, then $P = AM$ is a fully invariant submodule of M and $A = I_P$ by Lemma 3.4. Now if Q is a fully invariant submodule of M properly containing P then I_Q is an ideal of S properly containing A. This implies that $I_Q = S$ and so $Q = M$. Thus, P is a maximal fully invariant submodule of M and $P \in X_0(M)$. Conversely, suppose that $P \in X_0(M)$ then I_P is an ideal of S by Lemma 3.4. If A is an ideal of S properly containing I_P then AM is a fully invariant submodule of M properly containing P, so $AM = M$ and hence $A = S$ by Lemma 3.4. Now, we conclude that I_P is a maximal ideal of S. i.e., $I_P \in X_0(S)$. Let us assume it is true for ordinals less than α. We prove that it is true for α. If $A \in X_\alpha(S)$, then $P = AM$ is a prime submodule of M and $A = I_P$ by Lemmas 3.4 and 3.9. If Q is a prime submodule of M properly containing P, then I_Q is a prime ideal of S properly containing A and so $I_Q \in X_\beta(S)$ for some ordinal $\beta < \alpha$. By induction hypothesis, $Q \in X_\beta(M)$ and we have $P \in X_\alpha(M)$. Conversely, if $P \in X_\alpha(M)$ then I_P is a prime ideal of S by Lemma 3.9. If A is a prime ideal of S properly containing I_P then AM is a prime submodule of M properly containing P and hence $AM \in X_\beta(M)$ for some $\beta < \alpha$. Thus, $I_{AM} = A \in X_\beta(S)$ by induction. Hence, we conclude that $I_P \in X_\alpha(S)$ and we are done.
The proof of (2) is similar to the proof of (1) and, hence, it is omitted. □

The following result is now immediate.

Corollary 4.20. *Let M be an* FQS *module, $S = \mathrm{End}_R(M)$ and α be an ordinal number. Then, $X_\alpha(M) = X(M)$ if and only if $X_\alpha(S) = X(S)$.*

Proof. First, suppose that $X_\alpha(M) = X(M)$. If $A \in X(S)$ then $P = AM$ is a prime submodule of M and so $P \in X_\alpha(M)$. This implies that $I_P = A \in X_\alpha(S)$ by Lemma 4.19. This shows that $X_\alpha(S) = X(S)$. The converse is similar. □

In view of the previous corollary, we have the following result.

Theorem 4.21. *Let M be an* FQS *module and $S = \mathrm{End}_R(M)$. Then, M has classical Krull dimension if and only if S has classical Krull dimension and in this case $cl.k$-dim $M = cl.k$-dim S.*

It is well known that a ring R has the classical Krull dimension if and only if it satisfies the ACC on prime ideals, see [1, Proposition 1.4]. Thus, an FQS module M has classical Krull dimension if and only if S satisfies ACC on prime ideals, equivalently, M satisfies the ACC on prime submodules. Hence, we have the following result which is a counterpart of [1, Proposition 1.4].

Theorem 4.22. *Let M be an* FQS *module. Then, M has classical Krull dimension if and only if M satisfies the ACC on prime submodules.*

We recall that if R is a ring with Krull dimension, then R has the classical Krull dimension too and $cl.k\text{-}\dim R \leq k\text{-}\dim R$, see [18, Proposition 7.9]. We conclude this section with the next theorem, which is a counterpart of the above fact.

Theorem 4.23. *Let M be an* FQS *module with Krull dimension. Then, it also has classical Krull dimension and $cl.k\text{-}\dim M \leq k\text{-}\dim M$.*

Proof. The proof is clear by Theorems 4.3, 4.21 and [18, Proposition 7.9]. □

5. α-Short and α-DICC FQS Modules

An R-module M is called α-short if for each submodule N of M, either $n\text{-}\dim N \leq \alpha$ or $n\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property, see [13]. Clearly, each 0-short module is just a short module in the sense of [9]. Also, every $(\alpha+1)$-dual critical module is an α-short module. We note that if M is an α-short module, then every submodule and every factor module of M is β-short for some ordinal $\beta \leq \alpha$. In [13], the authors have shown that if M is an α-short module then M has Noetherian dimension and $n\text{-}\dim M = \alpha$ or $n\text{-}\dim M = \alpha + 1$. Moreover, they proved that a semiprime ring R is α-short if and only if $n\text{-}\dim R = \alpha$, see [13, Proposition 2.18]. O.A.S. Karamzadeh has informed us that the latter proposition is not valid for $\alpha = -1$ and the statement of the proposition may simply be corrected as follows: A ring R is α-short if and only if it is a division ring, where $\alpha = -1$ and when R is a semiprime non-division ring, it is α-short if and only if $n\text{-}\dim R = \alpha$, where $\alpha \geq 0$.
Clearly, the proof of the first part follows from the definition and that of the second part is exactly the proof of [13, Proposition 2.18], without even changing a single word. Now the following facts seem to be interesting. Before stating the next result, the reader is reminded that a module is simple if and only if it is -1-short, hence as observed above, a ring is a division ring if and only if it is -1-short, see also [13, Remark 1.15]. Considering the above comment of Karamzadeh, the following result is a natural connection between the α-shortness of an FQS module M and that of $S = \mathrm{End}_R(M)$ as a ring.

Theorem 5.1. *Let M be an* FQS *module and $S = \mathrm{End}_R(M)$, then M is an α-short module if and only if S is an α-short ring.*

Proof. Let us first get rid of the case when $\alpha = -1$. If M is -1-short then M is simple; hence, S is a division ring, by Schur's Lemma, which implies that it is -1-short, too. Conversely, if S is -1-short then it is a division ring. Now we may invoke part (1) of Theorem 4.2 to infer that M is simple and hence it is -1-short, too. For $\alpha \geq 0$, one can invoke Theorem 4.3, to complete the proof. □

The following result partially settles one of the questions which are raised in the comment preceding [13, Proposition 2.18].

Theorem 5.2. *Let M be a semiprime* FQS *module. Then, M is α-short if and only if $n\text{-}\dim M = \alpha$, where $\alpha \geq 0$.*

Proof. By Theorems 3.13 and 5.1, we conclude that M is a semiprime α-short module if and only if S is a semiprime α-short ring if and only if $n\text{-}\dim S = \alpha$, by the above correction of the statement of [13, Proposition 2.18] and this is also equivalent to $n\text{-}\dim M = \alpha$, by Theorem 4.3. □

The Double Infinite Chain Condition (DICC for short) was introduced by Contessa [10–12]. As a generalization of the concept of a DICC module, the notion of an α-DICC module was introduced by Karamzadeh and Motamedi [19]. An R-module M is called α-DICC if for any double infinite chain of submodules

$$\cdots \subseteq M_{-2} \subseteq M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

there exists an integer k such that $k\text{-}\dim \frac{M_{i+1}}{M_i} < \alpha$ for all $i \leq k$ or $n\text{-}\dim \frac{M_{i+1}}{M_i} < \alpha$ for all $i \geq k$ and α is the least ordinal with respect to this property. Clearly, a 0-DICC module is just a DICC module in the sense of Contessa.

The following fact is an immediate consequence of Theorem 4.3 and its easy proof is left to the reader.

Theorem 5.3. *Let M be an* FQS *module, $S = \mathrm{End}_R(M)$ and α be an ordinal number. M_R is an α-DICC module if and only if S is an α-DICC ring.*

Finally, let us recall that a semiprime ring R is α-DICC if and only if either $k\text{-}\dim R = \alpha$ or $n\text{-}\dim R = \alpha$, see [19, Corollary 1.1] and [19, Theorem 1.1]. This fact naturally raises the question of whether this is also true for a semiprime FQS modules. We conclude this paper with the next observation which shows the validity of this fact for semiprime FQS modules, too.

Theorem 5.4. *Let M be a semiprime* FQS *module. Then, M is an α-DICC module if and only if $k\text{-}\dim M = \alpha$ or $n\text{-}\dim M = \alpha$.*

Proof. By Theorems 3.13 and 5.3, M is a semiprime α-DICC module if and only if S is a semiprime α-DICC ring if and only if $k\text{-}\dim S = \alpha$ or $n\text{-}\dim S = \alpha$, by the above comment. This is also equivalent to either $k\text{-}\dim M = \alpha$ or $n\text{-}\dim M = \alpha$, by Theorem 4.3. □

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S. M. Javdannezhad and N. Shirali
Department of Mathematics
Shahid Chamran University of Ahvaz
Ahvaz
Iran
e-mail: shirali_n@scu.ac.ir;
nasshirali@gmail.com

S. M. Javdannezhad
e-mail: sm.javdannezhad@gmail.com

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